(1) For $x<-1$, $F(x)=\int_{-\infty}^{x} 0 \cdot dx = 0$

For $-1 \le x \le 1$, $F(x)=\int_{-\infty}^{-1} 0 \cdot dx + \int_{-1}^{x} \frac{1}{7}(4-3x^2)\,dx$

$$= \frac{1}{7}\left[4x - x^3\right]_{-1}^{x}$$

$$= \frac{1}{7}(3+4x-x^3)$$

For $1 < x \le 2$, $F(x)=\int_{-\infty}^{-1} 0 \cdot dx + \int_{-1}^{1} \frac{1}{7}(4-3x^2)\,dx$

$$+ \int_{1}^{x} \frac{1}{7}\,dx$$

$$= \frac{1}{7}\left[4x-x^3\right]_{-1}^{1} + \frac{1}{7}\left[x\right]_{1}^{x}$$

$$= \frac{1}{7}(5+x)$$

For $x>2$, $F(x)=\int_{-\infty}^{-1} 0 \cdot dx + \int_{-1}^{1} \frac{1}{7}(4-3x^2)\,dx$

$$+ \int_{1}^{2} \frac{1}{7}\,dx + \int_{2}^{x} 0 \cdot dx$$

$$= \frac{1}{7}\left[4x-x^3\right]_{-1}^{1} + \frac{1}{7}\left[x\right]_{1}^{2}$$

$$= 1$$

$$F(x) = \begin{cases} 0 & ; \quad x<-1 \\ \frac{1}{7}(3+4x-x^3) & ; \quad -1 \le x \le 1 \\ \frac{1}{7}(5+x) & ; \quad 1 < x \le 2 \\ 1 & ; \quad x>2 \end{cases}$$

(c) $P(X \leq 0) = F(0)$

$$= \frac{1}{7}(3 + 4\times 0 - 0)$$

$$= \underline{\underline{3/7}}$$

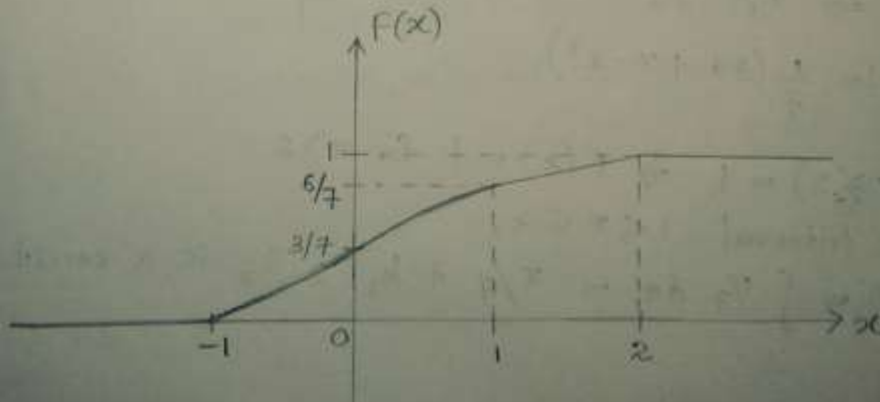
(d) $P(0.5 \leq X \leq 1.5) = P(X \leq 1.5) - P(X < 0.5)$

$$= F(1.5) - F(0.5)$$

$$= \frac{1}{7}(5 + 1.5) - \frac{1}{7}\left(3 + 4\times 0.5 + (0.5)^3\right)$$

$$= \frac{1}{7}\left(\frac{13}{2} - 5 + \frac{1}{8}\right)$$

$$= \underline{\underline{\frac{13}{56}}}$$



2 (a)

