Tutorial 05.

Section B.

Extra note.

A Diff. eq^n is an eq^n which contains derivatives of the unknown.

Consider $f(t,y)$ linear f^n, and ~~cond~~ consider diff. eq^n $\frac{dy}{dt} + p(t)y = g(t)$. & I.F. $= \mu(t)$.

Find $\mu(t) \Longrightarrow$

~~Let~~ Take $\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)\cdot g(t)$.

$\mu(t)$ is chosen s.t. the eq^n is integrable.
i.e. L.H.S. is the derivative of something.

Now we require $\Rightarrow \mu(t)\frac{dy}{dt} + \mu(t)p(t)y = (\mu(t)y)'$

$$\Rightarrow \mu(t)\frac{dy}{dt} + \underbrace{\mu(t)p(t)y}$$

$$= \mu(t)\frac{dy}{dt} + \frac{d\mu(t)}{dt}.y$$

which requires $\frac{d(\mu(t))}{dt} = \mu(t)p(t)$

$$\Rightarrow \frac{1}{\mu(t)}d(\mu(t)) = p(t)\,dt.$$

Integrating $\Longrightarrow \ln|\mu(t)| = \int p(t)\,dt$.
which gives the formula to compute $\mu(t)$,

$$\Rightarrow \mu(t) = \exp\left(\int p(t)\,dt\right)$$

$\therefore$ $\mu(t)$ is Integrating Factor.

1. $\frac{dy}{dt} + p(t)\,y = g(t)$; I.F. $\mu(t) = e^{\int p(t)dt}$.

(a). $\frac{dy}{dx} + 2xy = 0$

$p(x) = 2x$ & $g(x) = 0$

I.F. $\mu(x) = e^{\int p(x)dx} = e^{\int 2x\,dx} = e^{x^2}$ (drop the constant).

Now consider $\frac{dy}{dx} + 2xy = 0 \longrightarrow (1)$

$(1) \times \mu(x). \Rightarrow e^{x^2}\frac{dy}{dx} + 2xye^{x^2} = 0$

$\Rightarrow e^{x^2}dy + 2xe^{x^2}y\,dx = 0$

$\Rightarrow d\left(e^{x^2}y\right) = 0$

$e^{x^2}y = c$; c is a constant.

$y = \frac{c}{e^{x^2}}$ //

1 (b). $(e^x - 1)\frac{dy}{dx} + e^x y = 0$

$$\frac{dy}{dx} + \frac{e^x}{e^x - 1} y = 0 \longrightarrow ① \quad ; (e^x \neq 1).$$

I.F. $\mu(x) = e^{\int p(x)dx} = e^{\int \frac{e^x}{e^x-1}dx} = e^{\ln(e^x - 1)}$

$= (e^x - 1)$

$① \times \mu(x) \Rightarrow (e^x - 1)\frac{dy}{dx} + \frac{e^x y \cancel{(e^x - 1)}}{\cancel{(e^x - 1)}} = 0$

$$(e^x - 1)dy + e^x y\, dx = 0$$

$$d((e^x - 1)y) = 0$$

$$(e^x - 1)y = C \quad ; \text{ C is a constant.}$$

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Try :- $\frac{dy}{dx} + 2xy = 2e^{-x^2}$ ans: $e^{x^2}y = 2x + C.$

$$\frac{dy}{dx} + \cot(x)y = 5e^{\cos x}$$

ans: $y\sin x + 5e^{\cos x} = C$

(4) Separable Diff. eq^n.

(a) $\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}$; $y(0) = -1$.

$$\left[N(y)\frac{dy}{dx} = M(x) \Rightarrow N(y)dy = M(x)dx \atop \Rightarrow \int N(y)dy = \int M(x)dx \right]$$

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}$$

$$\int 2(y-1)dy = \int (3x^2+4x+2)\,dx$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + C \longrightarrow (1).$$

C is a constant

By Initial condition $y(0) = -1$

$\Rightarrow$ $y = -1$ when $x = 0$;

Hence by (1) $1 + 2 = C \Rightarrow C = 3$.

by (1)

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3$$

$$y^2 - 2y + 1 = x^3 + 2x^2 + 2x + 4$$

$$(y-1)^2 = x^3 + 2x^2 + 2x + 4$$

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}.$$

$$\Rightarrow y(x) = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}$$

(4) (a)

By I.C. $y(0) = -1$

$$y(0) = 1 \pm \sqrt{4} = 1 \pm 2 = -1$$

Hence explicity form is

$$y(x) = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}. //$$

Determine the interval of validity.

