Sequences

We know that the functions can be defined on any subsets of R. As the set of positive integers Z_+ is a subset of R, we can define a function on it in the following manner.
$f: Z_+ \to R$
$f(n) = a_n$

The range of this function is the infinite set $\{a_1, a_2, a_3, \ldots\}$. The set may not necessarily contain infinitely many distinct numbers because some of a_i's could be equal. That set $\{a_1, a_2, a_3, \ldots\}$ is called a sequence. We call a_r the r^{th} term of the sequence.

For our convenience, we shall denote the whole sequence $\{a_1, a_{2,} \ldots\}$ by $\{a_n\}$.

<u>Example 1</u>

Consider the sequence $\{a_n\}$ defined as $a_n = \frac{1}{n}$.

Here, $a_1 = 1$

$$a_2 = \frac{1}{2} = 0.5$$

$$a_3 = \frac{1}{3} = 0.333$$

$$a_4 = \frac{1}{4} = 0.25$$

One can see that as n gets bigger a_n is getting closer to 0. For example, $a_{15} = 0.06667$
$a_{20} = 0.05000$

Because of the behaviour we have seen with the sequence $\{a_n\}$, we suspect that by taking n large enough, we can make a_n as close to 0 as we wish. Because a_{20} is within 0.05 units of 0 and a_{21} is within 0.04762 units of 0, we can expect that a_{22} will be even closer. These observations prepare us for our first definition.

<u>Definition 1</u>

Let $\{a_n\}$ be a sequence. A number L is said to be the limit of $\{a_n\}$ if $\forall\, \varepsilon > 0, \exists\, n_0 \in Z_+$ such that $n > n_0 \Longrightarrow |a_n - L| < \varepsilon$
When this happens, we write $\lim_{n\to\infty} a_n = L$

Now, let us consider the sequence in example 1.

Let $\varepsilon > 0$.Then $\frac{1}{\varepsilon} \in R$.

By Archimedean property, $\exists\, n_0 \in Z_+$ such that $n_0 > \frac{1}{\varepsilon}$

$$n > n_0 \Longrightarrow n > \frac{1}{\varepsilon} \Longrightarrow \frac{1}{n} < \varepsilon$$

$$\Longrightarrow |a_n - 0| = \left|\frac{1}{n} - 0\right| < \varepsilon$$

$\therefore \lim_{n\to\infty} a_n = 0$

Definition 2

A sequence $\{a_n\}$ is said to be convergent if $\exists\, L \in R$ such that $\lim_{n\to\infty} a_n = L$. Because we don't consider ∞ as a real number, this L is understood to be finite.

The sequence in the example 1 is convergent. Not all the sequences are convergent. When a sequence is not convergent, it is said to be divergent.

Example 2 $\quad a_n = n$

Assume $\{a_n\}$ is convergent. Then $\exists\, L \in R$ such that $\lim_{n\to\infty} a_n = L$

Because $\frac{1}{2} > 0, \exists n_0 \in Z_+$ such that

$|n - L| < \frac{1}{2} \;\; \forall\, n > n_0$

$\therefore\ |(n_0 + 1) - L| < \frac{1}{2}$ and $|(n_0 + 3) - L| < \frac{1}{2}$

$2 = |(n_0 + 3 - L) - (n_0 + 1 - L)|$

$\quad \le |n_0 + 3 - L| + |n_0 + 1 - L| < 1$ – contradiction

$\therefore$The sequence $\{a_n\}$ is divergent.

Theorem 1

A convergent sequence has a unique limit.

Proof: Suppose $\exists$ a convergent sequence $\{a_n\}$ which doesn't have a unique limit.

$\therefore$ There are $L, M \in R$ such that $L \ne M$, $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = M$.

Then $|L - M| > 0$.

$\exists n_1 \in Z_+$ such that $n > n_1 \Longrightarrow |a_n - L| < \frac{|L - M|}{4}$

$\exists n_2 \in Z_+$ such that $n > n_2 \Longrightarrow |a_n - M| < \frac{|L - M|}{4}$

Let n_0 be such that $n_0 > n_1$ and $n_0 > n_2$.

$$\begin{aligned} \text{Then } |L - M| &= \left|L - a_{n_0} + a_{n_0} - M\right| \\ &\le \left|a_{n_0} - L\right| + \left|a_{n_0} - M\right| \\ &< \frac{|L - M|}{4} + \frac{|L - M|}{4} = \frac{|L - M|}{2} \end{aligned}$$
- contradiction

$\therefore$The limit is unique.

