The Number System

Concept of a Set

A collection of objects is called a set. A set is usually denoted by a capital English letter. For example, let us say that four men: David, Anthony, James and John are in a car. Then the set of men in that car is given by   A = { David, Anthony, James, John}

Suppose the ages of those four men are as follows:

David is	45
Anthony is	48
James is	36
John is	41

Then the set of ages of the men in that car is given by
B = { 45, 48, 36, 41 }

In this text, all the sets we consider will have nothing but numbers. The set B above has 4 elements. Hence, it is a finite set. A set could be finite or infinite.

Counting

In real life, counting is an essential activity. Entering a lecture hall, one may be interested in the number of students in that hall. Or one may be interested in the number of cars entering through a university gate during a particular hour.

Not only in our modern times, even before the advent of civilization, there was a need for counting. Ancient shepherds knew how to count the number of animals they had. A shepherd owning 30 sheep was considered richer than one owning 20 sheep.

The set { 1,2,3,………………..} was created as a result of this need. That set is also called the set of counting numbers. Note that this is an infinite set. It is possible that we may not have anything to count in a particular situation. For example, the lecture hall may be empty after lectures. To accommodate situations like that , the set of counting numbers was extended to the set of natural numbers N where
N = { 0,1,2,3,……………}

The Number Line

Imagine a horizontal line of infinite length and draw a portion of it on paper. Mark a point on it as 0. (Once we complete the construction of the number line, this 0 will be the center of that line. But , nobody can say how to divide a line of infinite length into two equal segments. Therefore, the exact location on which you place your zero doesn't really matter)

Now, mark the other natural numbers on this line so that the values increase as we move towards right.

```
-------------|-------|--------|-------|-------->
             0       1        2       3
```

For each $a \in N$, locate the position on the line which is 'a' units away from 0 but, on the left hand side. Mark that location as –a . The negative integers are created in this manner. Now , the construction of the number line is complete.

```
-----------------|--------|--------|--------|-------|--------|--------|----------------->
    .........–3       -2       -1       0       1       2       3......
```

The values appearing on this line are called the integers and the set of all those values is denoted by Z. We would refrain from making a statement like "a is greater than zero" at this point. The inequalities will be discussed in the next chapter and that type of statement can be made after learning the inequalities. For the time being, any number which appears to the right hand side of 0 on the number line is called a positive integer. If an integer appears to the left hand side of 0 on that line, that would be called a negative integer. Those sets will be denoted by the symbols

$$Z_+ = \{1, 2, 3, \ldots\ldots\}$$
$$Z_- = \{-1, -2, -3, \ldots\ldots\}$$

On those integers, we can perform two operations known as addition and multiplication. To perform the operation of addition, we will use the symbol + and to perform multiplication the symbol • will be used.

When a and b are two nonzero integers, there are 4 possibilities.

1. a is positive and b is positive
2. a is positive and b is negative
3. a is negative and b is positive
4. a is negative and b is negative

The number resulting from the addition of a and b is denoted by a + b . It is possible to explain how we would arrive at a + b for each of the 4 cases listed above. But , that would make this chapter unnecessarily long. As this text book is meant for the university students, a lengthy discussion over an elementary matter like that is not appropriate here. We will explain how to arrive at a + b only for case 1. If interested, the reader can check the other cases. But, we advice him/her not to waste too much time over this matter. To start this text, it is also acceptable to consider a + b as something already known to exist.

Now, let us assume that a and b are both positive. Mark a and b both on the number line.

```
-----------------|--------|--------|--------|-------|--------|--------|----------------->
                 0        1        2 .... ...a...... b       b+1      ............
```

Starting from the point b , measure a units to the right. Mark that location. That location will represent the number a+b.

```
                                                 <------ a units ------>
-----------------|--------|--------|--------|-------|-----------------------|------------>
                 0        1        2 .... ...a...... b                      a+b
```

The number resulting from the multiplication of a and b is denoted by a•b. Again, we shall explain how to arrive at a•b only for case 1. An interested reader may figure out how a•b is reached for the other cases. But, it is also acceptable to treat a•b as something already known to exist.

Assume that a and b are both positive. First mark the point b on the number line.

----------------|--------|--------|--------------|--------|----------------------->
0 1 2 …………. b …………..

Starting from the point b , measure b units in → direction. Mark that location as 2b which we shall call the 2 nd place.

Now, starting from the point 2b, measure b more units in → direction. Mark that location as 3b and name it as the 3rd place.
Continue in this manner until you reach the a th place. The location you have reached as the a th place will represent the number a · b.

b b b
…………………………...
|-------------|-------------|-------------|-------------------------|---------------------->
0 b 2b 3b…………………….ab

Symbols

Now, we shall introduce some symbols which will be needed throughout this text book.
$\in$ means belongs to. If we write $a \in$ S, we mean that *a* belongs to S.
For example, because 2 is an integer, we can write $2 \in Z$.

$\notin$ means doesn't belong to. Because -3 doesn't belong to the set of natural numbers, we can write $-3 \notin$ N.
$\forall$ means for all, for each (depending on the context). We can make a statement like "$\forall\, a \in Z_+$, *a* is located to the right of 0 on the number line."
$\exists$ means there is, there exists. We can make a statement like "$\forall\, a \in Z_+$, $\exists$ -*a* which is located to the left of 0 on the number line."

$\exists!$ means there is unique, there exists unique. When we constructed the number line, we have seen that for each $a \in Z_+$, the location of -*a* is uniquely determined on the left side of the number line. Hence, we can in fact improve the statement given above to this one. "$\forall\, a \in Z_+$, $\exists!$ -*a* which is located to the left of 0 on the number line." What we have seen here using our intuition is rigorously proven in theorem 2.

