1. Find $cos\theta$ where θ is the angle between:
   a. $u_1 = (1,3,-5,4)\ and\ u_2 = (2,-3,4-1)$ in $\mathbb{R}^4$.
   b. $A = \begin{bmatrix} 9 & 8 & 7 \\ 1 & 2 & 3 \end{bmatrix}\ and\ B = \begin{bmatrix} 6 & 5 & 4 \\ 1 & 2 & 5 \end{bmatrix}$ where $< A, B > tr(B^T A)$.
2. Find k so that $u = (1,2,k,4)$ and $v = (3,k,7,-4)$ in $\mathbb{R}^4$ are orthogonal.
3. Let W be the subspace of in $\mathbb{R}^5$ spanned by $u = (1,2,3,-1,2)$ and $v = (2,4,7,2,-1)$. Find a basis of the orthogonal complement $W^{\perp}$ of W.
4. Suppose $\{u_1, u_2, \dots u_r\}$ is an orthogonal set of vectors.Then

$$\| u_1 + u_2 + u_3 + \cdots u_r \|^2 = \| u_1 \|^2 + \| u_2 \|^2 + \cdots + \| u_r \|^2.$$

5. Consider the subspace U spanned by the vectors

$v_1 = (1,1,1,1), v_2 = (1,1,2,4)$ and $v_3 = (1,2,-4,-3)$ .Find

   a. An orthogonal basis of U b. an orthonormal basis of U.
6. Find the characteristic polynomial of $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & 3 & 9 \end{bmatrix}$
7. Find the minimal polynomial of the matrix $M = \begin{bmatrix} 9 & -1 & 5 & 7 \\ 8 & 3 & 2 & -4 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & -1 & 8 \end{bmatrix}$
8. Find the characteristic polynomial of each of the following

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 2 & 5 \\ 2 & 1 & 5 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 5 & 1 & 1 \\ 1 & 4 & 2 & 2 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & 2 & 3 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 3 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

9. Find the characteristic polynomial of each of the following linear operators.

   i $F: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x,y) = (3x + 5y\ ,2x - 7y)$.

   ii $A: V \to V$ defined by $A(f) = \frac{df}{dt}$,Where V is the space of functions with basis $S = \{sint\ , cost\}$.

10. Show that a matrix A and its transpose A^T have the same characteristic polynomial.

Let $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$

   a. Find all eigenvalues and corresponding eigenvectors.
   b. Find a nonsingular matrix P such that $D = P^{-1}AP$ is diagonal ,and P^{-1}.
   c. Find A^6.

11. Let $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$

   a. Find a maximum set of s of linearly independent eigenvectors of A.
   b. Find the characteristic polynomial of A.
   c. Is A diagonalizable? If yes find P such that $D = P^{-1}AP$ is diagonal.

12. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (2x + y - 2z, 2x + 3y - 4z, x + y - z)$.Find all eigenvalues of T, and find a basis of each eigenspace.Is T diagonalizable?if so ,find the basis s of $\mathbb{R}^3$ that diagonalizes T,and find its diagonal representation D.

13. Prove the following are equivalent

i. The scalar λ is an eigenvalue of A
ii. The matrix $\lambda I - A$ is singular.
iii. The scalar λ is a root of the charactorictic polynomial of A.

14. Let $B = \begin{bmatrix} 11 & -8 & 4 \\ -8 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix}$

a. Find all eigenvalues of B
b. Find a maximum set S of nonzero orthogonal eigenvectors of B.
c. Find an orthogonal matrix P such that $D = P^{-1}BP$ is orthogonal.

15. Find the minimal polynomial of each matrices.

$$M = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} \quad M' = \begin{bmatrix} 2 & 7 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix} \quad A = \begin{bmatrix} 2 & 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$$

16 Consider the vectors

$$\underline{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \underline{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \underline{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \underline{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \underline{w}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \underline{w}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

(a) Show that each of the sets $B = \{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ and $\hat{B} = \{\underline{w}_1, \underline{w}_2, \underline{w}_3\}$ is a basis for $\mathbb{R}^3$.