Theorem 2

Let $\{a_n\}, \{b_n\}$ be two sequences such that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$.

Then, $\lim_{n\to\infty} (a_n + b_n) = L + M$

Proof: Let $\varepsilon > 0$.
$\exists\, n_1 \in Z_+$ such that $n > n_1 \Longrightarrow |a_n - L| < \frac{\varepsilon}{2}$
$\exists\, n_2 \in Z_+$ such that $n > n_2 \Longrightarrow |b_n - M| < \frac{\varepsilon}{2}$
Let $n_0 = Max\,\{n_1, n_2\}$.

$$n > n_0 \Longrightarrow |(a_n + b_n) - (L + M)| \leq |a_n - L| + |b_n - M|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\therefore \lim_{n\to\infty}(a_n + b_n) = L + M$

Theorem 3
Let $\{a_n\}, \{b_n\}$ be 2 sequences such that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$.
Then the following claims are true.

i) $\lim_{n\to\infty}(ra_n) = rL \;\; \forall r \in R$

ii) $\lim_{n\to\infty}(a_n\, b_n) = LM$

iii) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ provided that $M \neq 0$

The proof is an exercise for the student.

Definition 3 Let $\{a_n\}$ be a sequence.
i) If $a_n \leq a_{n+1} \;\; \forall n$, then the sequence is said to be increasing.

ii) If $a_n \geq a_{n+1} \;\; \forall n$, then the sequence is said to be decreasing.

iii) If a sequence is either increasing or decreasing, then it is said to be a monotonic sequence.

If the strict inequalities are being used in i) or ii), then we say that the sequence is strictly increasing or strictly decreasing.

Definition 4 Let $\{a_n\}$ be a sequence.
i) If $\exists A \in R$ such that $a_n \leq A \;\; \forall n$, then $\{a_n\}$ is said to be bounded above.
ii) If $\exists B \in R$ such that $a_n \geq B \;\; \forall n$, then $\{a_n\}$ is said to be bounded below.
iii) If there are $A, B \in R$ such that $B \leq a_n \leq A \;\; \forall n$, then $\{a_n\}$ is said to be bounded.

Theorem 4 Every convergent sequence is bounded.

Proof: Let $\{a_n\}$ be a convergent sequence. Then $\exists L \in R$ such that $\lim_{n\to\infty} a_n = L$

$\therefore \exists n_0 \in Z_+$ such that $n > n_0 \Longrightarrow |a_n - L| < 1 \Longrightarrow |a_n| < |L| + 1$
Let $M = Max\left\{|a_1|, |a_2|, \ldots., \left|a_{n_0}\right|, |L| + 1\right\}$
Then $|a_n| \leq M \;\; \forall n$
$\therefore -M \leq a_n \leq M \;\; \forall n$
$\therefore \{a_n\}$ is bounded.

The student should realize that this theorem provides us with another way to make the claim we made in the example 2.

The converse of theorem 4 is false.
Let $a_n = (-1)^n$
Then $-2 < a_n < 2 \quad \forall n$
It will be easy proving that the sequence $\{a_n\}$ is divergent.

Suppose a sequence $\{a_n\}$ is bounded above. Then from the completeness axiom (chapter 2), we can say that the supremum of $\{a_n | \, n \in Z_+\}$ exists. In the same way, if $\{a_n\}$ is bounded below, we can say that the infimum of $\{a_n | \, n \in Z_+\}$ exists.

<u>Theorem 5</u> If $\{a_n\}$ is increasing and bounded above, then $\lim_{n\to\infty} a_n = L$ where $L = \sup\{a_n | n \in Z_+\}$.

Proof is a tutorial exercise.

<u>Theorem 6</u> If $\{a_n\}$ is decreasing and bounded below, then $\lim_{n\to\infty} a_n = M$ where $M = \inf\{a_n | \, n \in Z_+\}$.

Proof is a tutorial exercise.

<u>Example 3</u>

$b_n = 1 + \frac{1}{n}$

Clearly, $\{b_n\}$ is decreasing.

$$1 \le b_n \quad \forall n$$

So $\{b_n\}$ is bounded below.
The student should be able to prove that $1 = \inf\{b_n\}$
$\therefore$ From the above theorem,

$$\lim_{n\to\infty} b_n = 1$$

<u>Cauchy Sequences</u>

We know that in a convergent sequence the terms get closer and closer to its limit. It may also be possible to have a sequence whose terms get closer and closer together and, we should be able to explain the behaviour of those terms without even mentioning a limit. The following definition will explain such a behaviour.