The arrow $\Rightarrow$ is the implication arrow.
When P and Q are two statements, P $\Rightarrow$ Q means that the statement P implies the statement Q.
For example, we can write
***a* appears to the right hand of 0 $\Rightarrow$ *a* is positive.**
P$\Leftarrow$ Q means P is being implied by Q.
As an example we can write
***a* appears to the right hand side of 0 $\Leftarrow$ *a* is positive.**
The two sided arrow $\Leftrightarrow$ has the combined effect of both $\Rightarrow$ and $\Leftarrow$.
P $\Leftrightarrow$ Q means P implies Q and P is being implied by Q.
When P $\Leftrightarrow$ Q we can say that P and Q both mean the thing.
In such a situation, mathematicians say ' P is if and only if Q'
For example

a appears to the right hand side of 0 $\Leftrightarrow$ a is positive.
In words, we can say:
" a appears to the right hand side of 0 if and only if a is positive ".

As we continue our journey in text book the reader will realize that a symbol is needed to denote the set containing no objects. That set is called the empty set or null set. The symbol we use for that set will be Φ

The empty set shall not be mistaken for $\{0\}$. Because the number 0 belongs to {0}, {0} is not empty.
For any two sets A and B, the union of A and B is the set $A \cup B = \{x / x \in A \text{ or } x \in B\}$.
The intersection of A and B is the set $A \cap B = \{x / x \in A \text{ and } x \in B\}$.
When $A \cap B = \Phi$, A and B are said to be disjoint. In such a situation, the union $A \cup B$ is called a disjoint union.
It is clear that $Z = Z_+ \cup \{0\} \cup Z_-$. It should also be clear that it is a disjoint union.

This much knowledge about unions and intersections would be sufficient to complete chapter 1. After learning the material in chapter 2, a comprehensive definition of unions and intersections will be given.

Now, we are in a position to give our first definition.

<u>Definition1</u>

If a set A is located inside a set B, we say that A is a subset of B. using the material we have developed above, we can make following statement.
A is a subset of B if $x \in A \Rightarrow x \in B$.
When this happens, we write A $\subseteq$ B.

It is very easy to see that
$Z_+ \subseteq Z$
$\{0\} \subseteq Z$
$Z_- \subseteq Z$

<u>Definition 2</u>

i) **The set $S_1 = \{ 2m \mid m \in Z\}$ is called the set of even numbers.**
ii) The set $S_2 = \{ 2n+1 \mid n \in Z\}$ is called the set of odd numbers
It is very clear that both of these sets are infinite. It is also clear that Z is a disjoint union of S_1 and S_2.

Closed Operations

Let S be a set . Suppose $*$ is an operation defined on the members of S . We say that $*$ is a closed operation if for all a, b $\in$ S; $a * b \in S$

Example 1

Let S_1 be the set of even numbers. Then for all a,b $\in S_1$, a + b is also an even number. In otherwords, $a + b \in S_1$.
$\therefore$ The addition is a closed operation for S_1.

Example 2

Let S_2 be the set of odd numbers. Then for a, b $\in S_2$; we can write a = 2m + 1 and b = 2n + 1 where m, n $\in$ Z.
a + b = 2 (m + n) + 2 which is not an odd number.
$\therefore$ The addition is not a closed operation for S_2.

Clearly, when we add or multiply two integers, we will definitely get another interger. So, both + and · are closed operations for Z. We state that as the closure law for Z. Because the integers also obey several other laws, now we will list them all.

1. **Closure Law**

   $\forall$ **a, b $\in$ Z; a + b $\in$ Z and a · b $\in$ Z.**

2. **Associative Law**

   $\forall$ **a, b, c $\in$ Z;**

   a + (b + c) = (a + b) + c

   a · (b · c) = (a · b) · c

3. **Commutative Law**

   $\forall$ **a, b $\in$ Z ;**
   a + b = b + a
   a · b = b · a

4. **Distributive Law of Multiplication Over Addition**

   $\forall$ **a, b, c $\in$ Z**
   a · (b + c) = (a · b) + (a · c)

Question:

Can there be a distributive law of addition over multiplication?
If there is such a law , that would say $\forall$ a, b, c $\in$Z ;
a + (b · c) = (a + b) · (a + c) and this is easily seen as not to be the case.

5. **Law about Identities**

   i) $\exists$ 0$\in$Z such that a + 0 = a = 0 + a $\forall$ a $\in$ Z
   ii) $\exists$ 1$\in$Z such that a · 1 = a = 1 · a $\forall$ a $\in$ Z

6. **Law about an Additive Inverse**

   For each a $\in$ Z , $\exists$ - a $\in$ Z such that a + (-a) = 0 = (-a) + a

All 6 laws can be proven after a time consuming struggle with the number line. But, it would be inappropriate to go through that struggle in a text meant for university students. Hence, we shall accept those as 6 of the starting axioms for the subject. If we need to make an improvement to what is stated in any those six, that should be proven and such a proof would usually be at a level meant for this text.

We can make the following improvement to Law 5.

<u>Theorem 1</u>

- **i) The additive identity mentioned in Law 5 i) is unique.**
- **ii) The multiplicative identity mentioned in Law 5 ii) is unique.**

<u>Proof i)</u>

Suppose $0_1, 0_2$ are both additive identities with the property mentioned in Law 5i)
Because 0_1 is an additive identity and $0_2 \in Z$, $0_2 + 0_1 = 0_2$
Because 0_2 is an additive identity and $0_1 \in Z$, $0_1 = 0_2 + 0_1 = 0_2$
Hence, if there are two identities, they will be the same.
$\therefore$ Additive identity is unique.