Here $x^3 + 2x^2 + 2x + 4 \geq 0$

$$\Rightarrow x^2(x+2) + 2(x+2) \geq 0$$

$$\underbrace{(x^2+2)}_{>0}(x+2) \geq 0$$

$$\therefore \quad x + 2 \geq 0$$

$$\Rightarrow \quad x \geq -2 //$$

(6). $M(x,y) + N(x,y)\frac{dy}{dx} = 0 \longrightarrow (1)$

Now $I(x) = e^{\int g(x)dx}$; $g(x) = \left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\right)$

by Hint ; $I(x)M(x,y) + I(x)N(x,y)\frac{dy}{dx} = 0$

$$\Rightarrow \underbrace{e^{\int g(x)dx} M(x,y)}_{M'(x,y)} + \underbrace{e^{\int g(x)dx} N(x,y)}_{N'(x,y).}\frac{dy}{dx} = 0 \longrightarrow (2)$$

Let's check the requirments to be exact.

$$\frac{\partial M'}{\partial y} = \frac{\partial}{\partial y}\left(e^{\int g(x)\,dx} M\right) = e^{\int g(x)\,dx} \frac{\partial M}{\partial y} \longrightarrow (A).$$

$$\frac{\partial N'}{\partial x} = \frac{\partial}{\partial x}\left(e^{\int g(x)\,dx} N\right) = e^{\int g(x)\,dx} \frac{\partial N}{\partial x} + N e^{\int g(x)\,dx} g(x).$$

t.

$$= e^{\int g(x)\,dx}\left[\cancel{\frac{\partial N}{\partial x}} + \cancel{N}\cdot\frac{\left(\frac{\partial M}{\partial y} - \cancel{\frac{\partial N}{\partial x}}\right)}{\cancel{N}}\right]$$

$$= e^{\int g(x)\,dx} \frac{\partial M}{\partial y} \longrightarrow (B)$$

by (A) & (B) $\quad \frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x} \Rightarrow$ exact.

$\therefore$ (2)nd Diff eq^n is exact.

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Try :- Let $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ with $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$,

but $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = -f(y)$ is a function of y alone.

propose a mtd to find the sol^n.

Hint: Multiply both sides by $I(y) = e^{\int f(y)\,dy}$.

Extra: based on Q5.

$(2xy+1)dx + (x^2+4y)dy = 0$

$M\,dx + N\,dy = 0$

$\frac{\partial M}{\partial y} = 2x$, $\frac{\partial N}{\partial x} = 2x \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence exact.

Let $f(x,y)$ s.t. $df = M\,dx + N\,dy$.

$\Rightarrow df = (2xy+1)\,dx + (x^2+4y)\,dy$.

Consider $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

$\Rightarrow \frac{\partial f}{\partial x} = 2xy+1 \longrightarrow (1)$

& $\frac{\partial f}{\partial y} = x^2+4y \longrightarrow (2)$

by (1), $f = x^2y + x + \phi(y) \longrightarrow (3)$

(You can choose (1) or (2))

$\Rightarrow \frac{\partial f}{\partial y} = x^2 + 0 + \frac{\partial \phi(y)}{\partial y}$

$\frac{\partial f}{\partial y} = x^2 + \frac{\partial \phi(y)}{\partial y} = x^2+4y$ (by (2))

$\Rightarrow \frac{\partial \phi(y)}{\partial y} = 4y$.

$\Rightarrow \int \frac{\partial \phi}{\partial y}dy = \int 4y\,dy$.

$\Rightarrow \phi = \frac{4y^2}{2} + C = 2y^2 + C$; C is a constant.

$\longrightarrow$

Hence $f = x^2y + x + 2y^2 + C$ (by ③).

since $df = 0$ ($\because df = Mdx + Ndy = 0$)

$\Rightarrow$ f is a constant.