(b) Write down the matrix A_T of the linear transformation T given by $(\underline{e}_1) = \underline{v}_1$, $T(\underline{e}_2) = \underline{v}_2$ and $T(\underline{e}_3) = \underline{v}_3$, where $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ is the standard basis of $\mathbb{R}^3$.
Express $T(\text{x})$ for $x = (x, y, z)^T$ as a vector in $\mathbb{R}^3$ in terms of x, y, z.

(c) Write down the matrix A_S of the linear trans formation S given by $(\underline{v}_1) = \underline{e}_1$, $S(\underline{v}_2) = \underline{e}_2$, $S(\underline{v}_3) = \underline{e}_3$. What is the relationship between S and T?

17. (a) Let V be a subspace of R^n . Explain what means for a set of vectors in $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r\}$ in V to be

(i) linearly independent
(ii) a spanning set for V
(iii) a basis of V

Define the dimension of V .

Prove that if $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r\}$ is a basis for V , then every vector $\underline{v}$ in V can be expressed uniquely as a linear combination of $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r$.

(b) Let $\underline{u_1} = (1,1,1,2)$, $\underline{u_2} = (1,2,3,1)$ and $\underline{u_3} = (0,1,2,-1)$ be vectors in $\mathbb{R}^4$ and $U = span\{\underline{u_1}, \underline{u_2}, \underline{u_3}\}$.

(i) Find a basis for U. What is the dimension of U ?

(ii) Determine which of the vectors $\underline{v_1} = (2,1,0,5)$ and $\underline{v_2} = (1,2,1,1)$ belong to U. For the case when $\underline{v_i} \in U, i = 1,2$, express $\underline{v_i}$ as a linear combination of the vectors of the basis for U .

(iii) Let $\underline{w_1} = (2,0,0,3)$ and $\underline{w_2} = (1,0,1,0)$ and $W = span\{\underline{w_1}, \underline{w_2}\}$. Find the bases and their dimensions for W and $U + W$.

(iv) Use the dimension theorem to find the dimension of $U \cap W$.

18. Define the following terms for a vector space V over a field F :
(i) an inner product space
(ii) norm of a vector in V
(ii) orthonormal vectors

(a) Consider $u = (x_1, x_2)$ and $v = (y_1, y_2)$ in $\mathbb{R}^2$.
For what values of k is $\langle u, v \rangle = x_1 y_1 - 3x_1 y_2 - 3x_2 y_1 + kx_2 y_2$ an inner product on $\mathbb{R}^2$.

(b) Let $\{u_1, u_2, ..., u_r\}$ be an orthonormal set in V. Show that for any $v \in V$, the vector $w = v - (v, u_1)u_1 - ... - (v, u_r)u_r$ is orthogonal to each of the u_i.

(c) Obtain an orthonormal basis for the subspace of $\mathbb{R}^4$ generated by $(1,1,1,1), (1,-1,0,0)$ and $(0,0,1,-1)$ with respect to the standard inner product

19 (a) Let U and W be vector spaces over a field K.

(i) Explain what it means for $T : U \to W$ to be a linear transformation.

(ii) Define the Kernel, Ker$T$ and image , Im T of of T.

(iii) Show that T is one-one if and only if $\ker T = \{\underline{0}\}$.

(b) Determine whether there is a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1,1) = (-1,0,1)$, $T(1,0) = (3,2,1)$ and $T(3,1) = (5,0,-2)$

(c) Consider the basis $B = \{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ for $\mathbb{R}^3$; where $\underline{v_1} = (1,1,0)$, $\underline{v_2} = (1,0,1)$ and $\underline{v_3} = (0,1,1)$.