<u>Proof ii)</u>

Suppose 1 and $\bar{1}$ are both multiplicative identities with the property mentioned in Law 5 ii).
Because 1 is a multiplicative identity and $\bar{1} \in Z$, $\bar{1} \cdot 1 = \bar{1}$
Because $\bar{1}$ is a multiplicative identity and $1 \in Z$, $1 = \bar{1} \cdot 1 = \bar{1}$
$\therefore$ If there are two multiplicative identities, they will be the same.
$\therefore$ Multiplicative identity is unique.

We can also make an improvement to Law 6.

<u>Theorem 2</u>
The additive inverse mentioned in Law 6 is unique.

<u>Proof</u>: Let $a \in Z$.
Suppose $-a_1$ and $-a_2$ are both additive inverses for a. Because $-a_2$ is an additive inverse,
$a + (-a_2) = 0$
$\therefore -a_1 + (a + (-a_2)) = -a_1 + 0$
By Law 5 i), $-a_1 + 0 = -a_1$
By Law 2 (associative law), $-a_1 + (a + (-a_2)) = (-a_1 + a) + (-a_2)$
Because $-a_1$ is an additive inverse for a, $(-a_1) + a = 0$
Now,

$$\begin{aligned} -a_1 &= ((-a_1) + a) + (-a_2) \\ &= 0 + (-a_2) \\ &= -a_2 \text{ by Law 5 i)} \end{aligned}$$

Hence, the additive inverse is unique.

Note that we didn't mention about the existence of a multiplicative inverse. Up to this point, our set of integers has only two operations. Those are '+' and ' · '. Henceforth, the student is allowed to omit ' · ' when writing the product of two integers.

Now, we are in a position to define another operation. It is known as the subtraction and is denoted by ' – '.

<u>Definition 3</u>: For all $a, b \in Z$; $a - b = a + (-b)$
Some of the laws which were being satisfied by the addition will not be satisfied the subtraction.
In exercise 1, the student will check to see which ones among the first four laws are being satisfied by subtraction. Because Law 5 and Law 6 involve thinking at a little higher level, we shall discuss those now. Before we do that we need a simple lemma.

Lemma 1

$\forall a, b, c \in Z$; $a - b = c \Rightarrow a = b + c$

Proof:

$a - b = c \Rightarrow a + (-b) = c \Rightarrow (a + (-b)) + b = c + b$

$\Rightarrow a + ((-b) + b) = b + c$ by applying commutative law and associative law

$\Rightarrow a + 0 = b + c$ by Law 6

$\Rightarrow a = b + c$ by Law 5

Theorem 3

i) Law 5 is not satisfied by subtraction

ii) Law 6 is not satisfied by subtraction.

Proof i:

Suppose Law 5 is satisfied by subtraction.

Then

$\exists\, \alpha \in Z$ *such that* $a - \alpha = a = \alpha - a \quad \forall a \in Z$

$a - \alpha = a \Rightarrow a = \alpha + a$ *by lemma* 1

$\Rightarrow a + (-a) = (\alpha + a) + (-a)$

$\Rightarrow 0 = \alpha + (a + (-a))$

$= \alpha + 0 = \alpha$

$\alpha - a = a \Rightarrow \alpha = 2a$ *by lemma* 1

$\therefore\ 0 = 2a \quad \forall a \in Z$

This is not true. In short, we may say that "we have a contradiction."

$\therefore$ Law 5 is not satisfied by subtraction.

Proof ii)

If Law 6 is satisfied by subtraction, then $\forall a \in Z$, $\exists\, b \in Z$ such that $a - b = \alpha$ where α is the identity for subtraction.

But, by part i) of this theorem, we know that such an α doesn't exist.

$\therefore$ Law 6 is not satisfied by subtraction.

Note:

Regarding addition we recall that Law 6 is written as

$a + (-a) = 0 = (-a) + a$

A knowledgeable reader may have realized that we can write a similar statement regarding subtraction.

$a - a = 0 = a - a$

Had 0 been the identity for subtraction, this statement would have implied that the subtractive inverse of *a* is *a* itself. But, 0 is not the identity for subtraction.

A Logical Fact

Suppose P and Q are two statements where P $\Rightarrow$ Q is accepted as a known fact. Then we will also accept
"Q is false $\Rightarrow$ P is false" as a known fact.

This logical fact can in fact be proven and the proof is not beyond the level meant for chapter 1. But, such a proof is not necessary for a person studying Real Analysis and we will accept this logical fact without proof.

The proof by contradiction technique which we have already seen while proving theorem 3i) is a consequence of this logical fact.

When we want to prove something using this technique we assume the opposite of what we want to prove. Let that assumption be the statement P.

After that assumption, we make correct deductions. We will reach the conclusion Q using those correct deductions. So, P $\Rightarrow$ Q would be a correct statement.
But, if it is clear that Q is false , then according to the logical fact above
Q is false $\Rightarrow$ P is false
$\therefore$ The opposite of what we want to prove is false.

$\therefore$ What we want to prove is true.

Rational Numbers

For a , b $\in$Z; the quantity (a , b) is called an ordered pair. For example , (3 , 5) is an ordered pair. Among all these ordered pairs (a,b), the ones having b $\neq$ 0 will be of interest to us. They will be used to create a new set of numbers known as the rational numbers.

We say that the ordered pairs (a,b) and (c,d) are equivalent if ad = bc. We write this fact as (a,b) ~ (c,d) . For example, because $3 \cdot 10 = 5 \cdot 6$, we can write (3,5) ~ (6,10). The student will also see that $6 \cdot 25 = 10 \cdot 15$
Hence, we can in fact write (3,5) ~ (6,10) ~ (15,25)

We represent the ordered pair (a,b) or all the ordered pairs equivalent to (a,b) by a number $\frac{a}{b}$. In other words , this new type of numbers are created according to the following convention.

Convention

$$\frac{a}{b} = \frac{c}{d} \Leftrightarrow (a,b) \sim (c,d) \Leftrightarrow ad = bc$$

The new numbers $\frac{a}{b}, \frac{c}{d}$ etc. created in this manner are called the rational numbers. The number $\frac{a}{b}$ should be pronounced as " a over b ."
The set of all the rational numbers is denoted by Q.