Hence $x^2y + x + 2y^2 = \underline{\underline{c'}}$. ($c'$ is a constant)

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⑦. (a) $(x^2+y^2+x)dx + xy\,dy = 0$

$$M\,dx + N\,dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(x^2+y^2+x) = 2y$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(xy) = y$$

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{not exact.}$$

$$\therefore \text{Let } g(x) = \left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\right) = \frac{y}{xy} = \frac{1}{x}$$

I.F. $I(x) = e^{\int g(x)dx}$

$$\int g(x)dx = \int \frac{1}{x}dx = \ln|x|$$

I.F. $I(x) = e^{\ln|x|} = x$

Consider $(x^2+y^2+x)dx + xydy = 0$

Now multiply both sides by $I(x)$.

$$x(x^2+y^2+x)dx + x^2ydy = 0 \Leftarrow \text{(exact diff. eq}^n)$$

$$\Rightarrow x^3dx + xy^2dx + x^2dx + x^2ydy = 0$$

$$\Rightarrow \underbrace{xy^2dx + x^2ydy} + x^3dx + x^2dx = 0$$

$$\Rightarrow d\left(\frac{x^2y^2}{2}\right) + d\left(\frac{x^4}{4}\right) + d\left(\frac{x^3}{3}\right) = 0$$

$$\therefore \frac{x^2y^2}{2} + \frac{x^4}{4} + \frac{x^3}{3} = C \quad ; \quad C \text{ is a constant}$$

(d) $8ydx + 8xdy + x^2y^3(4ydx + 5xdy) = 0 \rightarrow 1$

$$\Rightarrow (8y + 4x^2y^4)dx + (8x + 5x^3y^3)dy = 0 \rightarrow 2$$

$$Mdx + Ndy = 0$$

$M(x,y) = 8y + 4x^2y^4$ & $N(x,y) = 8x + 5x^3y^3$

$$\frac{\partial M}{\partial y} = 8 + 16x^2y^3 \quad \& \quad \frac{\partial N}{\partial x} = 8 + 15y^3x^2$$

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \therefore \text{not exact.}$$

Consider; $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{16x^2y^3 - 15y^3x^2}{N}$

$$= \frac{x^2y^3}{N} \leftarrow \text{not the f}^n \text{ alone } x \text{ or } y.$$

Hence we can not use I.F in (Q6).

Now use the another I.F. $\mu = x^{\alpha} y^{\beta}$ when we have power of the x & y only.

$\therefore$ use I.F. $\mu = x^{\alpha} y^{\beta}$

(2) $\times\ x^{\alpha} y^{\beta}$

$$x^{\alpha}y^{\beta}(8y + 4x^2y^4)dx + x^{\alpha}y^{\beta}(8x + 5x^3y^3)dy = 0$$

$$\Big(\underbrace{8x^{\alpha}y^{\beta+1} + 4x^{\alpha+2}y^{\beta+4}}_{M}\Big)dx + \Big(\underbrace{8x^{\alpha+1}y^{\beta} + 5x^{\alpha+3}y^{\beta+3}}_{N}\Big)dy = 0$$

$$\frac{\partial M}{\partial y} = 8x^{\alpha}(\beta+1)y^{\beta} + 4x^{\alpha+2}(\beta+4)y^{\beta+3}$$

$$= 4x^{\alpha}y^{\beta}\left[2(\beta+1) + x^2y^3(\beta+4)\right] \longrightarrow (A)$$

$$\frac{\partial N}{\partial x} = 8y^{\beta}(\alpha+1)x^{\alpha} + 5y^{\beta+3}(\alpha+3)x^{\alpha+2}$$

$$= x^{\alpha}y^{\beta}\left[8(\alpha+1) + 5(\alpha+3)x^2y^3\right] \longrightarrow (B)$$

To exact $\Rightarrow \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

by (A) & (B) $\therefore$ $4x^{\alpha}y^{\beta}\left[2(\beta+1) + x^2y^3(\beta+4)\right] = x^{\alpha}y^{\beta}\left[8(\alpha+1) + 5(\alpha+3)x^2y^3\right]$

Considering coefficients on both sides.

$$4(\beta+4) = 5(\alpha+3) \longrightarrow (a)$$

& $$8(\beta+1) = 8(\alpha+1) \longrightarrow (b).$$

~~by (a) + (b)~~, by (b) $\Rightarrow$ $\alpha = \beta$.