Show that the linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^2$ such that $T(\underline{v_1}) = (2,1)$,

$T(\underline{v_2}) = (1,-1)$ and $T(\underline{v_3}) = (0,0)$ is

$$T(x,y,z) = \left(\frac{3}{2}x + \frac{1}{2}y - \frac{1}{2}z\,,\, y - z\right).$$

Find Kernel of T and image of T.

Verify the Rank –Nullity theorem.

(d) Determine whether the linear transformation $T : M_{2\times 2} \to \mathbb{R}$ defined by

$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a+b+c+d$ is one-one.

20 (a) Let A and B two square matrices. Show that AB and BA have same eigen values.

Consider the matrices $A = \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 0 \\ 6 & 1 \end{pmatrix}$.

(i)Compute AB and BA .

(ii)Find the eigen values of AB and hence deduce the eigenvalues of BA.

(iii)Find the eigenvectors of AB and BA and show that although that matrices have same eigenvalues, their eigen vectors are not same.

(iv)Determine which of them are diagonalizable.

(b) For the matrix $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & -6 \\ 2 & -2 & 3 \end{pmatrix}$ find P such that $P^{-1}AP$ is a diagonal matrix.

Hence compute A^2 and using the Cayley-Hamilton theorem find A^{-1}.

21 (a) Define a subspace W of a vector space V.

Let A be a $m \times n$ matrix and $U = \{\underline{x} \in \mathbb{R}^n \mid A\underline{x} = \underline{0}\}$. Show that U is a subspace of $\mathbb{R}^n$.

(b) Let $\underline{u_1} = (1,1,1,2)$, $\underline{u_2} = (1,2,3,1)$ and $\underline{u_3} = (0,1,2,-1)$ be vectors in $\mathbb{R}^4$ and

$U = \text{span}\{\underline{u_1}, \underline{u_2}, \underline{u_3}\}$.

(i) Find a basis for U. What is the dimension of U?

(ii) Determine whether the vectors $\underline{v_1} = (2,1,0,5)$ and $\underline{v_2} = (1,2,1,1)$ belong to U.For the case when $\underline{v_i} \in U$, $i = 1,2$, express $\underline{v_i}$ as a linear combination of the vectors of the basis for U.

(iii) Let $\underline{w_1} = (2,0,0,3)$, $\underline{w_2} = (1,0,1,0)$ and $W = \text{span}\{\underline{w_1}, \underline{w_2}\}$. Find the bases and their dimensions of W and $U + W$.

(iv) Use the dimension theorem to find the dimension of $U \cap W$.

22. (a) Given that $\begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}-3\\0\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\1\\0\end{pmatrix}$ are eigen vectors of the matrix

$A = \begin{pmatrix}1 & -2 & -6\\2 & 5 & 6\\-2 & -2 & -3\end{pmatrix}$.Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

(b) Consider the matrix $A = \begin{pmatrix}1 & 1 & 0 & 0\\-1 & -1 & 0 & 0\\-2 & -2 & 2 & 1\\1 & 1 & -1 & 0\end{pmatrix}$.

(i) Find the minimum polynomial of A.

23. Determine whether W is a subspace or not of the indicated vector space V.

(i) $W = \{(a, b, c): a + b + c = 0,\ a, b, c \in \mathbb{R}\}$, $V = \mathbb{R}^3$.

(ii) $W = \{(a, b, c): a^2 + b^2 + c^2 \leq 1,\ a, b, c \in \mathbb{R}\}$, $V = \mathbb{R}^3$.

(iii) $W = \left\{A: A^T = A,\ A = (a_{ij})_{3\times3}\right\}$, $V =$ all 3×3 matrices.

(iv) $W = \left\{A \in V : AT = TA, A = (a_{ij})_{3\times3}\right\}$ where T is a given matrix.

$V =$ all 3×3 matrices

(v) $W = \{f : f(7) = f(1)\}$, V=all functions from $\mathbb{R}$ to $\mathbb{R}$.