Because we have seen that (3,5) ~ (6,10) ~ (15,25); we know that $\frac{3}{5} = \frac{6}{10} = \frac{15}{25}$.

Hence, all 3 of these would be the same rational number.
Using the material we have developed up to this point, we can make the following statement.

$$Q = \left\{ \frac{a}{b} \mid a,b \in Z \quad \text{and} \quad b \neq 0 \right\}$$

We have already defined addition, subtraction and multiplication for integers. Using that knowledge, we can define those 3 operations for the rational numbers.

<u>Definition 4:</u>

Let a**, b, c, d** $\in Z$ **with** $b \neq 0, d \neq 0$

$$(i)\ \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

$$(ii)\ \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

$$(iii)\ \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Now, we are in a position to show that the Z we knew before is indeed a subset of Q. We will show it in the following manner.
For any a**, b** $\in$ **Z ;**
$\frac{a}{1} \in Q$ *and* $\frac{b}{1} \in Q$

According to the convention we made above,

$$\frac{a}{1} = \frac{b}{1} \Leftrightarrow a \cdot 1 = 1 \cdot b$$

$$\Leftrightarrow a = b$$

This also says that $\frac{a}{1}$ *and* $\frac{b}{1}$ **can be considered as distinct rational numbers when and only when** $a \neq b$**.**
So, the set $\left\{ \frac{a}{1} \mid a \in Z \right\}$ **which is a subset of Q is indeed a copy of integers.**
Furthermore, the student can convince himself / herself that the operations applied on $\left\{ \frac{a}{1} \mid a \in Z \right\}$ **has the same effect as the operations applied on Z.**
Let us take addition for example.
We have just seen that a rational number of the form $\frac{x}{1}$ **can be identified with x . We shall indicate that by writing** $\frac{x}{1} \leftrightarrow x$**.**

Now, for any a, b $\in Z$;

$\frac{a}{1} \leftrightarrow a$ **and** $\frac{b}{1} \leftrightarrow b$

According to definition 4 i),

$$\frac{a}{1} + \frac{b}{1} = \frac{a \cdot 1 + 1 \cdot b}{1 \cdot 1} = \frac{a+b}{1}$$

And, $\frac{a+b}{1} \leftrightarrow a + b$

$\therefore$ **When we performed the operation** $\frac{a}{1} + \frac{b}{1}$ **, we have in fact added the two integers a and b.**

If we perform any other operation we have learned up to this point on the set $\left\{ \frac{x}{1} \mid x \in Z \right\}$**,**

our conclusion would be the same.

Hence, we conclude that $\frac{a}{1} = a \;\; \forall\, a \in Z$

So, $Z \subseteq Q$

The six basic laws we gave for the integers will remain valid for the rational numbers. In fact, the Law 6 can be improved regarding the rational numbers. We shall give it now.

6. Law of Inverses

- **i)** $\forall$ **q** $\in$ **Q,** $\exists$ **- q** $\in$ **Q such that q + (-q) = 0 = (-q) + q**
- **ii)** $\forall$ **q** $\in$ **Q with q** $\neq 0$**,** $\exists$ $q^{-1} \in Q$ **such that** $q \,.\, q^{-1} = 1 = q^{-1} .\, q$

As we do not wish to lengthen this chapter by having long arguments over basic matters, we shall accept those 6 laws to be valid regarding the rational numbers. But, an interested reader can in fact prove those.

As one would expect, -q is called the additive inverse of q and q^{-1} is called the multiplicative inverse of q. It is possible to prove the uniqueness of q^{-1}.

As we have shown the reader how to handle such an issue before, at this point it would be more appropriate to include the proof among the exercises at the end of the chapter.
It will be very easy to see what this multiplicative inverse is under the usual notation. Before doing that, we will need the following lemma.

<u>Lemma 2</u>

For any $x \in Z - \{0\}$, $\frac{x}{x} = 1$

<u>Proof</u>: Take any $q \in Q$

Then $q = \frac{a}{b}$ **where** a, **b** $\in$ **Z with b** $\neq 0$

Let $x \in Z - \{0\}$

$a\,x\,b = a\,x\,b$

Because multiplication is commutative for integers,

$a\,x\,b = b\,x\,a$

By associative law,
$(ax)b = (bx)a$

By the convention about rational numbers, $\frac{ax}{bx} = \frac{a}{b}$

By definition 4iii

$$\frac{a}{b} \cdot \frac{x}{x} = \frac{a}{b}$$

$$\therefore q \cdot \frac{x}{x} = q \quad \forall q \in Q$$

Because the multiplicative identity is unique, $\frac{x}{x} = 1$

Let q $\in$ Q with q $\neq$ 0 . We can write q = $\frac{a}{b}$ **,** a **and b are integers with b $\neq$ 0. (As q $\neq$ 0, we also know that** $a \neq 0$ **.)**
Now,

$$\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = \frac{ab}{ab} = 1$$

$$\left(\frac{b}{a}\right)\left(\frac{a}{b}\right) = \frac{ab}{ab} = 1$$

$\therefore \frac{b}{a}$ **fulfills the requirements for** q^{-1} **. As** q^{-1} **is supposed to be unique,** $q^{-1} = \frac{b}{a}$ **and because** $a \neq 0$, $\frac{b}{a}$ **is indeed a member of Q.**

Now, because of this multiplicative inverse, we are in a position to define a division among rational numbers. We use the symbol $\div$ for this operation.

