Tutorial 01 – Section A

1. 2-way frequency table.

		Status of product received on time		
		Yes	No	
Status of satisfaction	Yes	$123{,}000 \times \frac{97.3}{100}$ = 119,679	123,000 − 119,679 = 3,321	150,000 − 27,000 = 123,000
	No	27,000 − 14,310 = 12,690	$27{,}000 \times \frac{53}{100}$ = 14310	$150{,}000 \times \frac{18}{100}$ = 27,000
		132,369	17,631	150,000

(a) Marginal probability

$$p(\text{satisfied with the experience}) = \frac{123{,}000}{150{,}000} = \underline{\underline{0.82}}$$

(b) Marginal probability

$$p(\text{received the product intime}) = \frac{132{,}369}{150{,}000} = \underline{\underline{0.88}}$$

(c) Joint probability

p(satisfied with the experience and did not receive the product in time)

$$= \frac{3{,}321}{150{,}000}$$

$$= \underline{\underline{0.022}}$$

② Let the no. of computer chips produced be 100,000.

		Status after applied testing procedure T		
		Good	Bad	
Type of chips	Non-defective	$98{,}000 \times \frac{99.2}{100} = 97{,}216$	$98{,}000 - 97{,}216 = 784$	$100{,}000 - 2{,}000 = 98{,}000$
	Defective	$2{,}000 - 1{,}994 = 6$	$2{,}000 \times \frac{99.7}{100} = 1{,}994$	$100{,}000 \times \frac{2}{100} = 2{,}000$
		97,222	2,778	100,000

Conditional probability = Joint prob. / Marginal prob.

p(the chip is functional given that T indicates bad)

$= \dfrac{784/100{,}000}{2{,}778/100{,}000}$ ← p(chip is functional and T indicates bad) / ← p(T indicates bad)

$= \underline{0.282}$

③ (a) X is a discrete random variable.

$$\therefore \sum_{x=0}^{4} P[X=x] = 1$$

$$\Rightarrow \frac{c}{4} + \frac{c}{3} + \frac{c}{3} + \frac{c}{2} + \frac{c}{12} = 1$$

$$\frac{18}{12} c = 1$$

$$\underline{\underline{c = 2/3}}$$

X	0	1	2	3	4
P(X)	1/6	2/9	2/9	1/3	1/18

$$E(x) = \sum_{x=0}^{4} x \cdot P[X=x]$$ ②

$$= 0 \cdot \frac{1}{6} + 1 \cdot \frac{2}{9} + 2 \cdot \frac{2}{9} + 3 \cdot \frac{1}{3} + 4 \cdot \frac{1}{18}$$

$$= \frac{2}{9} + \frac{4}{9} + 1 + \frac{2}{9}$$

$$= \frac{17}{9}$$

$$= \underline{\underline{1.89}}$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \sum_{x=0}^{4} x^2 P[X=x]$$

$$= 0 \cdot \frac{1}{6} + 1 \cdot \frac{2}{9} + 4 \cdot \frac{2}{9} + 9 \cdot \frac{1}{3} + 16 \cdot \frac{1}{18}$$

$$= \frac{2}{9} + \frac{8}{9} + 3 + \frac{8}{9}$$

$$= 5$$

$$Var(X) = 5 - \left(\frac{17}{9}\right)^2 = \frac{116}{81} = \underline{\underline{1.432}}$$

4 Let $A = \{\text{pass } A\}$ $P(A) = 4/5$ A, B, C, D are independent events.

$B = \{\text{pass } B\}$ $P(B) = 3/4$

$C = \{\text{pass } C\}$ $P(C) = 5/6$

$D = \{\text{pass } D\}$ $P(D) = 2/3$

$$p(\text{student qualifies}) = P(ABC\bar{D}) + P(ABD\bar{C}) + P(ACD\bar{B}) + P(ABCD)$$

$$= P(A)P(B)P(C)P(\bar{D}) + P(A)P(B)P(D)P(\bar{C}) + P(A)P(C)P(D)P(\bar{B}) + P(A)P(B)P(C)P(D)$$

$$= \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{1}{3} + \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{6} + \frac{4}{5} \cdot \frac{5}{6} \cdot \frac{2}{3} \cdot \frac{1}{4} + \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{2}{3}$$