24. Explain why the matrix $C = \begin{pmatrix}5 & 0 & 4\\a & -1 & b\\2 & 0 & 3\end{pmatrix}$ can be diagonalized for any values of $a, b \in \mathbb{R}$.

25. Find the eigenvalues of the matrices $A = \begin{pmatrix}1 & 1 & 1\\0 & 1 & -1\\1 & 0 & 2\end{pmatrix}$ and $B = \begin{pmatrix}-2 & 1 & -2\\-1 & 0 & 1\\2 & 1 & 2\end{pmatrix}$ and show that neither matrix can be diagonalized over the real numbers.

26. Let $= \begin{pmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{pmatrix}$. Find the characteristic polynomial and the minimum polynomial of the matrix.

27. Find the characteristic equation of the matrix $= \begin{pmatrix}3 & -1 & 2\\5 & -3 & 5\\1 & -1 & 2\end{pmatrix}$. Find the eigenvalues (which are integers) and corresponding eigenvectors of B. Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of the matrix B. Find an invertible matrix P and a diagonal matrix D such that $P^{-1}BP = D$. Check your answer for P by showing $BP = PD$. Then calculate P^{-1} and check that $P^{-1}BP = D$.

28. Diagonalize the matrix $A = \begin{pmatrix}0 & 0 & -2\\1 & 2 & 1\\1 & 0 & 3\end{pmatrix}$. Describe the eigenspace of each eigenvalue.

29. Decide which of the given matrices is similar to a diagonal matrix and which is not. If the matrix is diagonalizable , find both the diagonal matrix and non-singular matrix P that diagonalizes it.

(i) $\begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$ (ii) $\begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix}$ (iii) $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

(iv) $\begin{pmatrix} -1 & 0 & 2 \\ 1 & 2 & 1 \\ 2 & 0 & -1 \end{pmatrix}$ (v) $\begin{pmatrix} 2 & 0 & 1 \\ 3 & 3 & 3 \\ 1 & 0 & 2 \end{pmatrix}$ (vi) $\begin{pmatrix} 3 & 11 & 3 \\ 0 & -4 & -3 \\ 0 & 0 & 6 \end{pmatrix}$

(vii) $\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (viii) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -2 \\ 0 & 3 & 1 & 3 \\ 0 & -2 & 0 & -1 \end{pmatrix}$ (ix) $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & -2 \\ 0 & 3 & 2 & 3 \\ 0 & -2 & 0 & -1 \end{pmatrix}$.

30. Find the minimum polynomial of each of the following matrices:

(i) $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 8 & 9 & 9 \\ 3 & 2 & 3 \\ -9 & -9 & -10 \end{pmatrix}$ (iii) $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & -2 \\ 0 & 3 & 2 & 3 \\ 0 & -2 & 0 & -1 \end{pmatrix}$.

31. Find the minimum polynomials of the following matrices:

$$A = \begin{pmatrix} 2 & 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

32. For each of the following 2×2 matrices , find all eigenvalues and eigenspaces for each eigenvalue of each matrix ; if possible , diagonalize the matrix:

(a) $\begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$.

33. For each of the following 3×3 matrices , find all eigenvalues and eigenspaces for each eigenvalue of each matrix ; if possible , diagonalize the matrix:

(a) $\begin{pmatrix} -2 & 9 & -6 \\ 1 & -2 & 0 \\ 3 & -9 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -1 & -1 \\ 0 & 3 & 2 \\ -1 & 1 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

34. Consider the matrix $A = \begin{pmatrix} -10 & 6 & 3 \\ -26 & 16 & 8 \\ 16 & -10 & -5 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -6 & 16 \\ 0 & 17 & 45 \\ 0 & -6 & 16 \end{pmatrix}$.

(i) Show that A and B have the same eigenvalues.

(ii) Reduce A and B to the same diagonal matrix.