Definition 5 **For all** $q_1, q_2 \in Q$ **with** $q_2 \neq 0$ **;** $q_1 \div q_2 = q_1 \cdot q_2^{-1}$

For example,

$$\frac{4}{3} \div \frac{2}{7} = \frac{4}{3} \cdot \left(\frac{2}{7}\right)^{-1}$$

$$= \frac{4}{3} \cdot \frac{7}{2} = \frac{28}{6}$$

$$= \frac{14.2}{3.2} = \frac{14}{3} \cdot \frac{2}{2}$$

By lemma 2, $\frac{2}{2} = 1$

$$\therefore \frac{4}{3} \div \frac{2}{7} = \frac{14}{3}$$

Because $Z \subseteq Q$ **, we can perform division among integers by considering them as rational numbers. When we try to do that, we will discover something interesting.**

Lemma 3 **For any** a, **b,** $\in$ **Z with b $\neq$ 0,**

$a \div b = \frac{a}{b}$ *where* $\frac{a}{b}$ *is the rational number " a over b "*

Proof:

Let *a*, b$\in$ *Z* with b$\neq$ 0.

$$\left(\frac{b}{1}\right)^{-1} = \frac{1}{b}$$

$$\therefore a \div b = \left(\frac{a}{1}\right) \div \left(\frac{b}{1}\right)$$

$$= \frac{a}{1} \cdot \left(\frac{1}{b}\right) = \frac{a \cdot 1}{1 \cdot b} = \frac{a}{b}$$

With discovery, we will start calling the rational number $\frac{a}{b}$ the quotient of dividing *a* by b.

Regarding this number, *a* is called the numerator and b is called the denominator.
The following definition will generalize what we have seen above.

Definition 6

Let q_1, q_2 $\in$ Q with $q_2 \neq 0$. The quantity $q_1 \div q_2$ is called the quotient of dividing q_1 by q_2.

We shall denote it by $\frac{q_1}{q_2}$. Regarding this quotient, q_1 is called the numerator and q_2 is called the denominator.

This $\frac{q_1}{q_2}$ is indeed an ordinary rational number.

Write $q_1 = \frac{a}{b}$ *and* $q_2 = \frac{c}{d}$

$$\frac{q_1}{q_2} = q_1 \div q_2 = q_1 \cdot q_2^{-1}$$

$$= \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

$ad \in Z$ *and* $bc \in Z$.

Because $b \neq 0$ *and* $c \neq 0$, $bc \neq 0$.

So, $\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \frac{ad}{bc}$

It is clear that we can consider either $\frac{a}{b}$ or *ad* as the numerator.

If we say $\frac{a}{b}$ is the numerator, the corresponding denominator would be the $\frac{c}{d}$.

If we want to accept *ad* as the numerator, then bc would be the only choice for a denominator.

As an example, notice that

$$\frac{\frac{3}{4}}{\frac{2}{5}} = \frac{3}{4} \cdot \frac{5}{2} = \frac{15}{8}$$

<u>Mixed Fraction Notation</u>

This notation instantly gives us a feeling for the size of a rational number.

Let us take $q = \frac{7}{4}$ for example.

$$q = \frac{4 + 3}{4} = \frac{1\cdot 4 + 1\cdot 3}{1\cdot 4}$$

$$= \frac{1}{1} + \frac{3}{4} \quad by \quad definition\ 4i$$

$$= 1 + \frac{3}{4}$$

We shall indicate this fact by writing $q = 1\frac{3}{4}$. We say that we have presented q as a mixed fraction. If the number q is supposed to indicate the length of an object in meters, the notation $1\frac{3}{4}$ immediately tells us that the object is one and three quarters of meters long.

<u>The Real Numbers</u>

There are several ways to define the real numbers. We shall give a practical definition based on lengths of line segments because that is most suitable for a student at this level.

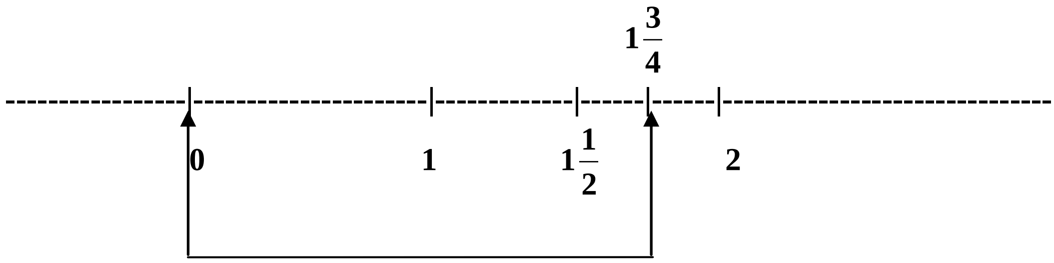
We shall start with the number line which we have constructed before.

```
------------|----------|----------|----------|----------|----------|----------->
           -2         -1          0          1          2          3……..
```

Let us consider the positive side of this line. We can speak of the length from 0 to each positive integer marked.

The length from 0 to 1 is 1 units, the length from 0 to 2 is 2 units etc.

If we measure the length from 0 to the point halfway between 1 and 2, it is $1\frac{1}{2}$ units. If we mark that point as $1\frac{1}{2}$ and measure the length from 0 to the point halfway between $1\frac{1}{2}$ and 2, that will be $1\frac{3}{4}$ units . Note that we can also cut a line segment of length $1\frac{3}{4}$ from another number line using this procedure. Then we place that segment on our line so that one end of the segment would be on 0. We mark the location of the other end as $1\frac{3}{4}$. So a new number $1\frac{3}{4}$ is created in this manner. A knowledgeable student will see that there is no end to this procedure.



The number line we have seen before is actually the real number line. To start with, we have marked it with only the integers. Now we are in a position to define all the real numbers.