$$= \frac{1}{6} + \frac{1}{15} + \frac{1}{9} + \frac{1}{3} = \frac{61}{90}$$

$$= \underline{\underline{0.678}}$$

$$E(x) = \sum_{x=0}^{4} x \cdot P[X=x]$$ (2)

$$= 0\cdot\frac{1}{6} + 1\cdot\frac{2}{9} + 2\cdot\frac{2}{9} + 3\cdot\frac{1}{3} + 4\cdot\frac{1}{18}$$

$$= \frac{2}{9} + \frac{4}{9} + 1 + \frac{2}{9}$$

$$= \frac{17}{9}$$

$$= \underline{\underline{1.89}}$$

$$Var(x) = E(x^2) - [E(x)]^2$$

$$E(x^2) = \sum_{x=0}^{4} x^2 P[X=x]$$

$$= 0\cdot\frac{1}{6} + 1\cdot\frac{2}{9} + 4\cdot\frac{2}{9} + 9\cdot\frac{1}{3} + 16\cdot\frac{1}{18}$$

$$= \frac{2}{9} + \frac{8}{9} + 3 + \frac{8}{9}$$

$$= 5$$

$$Var(x) = 5 - \left(\frac{17}{9}\right)^2 = \frac{116}{81} = \underline{\underline{1.432}}$$

(4) Let $A = \{\text{pass } A\}$ $\quad P(A) = 4/5$ $\quad$ A, B, C, D are independent events.

$B = \{\text{pass } B\}$ $\quad P(B) = 3/4$

$C = \{\text{pass } C\}$ $\quad P(C) = 5/6$

$D = \{\text{pass } D\}$ $\quad P(D) = 2/3$

$$p(\text{student qualifies}) = P(ABC\bar{D}) + P(ABD\bar{C}) + P(ACD\bar{B}) + P(ABCD)$$

$$= P(A)P(B)P(C)P(\bar{D}) + P(A)P(B)P(D)P(\bar{C}) + P(A)P(C)P(D)P(\bar{B}) + P(A)P(B)P(C)P(D)$$

$$= \frac{4}{5}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{3} + \frac{4}{5}\cdot\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{1}{6} + \frac{4}{5}\cdot\frac{5}{6}\cdot\frac{2}{3}\cdot\frac{1}{4} + \frac{4}{5}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{2}{3}$$

$$= \frac{1}{6} + \frac{1}{15} + \frac{1}{9} + \frac{1}{3} = \frac{61}{90}$$

$$= \underline{\underline{0.678}}$$

① (a) $f(x,y) = \ln(9 - x^2 - 9y^2)$

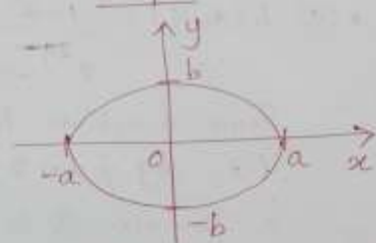
Since the logarithm is defined only for positive numbers,

$$9 - x^2 - 9y^2 > 0$$

$$\Rightarrow \frac{x^2}{9} + y^2 < 1$$

$$\Rightarrow \frac{x^2}{3^2} + \frac{y^2}{1^2} < 1$$

Ellipse

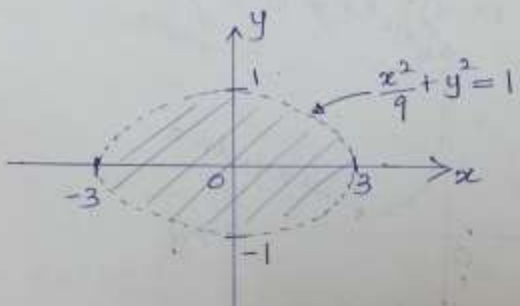


$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b)$$

center : $(0,0)$

vertices : $(\pm a, 0)$ on major axis

$(0, \pm b)$ on minor axis



∴ Domain is the set of points that lies interior of the ellipse $\frac{x^2}{9} + y^2 = 1$.

$$D = \left\{ \underset{\in \mathbb{R}^2}{(x,y)} : \frac{x^2}{9} + y^2 < 1 \right\}$$