(iii) Explain why there is an invertible matrix R such that $R^{-1}AR = B$.

35. Find A^8 and B^8 , where A and B are the two matrices in problem 3.

36. Suppose that $\theta \in \mathbb{R}$ is not an integer multiple of . Show that the matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ does not have an eigenvector in $\mathbb{R}^2$.

37. Consider the matrix $= \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$, where $\theta \in \mathbb{R}$.

(i) Show that A has an eigenvector in $\mathbb{R}^2$ with eigenvalue 1.

(ii) Show that any vector $\underline{v} \in \mathbb{R}^2$ perpendicular to the eigenvector in part(a) must satisfy $A\underline{v} = -\underline{v}$.

38. Let $a \in \mathbb{R}$ be non-zero. Show that the matrix $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ cannot be diagonalized.

39. Find the minimum polynomials of each of the following matrices:

$$A = \begin{pmatrix} \lambda & a \\ 0 & \lambda \end{pmatrix}, \quad B = \begin{pmatrix} \lambda & a & 0 \\ 0 & \lambda & a \\ 0 & 0 & \lambda \end{pmatrix}, \quad C = \begin{pmatrix} \lambda & a & 0 & 0 \\ 0 & \lambda & a & 0 \\ 0 & 0 & \lambda & a \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \text{ where } a \neq 0.$$

40. Define a subspace W of a vector space V .

(a) Let A be an $m \times n$ matrix and $U = \{\underline{x} \in \mathsf{R}^n \mid A\underline{x} = \underline{0}\}$. Show that U is a subspace of R^n.

If $A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ -3 & -6 & 0 & 6 \\ 1 & 2 & 2 & 0 \end{pmatrix}$, then show that the subspace $U = \{\underline{x} \in \mathsf{R}^4 \mid A\underline{x} = \underline{0}\}$ is given by $U = \{(2r-2s, s, -r, r) \mid r, s \in \mathsf{R}\}$.

(b) Show that the set of all polynomials of degree exactly 5 is not a subspace of P_5, the vector space of all polynomials of degree less or equal 5 .

(c) Suppose that U and W are subspaces of a vector space V .
Define the sum $V = U + W$ and the direct sum $V = U \oplus W$.

Let $W_1 = \{(a,b,0,0) \mid a,b \in \mathsf{R}\}$ and $W_2 = \{(0,0,c,d) \mid c,d \in \mathsf{R}\}$ are two subspaces of R^4. Show that $\mathsf{R}^4 = W_1 \oplus W_2$.

41. (a) Let V be a subspace of R^n. Explain what it means for a set of vectors in $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r\}$ in V to be

(iv) linearly independent
(v) a spanning set for V
(vi) a basis of V

Define the dimension of V.

Prove that if $\{\underline{v}_1, \underline{v}_2, ..., \underline{v}_r\}$ is a basis for V, then every vector $\underline{v}$ in V can be expressed uniquely as a linear combination of $\underline{v}_1, \underline{v}_2, ..., \underline{v}_r$.

(b) Let $\underline{u}_1 = (1,1,1,2)$, $\underline{u}_2 = (1,2,3,1)$ and $\underline{u}_3 = (0,1,2,-1)$ be vectors in R^4 and $U = span\{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$.

(i) Find a basis for U. What is the dimension of U?

(ii) Determine which of the vectors $\underline{v}_1 = (2,1,0,5)$ and $\underline{v}_2 = (1,2,1,1)$ belong to U. For the case when $\underline{v}_i \in U, i = 1,2$, express $\underline{v}_i$ as a linear combination of the vectors of the basis for U.

(iii) Let $\underline{w}_1 = (2,0,0,3)$ and $\underline{w}_2 = (1,0,1,0)$ and $W = span\{\underline{w}_1, \underline{w}_2\}$. Find the bases and their dimensions for W and $U + W$.