$\cancel{\Rightarrow \alpha = \beta}$. by (a) $\Rightarrow$ $\alpha = 1 = \beta$.

$\therefore \mu = x^{\alpha} y^{\beta} = xy$.

(1) $\times\, xy$

$\Rightarrow xy(8y\,dx + 8x\,dy) + x^3y^4(4y\,dx + 5x\,dy) = 0$

$\Rightarrow 8(xy^2\,dx + x^2y\,dy) + (4x^3y^5\,dx + 5x^4y^4\,dy) = 0$

$\Rightarrow 4\,d(x^2y^2) + d(x^4y^5) = 0$

$\Rightarrow \cancel{4x^2+y}\ 4x^2y^2 + x^4y^5 = c$; c is a constant.

(8) Bernoulli's eq^{n}

Consider form of B.E. $\frac{dy}{dx} + p(x)y = Q(x)y^{n}$; $n>1$

when $n=0$ $\Rightarrow$ first order diff eq^{n}

when $n=1$ $\Rightarrow$ variable separable diff. eq^{n}.

Use substitution $V = y^{1-n}$

(a). $xy - \frac{dy}{dx} = y^{3}e^{-x^{2}}$

Substitution; $\underline{V = y^{1-n}} = y^{1-3} = y^{-2}$

$$\frac{dV}{dx} = -2y^{-3}\frac{dy}{dx} \longrightarrow (A)$$

Now consider $\frac{dy}{dx} - xy = -y^{3}e^{-x^{2}} \longrightarrow (1)$

$(1) \times (-2y^{-3}) \Rightarrow -2y^{-3}\frac{dy}{dx} + 2xy^{-2} = 2e^{-x^{2}}$

by (A) $\Rightarrow$ $\frac{du}{dx} + 2xV = 2e^{-x^{2}}$

Now this is in the form of first order Differential eq^{n}.

Consider $\frac{dv}{dx} + 2xv = 2e^{-x^2} \longrightarrow (2)$

$p(x) = 2(x)$. $\quad Q(x) = 2e^{-x^2}$.

$\therefore$ I.F. $\mu(x) = e^{\int p(x)dx} = e^{\int 2x\,dx} = e^{x^2}$.

$\mu(x) \times (2)$.

$\Rightarrow \quad e^{x^2}\frac{dv}{dx} + 2xe^{x^2}v = 2$

$\Rightarrow \quad d(e^{x^2}v) = 2dx$

$\Rightarrow \quad e^{x^2}v = 2x + C$; c is a constant.

$\Rightarrow \quad e^{x^2}y^{-2} = 2x + C. \quad (\because v = y^{-2})$.

$\Rightarrow \quad \frac{1}{y^2}e^{x^2} = 2x + C$ //

9). Homogeneous Differential Equations.

If we have the form of $\frac{dy}{dx} = f(y/x)$, it is a Hom. Diff. eq^{n}.

ex: $$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1+(y/x)^2}$$

[Homogeneous functions.

If $\phi(\lambda x, \lambda y) = \lambda^n \phi(x,y)$; $\lambda \in \mathbb{R}$,

then $\phi(x,y)$ is a homogeneous f^{n} with order n.

ex: $x^3 - 3xy^2 \Rightarrow$ order 3.

$\sqrt{2x^2 + 3y^2} \Rightarrow$ order 1.

$\cos(x/y) \Rightarrow$ order 0

(a). $$\frac{dy}{dx} = \frac{x^2+y^2}{2xy}.$$

x^2+y^2 & $2xy$ are homogeneous with order 2

Substitution $V = y/x$.

Now $$\frac{dy}{dx} = \frac{1+(y/x)^2}{2(y/x)}$$ $x \neq 0$

~~By $v = y/x \Rightarrow y = vx$.~~

$$\frac{dy}{dx} = \frac{1+v^2}{2v} \longrightarrow (1).$$

By $v = y/x \Rightarrow y = vx$.