<u>Definition 5</u>

A sequence $\{a_n\}$ is said to be a Cauchy sequence if and only if $\forall \, \varepsilon > 0$, $\exists \, n_0 \in Z_+$ such that $m, n > n_0 \Rightarrow |a_m - a_n| < \varepsilon$

If the terms are getting closer and closer together, one would expect those to get closer and closer to a limit. In other words, one would expect such a sequence to be convergent. That fact needs to be proven and we will devote the next two theorems for that purpose.

Theorem 7
Every Cauchy sequence is bounded.

Proof: Let $\{a_n\}$ be a Cauchy sequence.
$\exists n_0 \in Z_+$ such that
$m, n > n_0 \Longrightarrow |a_m - a_n| < 1$
$\therefore \forall\, m > n_0, |a_m - a_{n_0+1}| < 1$
$\therefore\; m > n_0 \Longrightarrow |a_m| - |a_{n_0+1}| < 1$
$\Longrightarrow |a_m| < |a_{n_0+1}| + 1$

Let $M = \text{Max}\{|a_1|, |a_2|, \ldots\ldots, |a_{n_0}|, |a_{n_0+1}| + 1\}$
Then $|a_m| < M \;\; \forall\, m \in Z_+$
$\therefore\; \{a_m\}$ is bounded.

Theorem 8 Let $\{a_n\}$ be a sequence. Then $\{a_n\}$ is convergent if and only if $\{a_n\}$ is a Cauchy sequence.

Proof: Suppose $\{a_n\}$ is convergent.
Let $\varepsilon > 0$. Then $\exists n_0 \in Z_+$ such that

$n > n_0 \Longrightarrow |a_n - L| < \frac{\varepsilon}{2}$ where L is the limit of $\{a_n\}$.

Take m, n such that $m > n_0$, $n > n_0$.
$|a_m - a_n| = |a_m - L + L - a_n|$
$\leq |a_m - L| + |a_n - L| < \varepsilon$
$\therefore \{a_n\}$ is a Cauchy sequence.

Proving the converse is not necessary for engineering students.
Theorem 7 and some other material will be needed for that proof.

Because of this theorem, we can establish the convergence of a sequence without knowing the limit.

Example 4: Suppose there is a sequence with the property

$$a_{n+2} = \frac{a_n + a_{n+1}}{2} \quad \forall n \in Z_+$$

For this sequence,

$$a_{i+1} - a_i = \frac{a_{i-1} + a_i}{2} - a_i = \frac{a_{i-1} - a_i}{2}$$

$$= \frac{a_{i-1} - \left(\frac{a_{i-2} + a_{i-1}}{2}\right)}{2} = \frac{a_{i-1} - a_{i-2}}{2^2}$$

It follows that

$$|a_{i+1} - a_i| = \frac{|a_i - a_{i-1}|}{2} = \frac{|a_{i-1} - a_{i-2}|}{2^2}$$

$$= \text{...........} = \frac{|a_2 - a_1|}{2^{i-1}}$$

If $m = n$, then $|a_m - a_n| = 0$

So, without loss of generality, take $m > n$.

$$a_m - a_n = \sum_{i=n}^{m-1} (a_{i+1} - a_i) \Longrightarrow$$

$$|a_m - a_n| \leq \sum_{i=n}^{m-1} |a_{i+1} - a_i|$$

$$= |a_2 - a_1| \sum_{i=n}^{m-1} \left(\frac{1}{2}\right)^{i-1} < |a_2 - a_1| \left(\frac{1}{2}\right)^{n-2}$$

Let $\varepsilon > 0$.
If $a_2 = a_1$, then $|a_m - a_n| < \varepsilon$
So let $a_2 \neq a_1$
Choose n_0 such that

$$2^{n_0-2} > \frac{|a_2 - a_1|}{\varepsilon}$$

$$n > n_0 \Longrightarrow 2^{n-2} > \frac{|a_2 - a_1|}{\varepsilon} \Longrightarrow$$

$$\frac{|a_2 - a_1|}{2^{n-2}} < \varepsilon \Longrightarrow |a_m - a_n| < \varepsilon$$

$\therefore$ $\{a_n\}$ is a Cauchy sequence.

Hence, $\{a_n\}$ is convergent.

Note that we were able to establish the convergence without knowing the limit.

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