Definition 7

Take any point on the number line. We use the word 'length' to describe the length of the line segment from 0 to that point. The real number representing that point will be r where

r = length if the point is located to the right of 0

r = - (length) if the point is located to the left of 0

We denote the set of real numbers by R. Actually Q and R are more than just sets. They belong to a special category of sets known as fields. So, we can use the word " field of rational numbers" if we wish. However, the properties of a field will not be needed here. Hence, we shall not use that term here.

All six basic laws which are being satisfied by the rational numbers are also being satisfied by the set of real numbers.

We have proven the uniqueness of additive identity and multiplicative identity for integers. Also, we have proven the uniqueness of additive inverse for integers. The student will prove the uniqueness of multiplicative inverse for rational numbers in exercise 2.

Eventhough real numbers are more general than rational numbers or integers, a careful reader will see that the proofs mentioned above will work for all the real numbers. Hence, there is no need to make this chapter unnecessarily long by proving uniqueness properties for real numbers. With this remark, we would like to remind the reader that the identities mentioned in Law 5 and the inverses mentioned in Law 6 are unique for the set of real numbers R.

As with the rational numbers, the existence of an additive inverse makes it possible to define a subtraction and the existence of a multiplicative inverse guarantees a division.

We already know that 0 is the unique additive identity for real numbers. Now, we will prove another interesting fact about 0.

Theorem 4 For any $a \in R$

i. $a \cdot 0 = 0$

ii. $0 \cdot a = 0$

Proof:

Because 0 is the additive identity,

$0 + 0 = 0$

$\therefore\ a(0+0) = a\cdot 0$

$\therefore\ a\cdot 0 + a\cdot 0 = a\cdot 0$ **by using the distributive law .**

Now, by closure law, $a\cdot 0 \in R$

Hence, $\exists\ -a\cdot 0 \in R$ **such that** $-a\cdot 0 + a\cdot 0 = 0$

Now, $-a\cdot 0 + (a\cdot 0 + a\cdot 0) = -a\cdot 0 + a\cdot 0$

By associativity,

$(-a\cdot 0 + a\cdot 0) + a\cdot 0 = -a\cdot 0 + a\cdot 0$

$\therefore\ 0 + a\cdot 0 = 0$

$\therefore\ a\cdot 0 = 0$

There is nothing to prove in part ii. Because by commutative law, $0\cdot a = a\cdot 0$

<u>Theorem 5:</u>

For any $x \in R$,

i) $(-1)x = -x$

ii) $x(-1) = -x$

<u>Proof i :</u> **Because -1 is the additive inverse of 1 ,** $1 + (-1) = 0$

$\therefore \{1 + (-1)\}x = 0\ .\ x$

$\therefore 1.x + (-1)x = 0$ **by theorem 4**

$\therefore x + (-1)x = 0$ **because 1 is the multiplicative identity**

$\therefore -x + (x + (-1)x) = -x + 0 = -x$

$\therefore (-x + x) + (-1)x = -x$ **by associative law**

$\therefore 0 + (-1)x = -x$

$\therefore (-1)x = -x$

ii) is trivial because of the commutative law.

<u>Theorem 6:</u>

Let $a, b \in R$. **Then**

i) $-(-a) = a$

ii) $(-a)b = a(-b) = -ab$

iii) $(-a)(-b) = ab$

The proof is left as an exercise for the student.

<u>Definition 8:</u> Let $m \in Z_+$. **For any** $a \in R$, **the** m^{th} **power of** a **is defined as**

$$a^m = \underbrace{a\cdot a\cdot a\cdots\cdots a}_{m\text{ – times}}$$

For example, $5^2 = 5\cdot 5 = 25$

$5^3 = 5\cdot 5\cdot 5 = 125$

We can also define a^m **when** $m \in Z\_ \cup \{0\}$

Definition 9 : **Suppose $a \in R$**

i) $a^0 = 1$

ii) If $m \in Z_-$, then

$$a^m = \frac{1}{a^{-m}}$$

(Because - $m \in Z_+$, a^{-m} can be calculated according to definition 8)

Definition 10: **Let y be a positive real number. If $\exists$ x $\in$ R such that y = x^2, then we will say that x is a square root of y. If x is positive , then we shall write x = $\sqrt{y}$. And, if x is negative, we shall write x = - $\sqrt{y}$**

5^2 = 25 and 5 is positive.

$\therefore$ 5 = $\sqrt{25}$

By theorem 6iii),

(-5) (-5) = $5 \cdot 5$ = 25

$\therefore$ $(-5)^2$ = 25 and -5 is negative.

$\therefore$ -5 = - $\sqrt{25}$

In chapter 3, we will generalize both definition 8 and definition 10. For the purpose of the first two chapters of this text , those 2 definitions would be sufficient.

Now, we shall return to the integers to discuss some properties relevant only for integers.

Definition 11:

Let a, b $\in$ Z. We say that a divides b (and write this fact as $a|b$) if $\exists$ c $\in$ Z such that b = ac. We also say that a and c are factors of b.

$$7 \cdot 4 = 28$$

$$\therefore 4 | 28 \textit{ and } 7 | 28$$

We also say that 4 and 7 are factors of 28.

Definition 12 :

Let p be an integer not equal to 1. p is said to be a prime number if p doesn't have any factors other than 1 and p itself.

Example:

2,3,5 are prime numbers.

Prime Factorization

Any integer can be factored into a product of prime powers.

For example, 200 = $2^3 \cdot 5^2$

525 = $3 \cdot 5^2 \cdot 7$

In general, for a$\in$ Z, we can have a = $p_1^{\alpha_1} p_2^{\alpha_2} ....p_r^{\alpha_r}$

Theorem 7: **Let p be a prime number. Then for any a,b $\in$ Z;**

$$p|ab \Rightarrow p|a \text{ or } p|b$$

Proof: Write

$a = p_1^{\alpha_1} p_2^{\alpha_2} .... p_r^{\alpha_r}$ **and**

$b = q_1^{\beta_1} q_2^{\beta_2} ......q_s^{\beta_s}$

Assume p | ab.