(b) $g(x,y,z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2 - 16}}$

Since the squre root of a negative number is not real and division by zero is not defined,

$$x^2 + y^2 + z^2 - 16 > 0$$

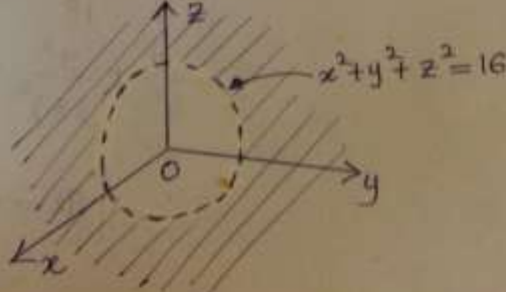
$$\Rightarrow x^2 + y^2 + z^2 > 16$$

sphere

$x^2 + y^2 + z^2 = R^2$

center : $(0,0)$

radius : R



$\therefore$ Domain is the set of points that lies completely outside the sphere $x^2+y^2+z^2=16$.

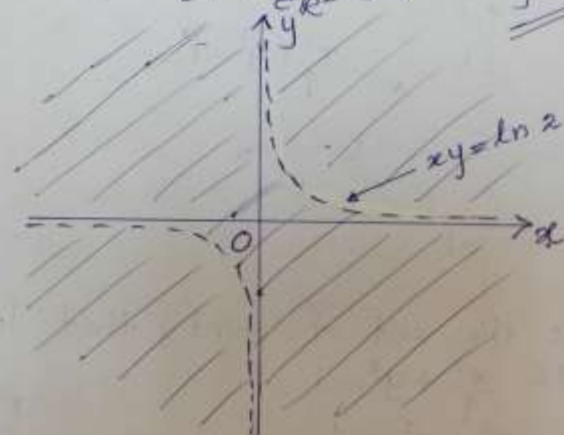
$$D=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2>16\}$$

• (c) $h(x,y)=\dfrac{1-x}{e^{xy}-2}$

Since division by zero is not defined, $e^{xy}-2\neq 0$

i.e. $xy\neq \ln 2$

$\therefore$ Domain $D=\{(x,y)\in\mathbb{R}^2 : xy\neq \ln 2\}$



• (d) $f(x,y)=\dfrac{\ln(x-y)}{\sqrt{18-2x^2-y^2}}$

The logarithm is defined only for positive numbers. And the square root of a negative number is not real and division by zero is not defined.

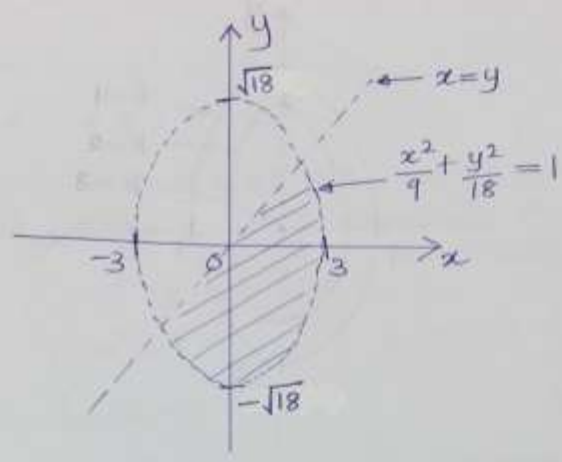
$\therefore$ we've $x-y>0$ and $18-2x^2-y^2>0$

$\Rightarrow x>y$

$$\Rightarrow 2x^2+y^2<18$$

$$\Rightarrow \frac{x^2}{9}+\frac{y^2}{18}<1$$

$$\Rightarrow \frac{x^2}{3^2}+\frac{y^2}{(\sqrt{18})^2}<1$$



$\therefore$ Domain $D = \left\{ \underset{\in \mathbb{R}^2}{(x,y)} : \frac{x^2}{9} + \frac{y^2}{18} < 1 \text{ and } x > y \right\}$

(2) (a) $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$

$\Rightarrow z = 2\left(1 - x^2 - \frac{y^2}{9}\right)^{1/2}$

Let $f(x,y) = 2\left(1 - x^2 - \frac{y^2}{9}\right)^{1/2}$.

$1 - \frac{z^2}{4} = x^2 + \frac{y^2}{9} \geq 0$

$\Rightarrow z^2 \leq 4$

$\Rightarrow -2 \leq z \leq 2$

Range $R = \{ z : z \in [-2, 2] \}$

Level curves of $f(x,y)$ are curves with equations $f(x,y) = k$, where k is a constant in the range of f.