(vii) Use the dimension theorem to find the dimension of $U \cap W$.

42 Show that $V = \mathbb{R}^2$ is not a vector space over $\mathbb{R}$ with respect to the following operations:

(i) $(a, b) + (c, d) = (a, b)$ and $k(a, b) = (ka, kb)$.

(ii) $(a, b) + (c, d) = (a + c, b + d)$ and $k(a, b) = (k^2 a, k^2 b)$

43 Determine whether W is a subspace or not of the indicated vector space V.

(i) $W = \{(a, b, c): a + b + c = 0,\ a, b, c \in \mathbb{R}\}$, $V = \mathbb{R}^3$.

(ii) $W = \{(a, b, c): a^2 + b^2 + c^2 \leq 1,\ a, b, c \in \mathbb{R}\}$, $V = \mathbb{R}^3$.

(iii) $W = \left\{A: A^T = A,\ A = (a_{ij})_{3\times3}\right\}$, $V =$ all 3×3 matrices.

(iv) $W = \left\{A \in V : AT = TA, A = (a_{ij})_{3\times3}\right\}$ where T is a given matrix. $V =$ all 3×3 matrices

(v) $W = \{f : f(7) = f(1)\}$, V=all functions from $\mathbb{R}$ to $\mathbb{R}$.

44

(a)Define a subspace W of a vector space V.

Let A be a $m \times n$ matrix and $U = \{\underline{x} \in \mathbb{R}^n \mid A\underline{x} = \underline{0}\}$. Show that U is a subspace of $\mathbb{R}^n$.

(b) Let $\underline{u}_1 = (1,1,1,2)$, $\underline{u}_2 = (1,2,3,1)$ and $\underline{u}_3 = (0,1,2,-1)$ be vectors in $\mathbb{R}^4$ and $U = \text{span}\{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$.

(v) Find a basis for U. What is the dimension of U?

(vi) Determine whether the vectors $\underline{v}_1 = (2,1,0,5)$ and $\underline{v}_2 = (1,2,1,1)$ belong to U.For the case when $\underline{v}_i \in U$, $i = 1,2$, express $\underline{v}_i$ as a linear combination of the vectors of the basis for U.

(vii) Let $\underline{w}_1 = (2,0,0,3), \underline{w}_2 = (1,0,1,0)$ and $W = \text{span}\{\underline{w}_1, \underline{w}_2\}$. Find the bases and their dimensions of W and $U + W$.

(viii) Use the dimension theorem to find the dimension of $U \cap W$.

45

a). Examine the linear independence or dependence of the following sets of vectors.

I. $\{[2 - 1\ 4], [0\ 1\ 2], [6 - 1\ 14], [4\ 0\ 12]\}$

II. $\{[3\ 2\ 4], [1\ 0\ 2], [1 - 1 - 1]\}$

III. $\{[1\ 2\ 3\ 4], [0\ 1 - 1\ 2], [1\ 5\ 1\ 8], [3\ 7\ 8\ 14]\}$

b). Show that the following sets of vectors constitute a basis of $\mathbb{R}^3$

I. $\{[2\ 3\ 4], [0\ 1\ 2], [-1\ 1\ -1]\}$

II. $\{[1 - 1\ 0], [3\ 0\ 0], [0\ \ 2\ 1]\}$

46. Consider the following subspaces of $\mathbb{R}^3$.

$$U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| x - y + 2z = 0 \right\}, W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| 3x + 2y + z = 0 \right\}$$

Find a basis of the subspace $U \cap W$.

For what value(s) of c are the vectors $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} c \\ 1 \\ 5 \end{pmatrix}$ linearly independantant? If c=0 ,explain why the set $B = \{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$

47. Consider $A = \begin{pmatrix} 4 & -3 & -3 \\ 5 & -4 & -5 \\ -1 & 1 & 2 \end{pmatrix}, v = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

find the rank of the matrix A.Find a general solution of the system of equation $AX = 0$.write down a basis for the null space of A,N(A).

determine whether the function $T: \mathbb{R}^2 \to \mathbb{R}^2$ which is defined by $T(x, y) = (x^2, y)$ is linear?