Then $\exists\ c \in Z$ such that

$$pc = p_1^{\alpha_1} p_2^{\alpha_2} .... p_r^{\alpha_r}\ q_1^{\beta_1} q_2^{\beta_2} ......q_s^{\beta_s}$$

Because p is prime, p has to be either a member of

$\{ p_{1,} p_{2,} ...... p_r \}$or $\{ q_{1,} q_{2,} ......q_s \}$

$\therefore$ p | a or p | b

It is clear that there are infinitely many integers. A good question is that " Do we have infinitely many prime numbers?"

The answer to this question is yes. We will give a proof by contradiction.

<u>Theorem 8</u>

There are infinitely many prime numbers.

<u>Proof</u>

Suppose there are only finitely many primes. Label them as $p_1, p_2,………,p_n$

Let $a = p_1p_2p_3…………p_n + 1$

If *a* is not prime, then *a* should have a factor b which in turns should have a prime factor in it. Let us say p_j .

So, p_j | b and b | *a*

$\therefore$ p_j | *a* ------------------(1)

Clearly, p_j | $p_1p_2p_3$………………. p_n---------------------(2)

By (1) and (2), p_j | 1 ----- contradiction.

$\therefore$*a* is a prime number.

This is a contradiction because we assumed that $\{ p_1, p_2,………,p_n \}$ are the only possible primes.

$\therefore$ There are infinitely many primes .

Studying in depth about the prime numbers is not needed in a textbook on Real Analysis. However, a fundamental knowledge about primes will be needed. Study of prime numbers belongs to a particular branch of mathematics known as Number Theory.

It has been a fact about Number Theory that even a very simple question needs the advanced tools to investigate. And, some questions remain unanswered even after using the best available tools. We will give you just one illustration to make our point. Before we do that, we need to explain what the twin primes are.

<u>Definition 13</u>: If the integers p and p+2 are both primes, then they are called twin primes.

The followings are examples of twin primes:

5 and 7, 11 and 13, 17 and 19, 29 and 31.

At this point, any intelligent student should be asking himself/herself the following question.

Twin Prime Problem:

Are there infinitely many twin primes?

This question was first raised around 1710. Since then many of the best brains on the earth have tried to solve it but, nobody has found a solution.

Most of the number theorists believe that the answer should be " yes ". In that case, somebody should be able to prove the existence of infinitely many twin primes either using a proof by contradiction or some other means. But, after nearly three centuries of effort, it hasn't happened yet.

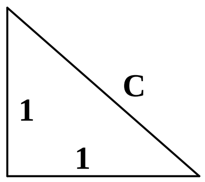
A minority among number theorists believe that the answer should be "no" . If somebody wants to follow that route, finding an upper bound above which twin primes don't exist would be a good idea. (We know that there is no upper bound above which primes don't exist. Look at theorem 8.) Nobody was able to prove it in that direction either.

If you can find a solution to this problem, you will be very famous. But, we recommend you not to try it.

Irrational Numbers

There are real numbers which can't be written as the quotient of two integers. In other words, not all the real numbers are rational numbers. A real number which can't be written as the quotient of two integers is called an irrational number.

We shall establish the existence of one particular irrational number in the following manner. Let us consider a rectangle having each side of length 1 unit.



According to the Pythogoras theorem, $C^2 = 1^2 + 1^2 = 2$

$\therefore \quad C = \sqrt{2}$

We have seen before that the length of any line segment becomes a real number. Hence, there is a real number which is equal to $\sqrt{2}$.

It can be proven that this $\sqrt{2}$ is an irrational number. Hence, there exists at least one irrational number.

We shall not waste time to prove that $\sqrt{2}$ is irrational because we can prove a more general result. Before we do that, we need to see what is meant by the word reduced form for a quotient.

A quotient $\frac{b}{a}$ is said to be of the reduced form if a and b don't have any common factors. For example, $\frac{12}{45}$ is not of the reduce form because $\frac{12}{45}=\frac{3\cdot 4}{3\cdot 15}$.

But, after canceling out the common factor 3, $\frac{4}{15}$ is of the reduced form. In fact, any quotient can be put into the reduced form. Definition 10 of chapter 2 will be relevant regarding the reduced forms.

<u>Theorem 9</u>: For any prime number p, $\sqrt{p}$ is an irrational number.

Proof: Let p be prime. Suppose $\sqrt{p}$ is a rational number.

Then $\sqrt{p} = \frac{a}{b}$ where a,b $\in$Z and we can say that this quotient is of the reduced form.

$\therefore$ **p** $= \frac{a^2}{b^2}$

$\therefore pb^2 = a^2 \Rightarrow p \mid a^2$

$\therefore p \mid a$ **because p is prime.**

$\therefore a = px$ **for some** $x \in Z$

$\therefore a^2 = p^2x^2$

$\therefore p^2x^2 = pb^2$

$\therefore px^2 = b^2 \Rightarrow p \mid b^2$

$\therefore p \mid b$ **because p is prime.**

Now, $p \mid a$ **and** $p \mid b$ **.**

$\therefore \frac{a}{b}$ **is not of the reduced form ----contradiction.**

$\therefore \sqrt{p}$ **is an irrational number.**

Decimals

Consider the real number

$r_1 = 15\frac{3}{4}$

$r_1 = 15 + \frac{3}{4}$

$= 10 + 5 + \frac{75}{100}$

$= 10 + 5 + \frac{7}{10} + \frac{5}{100}$

$= 1(10^1) + 5(10^0) + 7(10^{-1}) + 5(10^{-2})$

What we wrote above is called the decimal expansion of r_1. This expansion provides us with another way to write the number $15\frac{3}{4}$.