$k = 0 \Rightarrow x^2 + \frac{y^2}{9} = 1$

$\frac{x^2}{1} + \frac{y^2}{9} = 1$

$k = 1 \Rightarrow 2\sqrt{1 - x^2 - \frac{y^2}{9}} = 1$

$1 - x^2 - \frac{y^2}{9} = \frac{1}{4}$

$x^2 + \frac{y^2}{9} = \frac{3}{4}$

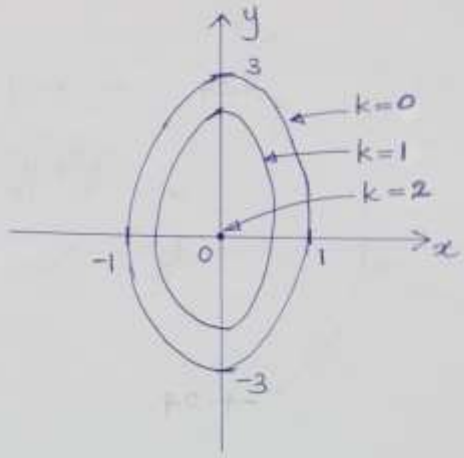
$\frac{x^2}{3/4} + \frac{y^2}{27/4} = 1$

$k = 2 \Rightarrow 2\sqrt{1 - x^2 - \frac{y^2}{9}} = 2$

$1 - x^2 - \frac{y^2}{9} = 1$

$x^2 + \frac{y^2}{9} = 0$

$\Rightarrow (x,y) \equiv (0,0)$



•(b) $z = 4x^2 + y^2$

Let $f(x,y) = 4x^2 + y^2$

Range $R = \{ z : z \in [0, \infty) \}$

Level curves of $f(x,y)$ are curves with equations $f(x,y) = k$, where k is a constant in the range of f.

$k = 0 \Rightarrow 4x^2 + y^2 = 0 \Rightarrow (x,y) = (0,0)$

$k = 1 \Rightarrow 4x^2 + y^2 = 1$

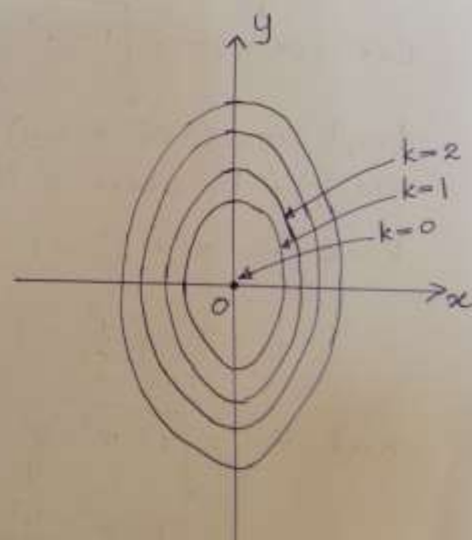
$$\frac{x^2}{1/4} + \frac{y^2}{1} = 1$$

$k = 2 \Rightarrow 4x^2 + y^2 = 2$

$$\frac{x^2}{1/2} + \frac{y^2}{2} = 1$$

$k = 3 \Rightarrow 4x^2 + y^2 = 3$

$$\frac{x^2}{3/4} + \frac{y^2}{3} = 1$$



• (c) $z = y^2 - x^2$

Let $f(x,y) = y^2 - x^2$

Range $R = \{z : z \in \mathbb{R}\}$

Level curves of $f(x,y)$ are curves with equations $f(x,y) = k$, where k is a constant in the range of f.