$\text{cov}(X,Y)$ is defined as

$$\text{cov}(X,Y) = E\{[X-E(X)][Y-E(Y)]\}$$

Then, $\text{Cov}(X,Y) = E\{XY - X.E(Y) - Y.E(X) + E(X).E(Y)\}$

$$= E(XY) - E(Y).E(X) - E(X).E(Y) + E(X).E(Y)$$

$$= E(XY) - E(X).E(Y)$$

(b) $\text{Cov}(cX, Y) = E\{[cX - E(cX)][Y - E(Y)]\}$

$$= E\{c[X - E(X)][Y - E(Y)]\}$$

$$= c.\, E\{[X - E(X)][Y - E(Y)]\}$$

$$= c.\, \text{Cov}(X, Y)$$

(c) $\text{Cov}(aX + bY,\ cW + dV)$

$$= E\{[(aX + bY) - E(aX + bY)][(cW + dV) - E(cW + dV)]\}$$

$$= E\{[aX + bY - aE(X) - bE(Y)][cW + dV - cE(W) - dE(V)]\}$$

$$= E\{[a(X - E(X)) + b(Y - E(Y))][c(W - E(W)) + d(V - E(V))]\}$$

$$= E\{ac(X - E(X))(W - E(W)) + ad(X - E(X))(V - E(V)) + bc(Y - E(Y))(W - E(W)) + bd(Y - E(Y))(V - E(V))\}$$

$$= ac.E[(X - E(X))(W - E(W))] + ad.E[(X - E(X))(V - E(V))] + bc.E[(Y - E(Y))(W - E(W))] + bd.E[(Y - E(Y))(V - E(V))]$$

$$= ac.\,\text{Cov}(X, W) + ad.\,\text{Cov}(X, V) + bc.\,\text{Cov}(Y, W) + bd.\,\text{Cov}(Y, V)$$

(3)

$f_{X,Y}(x,y)$		Y: -1	0	1	$f_X(x)$
X	1	0.15	0.27	0.25	0.67
	2	0.05	0.08	0.20	0.33
$f_Y(y)$		0.20	0.35	0.45	1

(a) $Var(X) \overset{def}{=} E\left[(X-E(X))^2\right]$

$$= E(X^2) - [E(X)]^2$$

$$E(X) = \sum_i x_i\, f_X(x_i)$$

$$= 1 \times 0.67 + 2 \times 0.33$$

$$= 1.33$$

$$E(X^2) = \sum_i x_i^2 f_X(x_i)$$

$$= 1^2 \times 0.67 + 2^2 \times 0.33$$

$$= 1.99$$

$$Var\, X = 1.99 - (1.33)^2$$

$$= \underline{\underline{0.2211}}$$

(b) $Var(Y) = E\left[(Y-E(Y))^2\right]$

$$= E(Y^2) - [E(Y)]^2$$

$$E(Y) = \sum_i y_i\, f_Y(y_i)$$

$$= (-1) \times 0.2 + 0 \times 0.35 + 1 \times 0.45$$

$$= 0.25$$

$$E(Y^2) = \sum_i y_i^2 f_Y(y_i)$$
$$= (-1)^2 \times 0.2 + 0 \times 0.35 + 1^2 \times 0.45$$
$$= 0.65$$
$$Var(Y) = 0.65 - (0.25)^2$$
$$= \underline{\underline{0.5875}}$$

(c) $Cov(X,Y) = E[(X - E(X))(Y - E(Y))]$
$$= E(XY) - E(X)E(Y)$$
$$E(XY) = \sum_i \sum_j x_i y_j f_{X,Y}(x_i, y_j)$$
$$= (1)(-1)(0.15) + (1)(0)(0.27) + (1)(1)(0.25)$$
$$+ (2)(-1)(0.05) + (2)(0)(0.08) + (2)(1)(0.2)$$
$$= 0.4$$
$$Cov(X,Y) = 0.4 - (1.\dot{3}3)(0.25)$$
$$= \underline{\underline{0.0675}}$$

$$Var(X) + Var(Y) + 2.Cov(X,Y)$$
$$= 0.2211 + 0.5875 + 2 \times 0.0675$$
$$= 0.9436$$
$$Var(X+Y) = E[((X+Y) - E(X+Y))^2]$$
$$= E[(X+Y)^2] - [E(X+Y)]^2$$
$$= E[X^2 + Y^2 + 2XY] - [E(X) + E(Y)]^2$$
$$= E(X^2) + E(Y^2) + 2E(XY) - [E(X) + E(Y)]^2$$
$$= 1.99 + 0.65 + 2 \times 0.4 - [1.33 + 0.25]^2$$
$$= 0.9436$$
$$\therefore Var(X+Y) = \underline{\underline{Var(X) + Var(Y) + 2.Cov(X,Y)}}$$

Tutorial 02 - Section C

1. Given the data set $\{(1,1), (2,1), (3,2), (4,3)\}$.

Let $(x_0, y_0) = (1,1)$, $(x_1, y_1) = (2,1)$,
$(x_2, y_2) = (3,2)$ and $(x_3, y_3) = (4,3)$.