The decimal expansion of r_1 which we have seen above gives us the idea to write r_1 as r_1 =15.75. Once it is done, 15.75 is called the decimal form of $15\frac{3}{4}$.

Now, consider the real number $r_2 = 215\frac{714}{1000}$

As before, we will see that

$r_2 = 2(10^2) + 1(10^1) + 5(10^0) + 7(10^{-1}) + 1(10^{-2}) + 4(10^{-3})$ and, this would be the decimal expansion of r_2 . According to this expansion,

$r_2 = 215.714$

Regarding r_2,

1st decimal place = 7
2nd decimal place = 1
3rd decimal place = 4

Suppose the decimal expansion of a general real number r can be given as

$r = a_n(10^n) + a_{n-1}(10^{n-1}) + ... + a_1(10^1) + a_0(10^0) + b_1(10^{-1}) + b_2(10^{-2}) + ... + b_m(10^{-m})$

where $\{a_0, a_1, a_2, ..., a_n\}$ and $\{b_1, b_2, ..., b_m\}$ are all integers.

According to the notation we have developed here, the corresponding decimal form will be

$r = a_n a_{n-1} a_{n-2} .... a_1 a_0 . b_1 b_2 .... b_m$

The decimal form 215.714 terminates at the 3rd decimal place. Hence, we call it a terminating decimal. But, if we consider $\frac{1}{3}$ = 0.33333 …………, it is clearly a non – terminating decimal. Because 3 is being repeated all the way, it can be called a repeating decimal. To indicate this fact, we will write $\frac{1}{3} = 0.\bar{3}$

The same notation will help us even when a string of integers are being repeated in a decimal form. As an example, consider 4.27857857857……………. We can write this number as $4.27\overline{857}$.

The line over 857 indicates that only that portion is being repeated. Now, we will notice something interesting about this number.

Let $x = 4.27\overline{857}$ Then

$$100{,}000x = 427{,}857.\overline{857}$$
$$100x = 427.\overline{857}$$

$$99{,}900x = 427{,}857.\overline{857} - 427.\overline{857} = 427{,}430$$
$$\therefore \quad x = \frac{427{,}430}{99{,}900} \in \mathbf{Q}$$

So, the number $427.\overline{857}$ which appeared formidable is indeed a harmless rational number.

Regarding r = 215.714,   we have 1000r = 215714

$$\therefore \quad r = \frac{215{,}714}{1000} \in Q$$

So, we have seen a repeating decimal being put into the rational form and then we have seen a terminating decimal being put into the rational form. The same can be done regarding an arbitrary repeating decimal and an arbitrary terminating decimal. The proof will be left as an exercise for the student.

We know that $\sqrt{2}$ is irrational. To the first 20 decimal places, $\sqrt{2}$ is given as
$\sqrt{2}$ = 1.41421356237309504880……
It appears that there is no repeating portion. According to exercise 15, there should not be such a portion. The same exercise would tell us that even if we check one billion decimal places, $\sqrt{2}$ wouldn't terminate.

Exercises

1. **Check to see which ones among the four basic laws are being satisfied by the operation of subtraction .**

2. **For any non-zero rational number q, prove that q^{-1} mentioned in Law 6 is unique.**

3. **For any $a \in Z$, prove that**
   (i) a is even $\Leftrightarrow a^2$ is even
   (ii) a is odd $\Leftrightarrow a^2$ is odd.

4 **Prove the following cancellation laws.**
   i) $ab = ac$ with $a \neq 0 \Rightarrow b = c$
   ii) $a + b = a + c \Rightarrow b = c$

5. **Prove theorem 6.**

6. **Let $a, b \in R$. Show that $ab = 0 \Rightarrow a = 0$ or $b = 0$**

7. **Suppose $a, b \in R$ such that $a \neq 0$ and $b \neq 0$. Prove that**
   i) $(ab)^{-1}$ exists
   ii) $(ab)^{-1} = a^{-1}b^{-1}$

8. **Generalize the result in exercise 7 as follows.**
   Let $a_1\ a_2$, ………………, $a_n \in$ R such that $a_i \neq 0\ \forall a_i$. Then
   - i) $(a_1\ a_2 \ldots\ldots\ldots\ldots\ a_n)^{-1}$ **exists**
   - ii) $(a_1 a_2 \ldots\ldots . a_n)^{-1} = a_1^{-1} a_2^{-1} \ldots\ldots \ldots . a_n^{-1}$

9. **Prove or disprove the followings.**
   - i) **If x is a non- zero rational number and y is an irrational number, then xy is irrational.**
   - ii) **If x is rational and y is irrational, then** $x + \sqrt{3}\ y$ **is irrational.**
   - iii) **If a is rational and b is irrational, then a + 5b is irrational.**

10. **Suppose x,y,u,v∈Q with x ≠u such that** $x + \sqrt{y} = u + \sqrt{v}$.

    Prove that $\sqrt{y} \in Q$ *and* $\sqrt{v} \in Q$.

11. **Let x be the number of primes below 100 and let y be the number of twin primes below 100. Obtain a relationship of the form ax = by where a and b are integers.**

12. **Let p_1,p_2………………..,p_n be distinct primes. Prove that** $\sqrt{p_1\ p_2 \ldots\ldots p_n}$ **is irrational.**

13. **Suppose** $a\sqrt{2} + b\sqrt{3} = 0$ **with a ∈Q, b∈Q. Prove that a = b = 0**

14. **Express the following decimals as rational numbers.**
    - i) **2.7143935**
    - ii) **3.516325832583258………………..**
    - iii) **7.25631926926926…………………**
    - iv) **1.4142135623762376237………….**

    The answer to part iv) would be a rational number extremely close to $\sqrt{2}$

15. **Prove the following claims:**
    - a. **Any terminating decimal is a rational number.**
    - b. **Any repeating decimal is a rational number.**