$k=0 \Rightarrow y^2 = x^2 \Rightarrow y = \pm x$

$k=1 \Rightarrow y^2 - x^2 = 1$

$\frac{y^2}{1} - \frac{x^2}{1} = 1$

$k=2 \Rightarrow y^2 - x^2 = 2$

$\frac{y^2}{2} - \frac{x^2}{2} = 1$

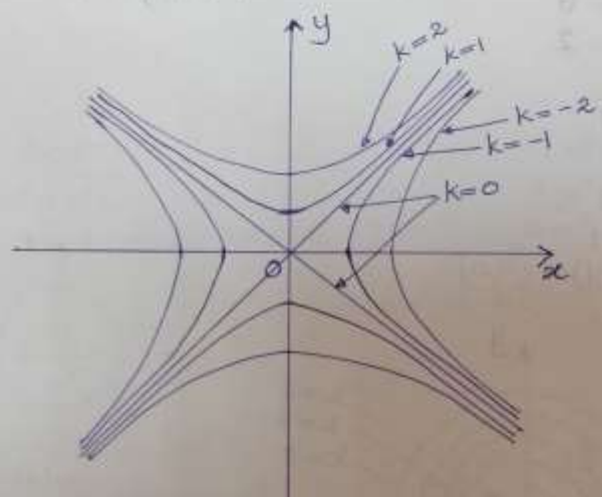
$k=-1 \Rightarrow y^2 - x^2 = -1$

$x^2 - y^2 = 1$

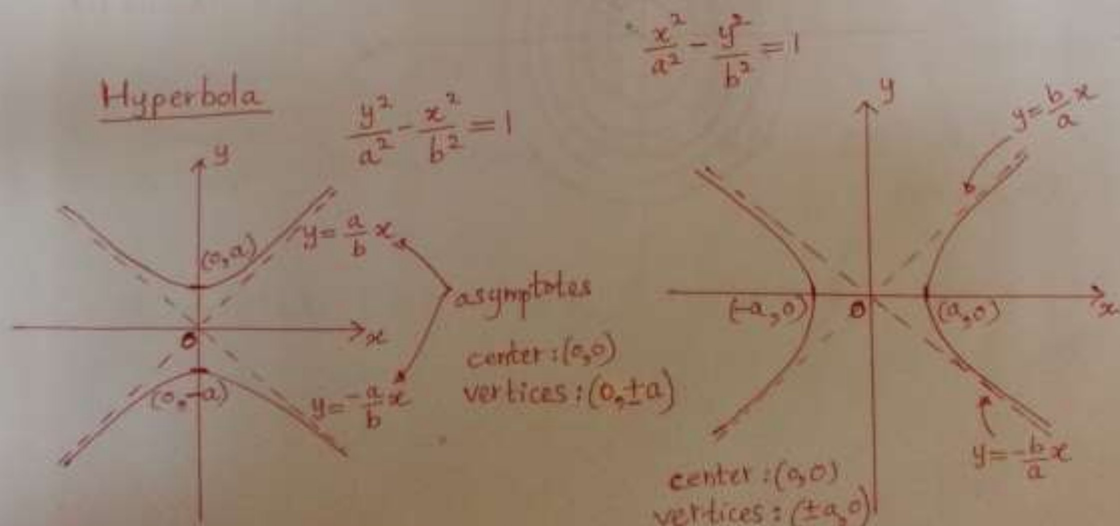
$\frac{x^2}{1} - \frac{y^2}{1} = 1$

$k=-2 \Rightarrow y^2 - x^2 = -2$

$\frac{x^2}{2} - \frac{y^2}{2} = 1$



Hyperbola



(d) $x^2+y^2+z=8$

$z = 8-x^2-y^2$

Let $f(x,y) = 8-x^2-y^2$

$x^2+y^2 \geq 0$

$-(x^2+y^2) \leq 0$

$8-x^2-y^2 \leq 8$

Range $R = \{z : z \in (-\infty, 8]\}$

Level curves of $f(x,y)$ are curves with equations $f(x,y)=k$, where k is a constant in the range of f.