The Lagrange interpolating polynomial is

$$p(x) = \sum_{k=0}^{3} y_k \cdot l_k(x)$$

where $$l_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^{3} \left(\frac{x - x_i}{x_j - x_i} \right)$$

$$l_0(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)}$$

$$= \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)}$$

$$= \frac{x^3 - 9x^2 + 26x - 24}{(-6)}$$

$$l_1(x) = \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)}$$

$$= \frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)}$$

$$= \frac{x^3 - 8x^2 + 19x - 12}{2}$$

$$l_2(x) = \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)}$$

$$= \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)}$$

$$= \frac{x^3 - 7x^2 + 14x - 8}{(-2)}$$

$$l_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$= \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)}$$

$$= \frac{x^3-6x^2+11x-6}{6}$$

$$p(x) = y_0 \cdot l_0(x) + y_1 \cdot l_1(x) + y_2 \cdot l_2(x) + y_3 \cdot l_3(x)$$

$$= 1 \times \frac{(x^3-9x^2+26x-24)}{(-6)} + 1 \times \frac{(x^3-8x^2+19x-12)}{2}$$

$$+ 2 \times \frac{(x^3-7x^2+14x-8)}{(-2)} + 3 \times \frac{(x^3-6x^2+11x-6)}{6}$$

$$= \frac{1}{6}\left(-x^3+9x^2-20x+18\right)$$

Roots of $y^2-y-1=0$ are $y=1.618$ and $y=-0.618$.

$\therefore\ \phi = 1.618 > 0$ and $\psi = -0.618 < 0$.

Given, $f(x) = \frac{1}{\sqrt{5}}\left(\phi^x - (-\psi)^x \cdot \cos \pi x\right)$

The error, $R = \dfrac{f^{(4)}(\xi)}{4!} \cdot \prod_{i=0}^{3} (x-x_i)$

where $\xi \in (x_0, x_3) \equiv (1, 4)$.

Maximum error,

$$|R| = \left| \frac{f^{(4)}(\xi)}{4!} \cdot (x-x_0)(x-x_1)(x-x_2)(x-x_3) \right|$$

$$= \left| \frac{f^{(4)}(\xi)}{4!} \cdot (x-1)(x-2)(x-3)(x-4) \right|$$

$$= \frac{|f^{(4)}(\xi)|}{4!} \cdot |x^4-10x^3+35x^2-50x+24|$$

Let $g(x) = x^4 - 10x^3 + 35x^2 - 50x + 24$

$\Rightarrow g'(x) = 4x^3 - 30x^2 + 70x - 50$

For critical points,

$g'(x) = 0$

$\Rightarrow 4x^3 - 30x^2 + 70x - 50 = 0$

The critical points are

$x = 1.382$, $x = 3.618$, $x = 2.5$

For $x = 1.382$, $|g(x)| = 0.9999$

For $x = 3.618$, $|g(x)| = 0.9999$

For $x = 2.5$, $|g(x)| = 0.5625$

$\therefore \max |g(x)| = 0.9999$

$$f(x) = \frac{1}{\sqrt{5}}\left(\phi^x - (-\psi)^x \cdot \cos \pi x\right)$$

$$f^{(1)}(x) = \frac{1}{\sqrt{5}}\left(\ln\phi \cdot \phi^x\right) - \frac{1}{\sqrt{5}}\Big[-\pi(-\psi)^x \cdot \sin \pi x + \ln(-\psi) \cdot (-\psi)^x \cdot \cos \pi x\Big]$$

$$f^{(2)}(x) = \frac{1}{\sqrt{5}}\left((\ln\phi)^2 \cdot \phi^x\right) - \frac{1}{\sqrt{5}}\left\{\left[-\pi^2 + \left(\ln(-\psi)\right)^2\right] \cdot (-\psi)^x \cdot \cos \pi x - 2\pi \cdot \ln(-\psi) \cdot (-\psi)^x \cdot \sin \pi x\right\}$$