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\frac{1+v^2}{2v} = v + x\frac{dv}{dx} \qquad (\text{by } (1))$$

$$\frac{1-v^2}{2v} = x\frac{dv}{dx}$$

$$\Rightarrow \frac{1}{x}\,dx = \frac{2v}{1-v^2}\,dv. \qquad ;\ 1 \neq v^2$$

$$\Rightarrow \ln|x| + c = -\ln|1-v^2|$$

$$\Rightarrow \ln|x(1-v^2)| = c \qquad ;\ c \text{ is a constant.}$$

$$\Rightarrow |x(1-v^2)| = c'$$

$$x\left(1 - \frac{y^2}{x^2}\right) = c''$$

$$x^2 - y^2 = x c'' \qquad c'' \text{ is a constant}$$

9 (c). $\dfrac{dy}{dx} = \dfrac{y^2 + x^4}{x^3}$

Let $y = z^m$.

$$\frac{dy}{dx} = m z^{m-1} \frac{dz}{dx}$$

By substituting, $m z^{m-1} \dfrac{dz}{dx} = \dfrac{(z^m)^2 + x^4}{x^3}$

$$\frac{dz}{dx} = \frac{z^{2m} + x^4}{m z^{m-1} x^3}$$

For homogeneous $\Rightarrow$ ~~$2m = 4$ & $m+2 = 4$~~.

$2m = 4$ & $m - 1 + 3 = 4$

$\Rightarrow m = 2$.

$$\therefore \frac{dz}{dx} = \frac{z^4 + x^4}{2 z x^3} = \frac{(z/x)^4 + 1}{2\frac{z}{x}} \longrightarrow (1)$$

Use substitution

$v = z/x \Rightarrow z = vx \Rightarrow x\dfrac{dv}{dx} + v = \dfrac{dz}{dx} \longrightarrow (2)$

by (1) & (2) $\quad \dfrac{v^4+1}{2v} = x\dfrac{dv}{dx} + v \Rightarrow \dfrac{v^4 - 2v^2 + 1}{2v} = x\dfrac{dv}{dx}$

$\Rightarrow \dfrac{1}{x}dx = \dfrac{2v}{(v^2-1)^2} dv$

$\Rightarrow \ln|x| + c = -(v^2-1)^{-1} = \dfrac{-1}{v^2-1} = \dfrac{1}{1-v^2}$

$\Rightarrow \dfrac{1}{1-(z/x)^2} = \ln|x| + c \Rightarrow \dfrac{x^2}{x^2 - z^2} = \ln|x| + c$ //

Bisection method.

01) Find the starting values a & b s.t.

$f(a) \cdot f(b) < 0$

02) Genarate the sequence $\{C_n\}$ by bisecting the root enclosing intervals $c = \frac{a+b}{2}$

03) If $f(c) \cdot f(b) < 0$ let $a = c$

o/w let $b = c$

04) $c = \frac{a+b}{2}$; continue.

Section c

04)

a) $f(x) = x - e^{-x}$

let $a = 0$. $b = 1$

if $f(a) = a - e^{-a} = -1$

$f(b) = b - e^{-b} = 1 - e^{-1}$

$f(a) \cdot f(b) = -0.6321 < 0$

by intermidete theorem there is a real root between a & b

$\therefore$ interval $[0, 1]$

b) If we want to get corret approximation we do the sinarigo until $\frac{|b-a|}{2^n} < 10^{-3}$

$(\frac{1}{2})^n |b-a| < 10^{-3}$

$(\frac{1}{2})^n (1-0) < 10^{-3}$

$n \approx 10$

c) $a = 0$, $b = 1$

$c = \frac{a+b}{2} = \frac{0+1}{2} = \frac{1}{2}$

$f(a) \cdot f(c) = (-1)(\frac{1}{2} - e^{-1/2})$

$= +0.1065 > 0$

$\therefore$ root between b & c

$\therefore a = c$

New $a = \frac{1}{2}$, $b = 1$

n	a	b	c_n	$f(c_n)$	$\frac{\lvert b-a \rvert}{2}$
1	0	1	1/2	-0.1065	0.5
2	0.5	1	0.75	0.277	0.25
3	0.5	0.75	0.625	0.8973	0.125
4	0.5	0.625	0.5625	-7.28×10^{-3}	0.0625
5	0.5625	0.625	0.59375		0.03125
6	⋮	⋮	⋮	⋮	⋮
⋮			c'	$f(c')$	
10					

Approximated real root for the function $f(x) = x - e^{-x}$ is c', which lies on interval $[0, 1]$