$k=6 \Rightarrow 8-x^2-y^2 = 6$

$x^2+y^2 = 2$

$k=7 \Rightarrow 8-x^2-y^2 = 7$

$x^2+y^2 = 1$

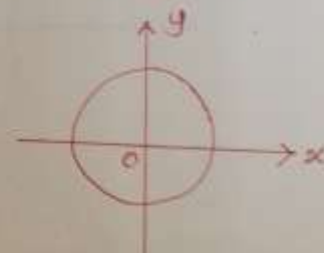
$k=8 \Rightarrow 8-x^2-y^2 = 8$

$x^2+y^2 = 0$

$\Rightarrow (x,y) \equiv (0,0)$

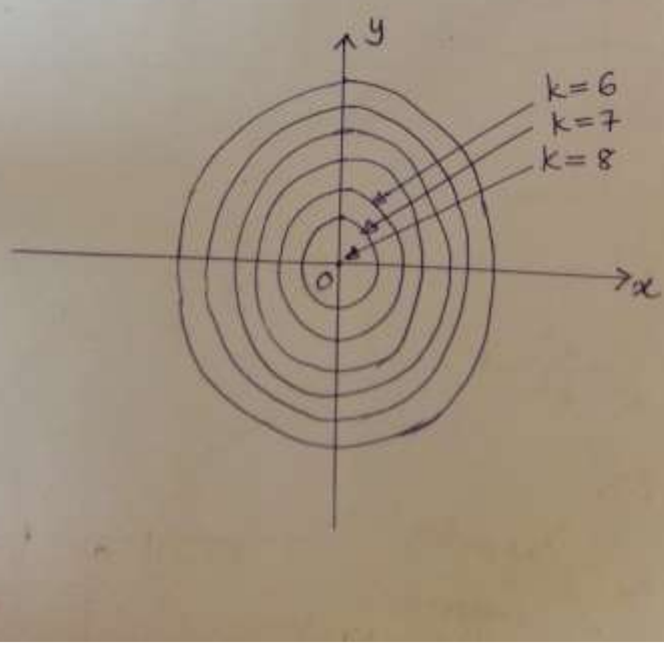
circle

$x^2+y^2=r^2$



center : $(0,0)$

radius : r



(d) $x^2+y^2+z=8$

$z = 8-x^2-y^2$

Let $f(x,y) = 8-x^2-y^2$

$x^2+y^2 \geq 0$

$-(x^2+y^2) \leq 0$

$8-x^2-y^2 \leq 8$

Range $R = \{z : z \in (-\infty, 8]\}$

Level curves of $f(x,y)$ are curves with equations $f(x,y)=k$, where k is a constant in the range of f.

$k=6 \Rightarrow 8-x^2-y^2 = 6$

$x^2+y^2 = 2$

$k=7 \Rightarrow 8-x^2-y^2 = 7$

$x^2+y^2 = 1$

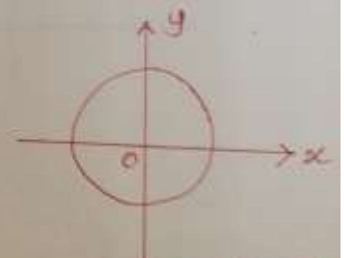
$k=8 \Rightarrow 8-x^2-y^2 = 8$

$x^2+y^2 = 0$

$\Rightarrow (x,y) \equiv (0,0)$

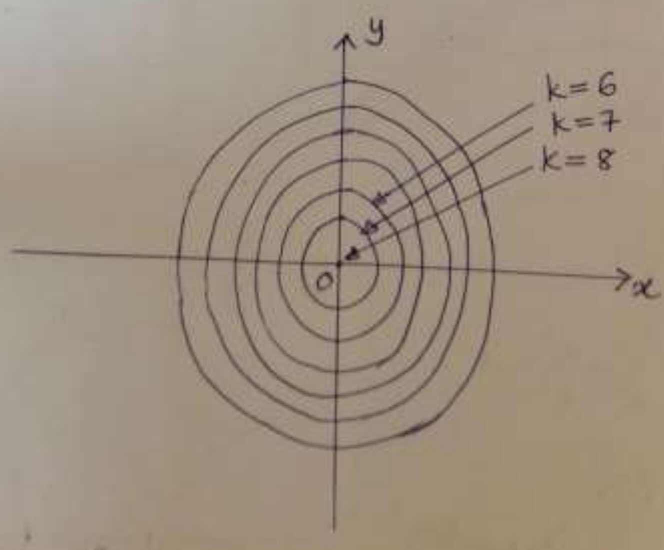
Circle

$x^2+y^2 = r^2$



center : $(0,0)$

radius : r



3

Let f be non-zero linear function of two variables with real coefficients.

Then, define $f(x,y) = ax + by + c$, where $a, b, c \in \mathbb{R}$.

We have to show that

$$\forall r \in \mathbb{R},\ \exists (x,y) \in \mathbb{R}^2 \text{ s.t. } f(x,y) = r$$

Let $f(x,y) = r$ where $r \in \mathbb{R}$

Then $ax + by + c = r$

$$\Rightarrow x = \frac{r-c-by}{a} \ ;\ a \neq 0$$

$$f\left(\frac{r-c-by}{a}, y\right) = a\frac{(r-c-by)}{a} + by + c$$
$$= r$$

$\therefore$ For $a \neq 0$,
$\forall r \in \mathbb{R}, \exists \left(\frac{r-c-by}{a}, y\right) \in \mathbb{R}^2$ s.t. $f(x,y) = r$.