$$f^{(3)}(x) = \frac{1}{\sqrt{5}}\left((\ln\phi)^3 \cdot \phi^x\right) - \frac{1}{\sqrt{5}}\left\{\left[\left(\ln(-\psi)\right)^2 - 3\pi^2\right] \cdot \ln(-\psi) \cdot (-\psi)^x \cdot \cos \pi x - \pi\left[3\left(\ln(-\psi)\right)^2 - \pi^2\right](-\psi)^x \cdot \sin \pi x\right\}$$

$$f^{(4)}(x) = \frac{1}{\sqrt{5}}\left[(\ln\phi)^4 \cdot \phi^x\right]$$

$$-\frac{1}{\sqrt{5}}\cdot(-\psi)^x\left\{\left[\left[(\ln(-\psi))^2 - \pi^2\right]^2 - 4\pi^2(\ln(-\psi))^2\right]\cdot\cos\pi x - 4\pi\ln(-\psi)\left[(\ln(-\psi))^2 - \pi^2\right]\cdot\sin\pi x\right\}$$

For $x=1$, $|f^{(4)}(1)| = 23.19$ For $x=2$, $|f^{(4)}(2)| = 14.24$

For $x=4$, $|f^{(4)}(4)| = 5.2987$ For $x=3$, $|f^{(4)}(3)| = 8.94$

$\therefore \max|f^{(4)}(\xi)| < 23.19 \quad (1<\xi<4)$

$$\therefore \max|R| < \frac{23.19 \times 0.9999}{4!}$$

$$= 0.966$$

3. Given the data set $\{(1,1),(2,1),(3,2),(5,5)\}$.

Let $x_0 = 1$, $p(x_0) = 1$

$x_1 = 2$, $p(x_1) = 1$

$x_2 = 3$, $p(x_2) = 2$

$x_3 = 5$, $p(x_3) = 5$

Consider the iterative process,

$$p(x) = A_1 + p_1(x).(x - x_0)$$
$$p_1(x) = A_2 + p_2(x).(x - x_1)$$
$$p_2(x) = A_3 + p_3(x).(x - x_2)$$
$$p_3(x) = A_4 + p_4(x).(x - x_3)$$

Take $x = x_0$,

$$p(x_0) = A_1$$
$$1 = A_1$$

Then, $p(x) = 1 + p_1(x).(x - x_0)$

Take $x = x_1$,

$$p(x_1) = 1 + p_1(x_1).(x_1 - x_0)$$
$$p(x_1) = 1 + A_2.(x_1 - x_0)$$
$$1 = 1 + A_2.(2-1)$$
$$A_2 = 0$$

Then, $p(x) = 1 + [0 + p_2(x).(x - x_1)](x - x_0)$

$$p(x) = 1 + p_2(x).(x - x_1)(x - x_0)$$

Take $x = x_2$,

$$p(x_2) = 1 + p_2(x_2)(x_2 - x_1)(x_2 - x_0)$$
$$p(x_2) = 1 + A_3.(x_2 - x_1)(x_2 - x_0)$$
$$2 = 1 + A_3.(3-2)(3-1)$$
$$A_3 = 1/2$$

Then, $p(x) = 1 + \left[\tfrac{1}{2} + P_3(x).(x - x_2) \right](x - x_1)(x - x_0)$

Take $x = x_3$,

$$p(x_3) = 1 + \left[\tfrac{1}{2} + P_3(x_3).(x_3 - x_2) \right](x_3 - x_1)(x_3 - x_0)$$

$$p(x_3) = 1 + \left[\tfrac{1}{2} + A_4.(x_3 - x_2) \right](x_3 - x_1)(x_3 - x_0)$$

$$5 = 1 + \left[\tfrac{1}{2} + A_4.(5 - 3) \right](5 - 2)(5 - 1)$$

$$4 = \left(\tfrac{1}{2} + 2.A_4 \right).12$$

$$A_4 = -\tfrac{1}{12}$$

Then, $p(x) = 1 + \left[\tfrac{1}{2} + \left[\left(-\tfrac{1}{12}\right) + \underbrace{P_4(x)}_{=0}.(x - x_3) \right](x - x_2) \right].(x - x_1)(x - x_0)$

Since the interpolation done using only four nodes,

$$p(x) = 1 + \left[\frac{1}{2} - \frac{1}{12}(x - x_2) \right](x - x_1)(x - x_0)$$