48. Let F be the vector space of all functions from $\mathbb{R}$ in to $\mathbb{R}$ with pointwise addition and scalar multiplication.

(a) Which of the following sets are subspaces of F

$S_1 = \{f \in F | f(0) = 1\}$ $S_2 = \{f \in F \,| f(1) = 0\}$

(b) Show that the set $s_3 = \{f \in F | \text{f is differentiable and } f' - f = 0\}$ is a subspace of F.

(c)

49. Let U=span$\{(1,3,-2,2,3), (1,4,-3,4,2), (2,3,-1,-2,9)\}$

V=span$\{(1,3,0,2,1), (1,5,-6,6,3), (2,5,3,2,1)\}$

a).Find a basis and the dimension of $U + W$.

b).Find a homogeneous system whose solution space is U.

c). Find a homogeneous system whose solution space is W.

50. Prove that the set $H = \{\begin{pmatrix} 2t \\ t \\ 3t \end{pmatrix} : t \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.
Show that every vector $\underline{w} \in H$ is a unique linear combination of the vectors $v_1 = (1,0,-1)^T$, $v_2 = (0,1,5)^T$. Is $\{v_1, v_2\}$ a basis of the subspace H? if yes ,state why.if no,write down a basis of H.state the dimension of H

51. Find a basis for the range and a basis for the kernel of the following linear transformations:

(i) $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 3x_2 \\ x_2 \end{bmatrix}$

(ii) $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ -x_2 \\ 0 \end{bmatrix}$

(iii) $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_3 - x_2 \\ 0 \\ x_3 - x_1 \end{bmatrix}$

(iv) $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_2 + x_3 \\ x_3 - x_1 \end{bmatrix}$

(v) $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 8 & 1 \\ -1 & 0 & 5 & 0 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

Also confirm the Rank-Nullity theorem for the above transformations.

52. Suppose that U is a vector space of dimension 3 with a basis $B = \{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$. Also suppose that
$T: U \to U$ is a linear transformation with $T(\underline{u}_1) = \underline{u}_3$, $T(\underline{u}_2) = \underline{u}_2$ and $T(\underline{u}_3) = \underline{u}_1$.
Prove that Ker $T = \{\underline{0}\}$.

53. Prove that following are equivalent:
(i) The kernel of the linear transformation $T: U \to V$ is equal to $\{\underline{0}_U\}$.
(ii) T is one-one.
(iii) The set of vectors $\{T(\underline{u}_i) \mid i = 1,2, \ldots, n\}$ is a basis for the range of T in V.

54. Find the row rank of A by finding a row echelon form for U. Then find the column rank of A by finding a row echelon form for A^T.

(i) $\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 7 \\ 1 & 2 & 3 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 3 \end{pmatrix}$ (iv) $\begin{pmatrix} 1 & 2 & -1 \\ -1 & 4 & 3 \\ 3 & 0 & 1 \end{pmatrix}$

(v) $\begin{pmatrix} 1 & 0 & -1 & 0 \\ 3 & -1 & 1 & 2 \\ 3 & 0 & 1 & 0 \end{pmatrix}$ (vi) $\begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$ (vii) $\begin{pmatrix} -1 & 1 & 0 & 5 \\ 0 & 1 & -2 & -1 \\ 1 & 0 & -3 & 0 \\ -1 & 0 & -2 & 0 \end{pmatrix}$

55). Prove that the set of all solutions to the two simultaneous linear equations $x_1 - 2x_2 + 5x_3 = 0$ and $x_1 + x_2 - x_3 = 0$ forms a subspace W of $\overline{\mathbb{R}}^3$. Find a basis of W .What is the dimension of W?