And also, we've $y = \frac{r-c-ax}{b}$; $b \neq 0$

$$f\left(x, \frac{r-c-ax}{b}\right) = ax + b\frac{(r-c-ax)}{b} + c$$
$$= r$$

$\therefore$ For $b \neq 0$,
$\forall r \in \mathbb{R}, \exists \left(x, \frac{r-c-ax}{b}\right) \in \mathbb{R}^2$ s.t. $f(x,y) = r$.

~~For $a = b = 0$, $f(x,y) = c = r$ i.e. f is a constant function.~~
~~$\therefore$ it is obvious that $\forall r \in \mathbb{R}, \exists (x,y) \in \mathbb{R}^2$ s.t. $f(x,y) = r$.~~

So, conclude that $\forall r \in \mathbb{R}, \exists (x,y) \in \mathbb{R}^2$ s.t. $f(x,y) = r$

④ Let f be a f^n of two variables.
The level curves of f are the curves with equations $f(x,y) = k$, where k is a constant in the range of f.

Let $f(x,y) = k_1$ and $f(x,y) = k_2$ be two level curves of f where $k_1 \neq k_2$.

Need to show that $f(x,y) = k_1$ and $f(x,y) = k_2$ cannot intersect.

Assume that $f(x,y) = k_1$ and $f(x,y) = k_2$ can intersect.
Then, $\exists\, (x_0, y_0) \in \mathbb{R}^2$ s.t. $f(x_0, y_0) = k_1$ and
$f(x_0, y_0) = k_2$

$$\Rightarrow f(x_0, y_0) = k_1 = k_2$$

This is a contradiction.
$\therefore$ $f(x,y) = k_1$ and $f(x,y) = k_2$ cannot intersect
i.e. the distinct curves cannot intersect.

Tutorial 01 - Section C

(1)(a) Consider the f^n $f(x) = \ln(1+x)$.

We've shown that,

$$T_n(x,0) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \cdot x^k$$

And, range of convergence for $\ln(1+x)$ is $(-1, 1]$.

where $f^{(n)}(x) = \dfrac{(-1)^{n-1}(n-1)!}{(1+x)^n}$

Let $x = -0.98 < 0$. Then, $\ln(1-0.98) = \ln(0.02)$.

So, $T_n(-0.98, 0) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \cdot (-0.98)^k$

Consider the integral form of the Remainder,

$$R_n(x,a) = \frac{1}{n!} \int_a^x f^{(n+1)}(t) \cdot (x-t)^n \, dt$$

Then, $R_n(-0.98, 0) = \dfrac{1}{n!} \displaystyle\int_0^{-0.98} \frac{(-1)^n n!}{(1+t)^{n+1}} \cdot (-0.98-t)^n \, dt$

$$= (-1)^{2n} \int_0^{-0.98} \frac{(0.98+t)^n}{(1+t)^{n+1}} \, dt$$

$$|R_n(-0.98,0)| = \left| \int_0^{-0.98} \frac{(0.98+t)^n}{(1+t)^{n+1}} \, dt \right| < 10^{-6}$$

By finding such n, we've $n = 560$.

$$\therefore \; T_n(-0.98, 0) = \sum_{k=1}^{560} \frac{(-1)^{k-1}}{k} \cdot (-0.98)^k$$

① (a) Consider the f^n $f(x) = \ln(1+x)$.

We've shown that,

$$T_n(x,0) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \cdot x^k$$

And, range of convergence for $\ln(1+x)$ is $(-1, 1]$.

where $f^{(n)}(x) = \dfrac{(-1)^{n-1}(n-1)!}{(1+x)^n}$

Let $x = -0.98 < 0$. Then, $\ln(1-0.98) = \ln(0.02)$.

So, $T_n(-0.98,0) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \cdot (-0.98)^k$

Consider the integral form of the Remainder,

$$R_n(x,a) = \frac{1}{n!} \int_a^x f^{(n+1)}(t) \cdot (x-t)^n \, dt$$

Then, $R_n(-0.98,0) = \dfrac{1}{n!} \displaystyle\int_0^{-0.98} \frac{(-1)^n n!}{(1+t)^{n+1}} \cdot (-0.98-t)^n \, dt$

$$= (-1)^{2n} \int_0^{-0.98} \frac{(0.98+t)^n}{(1+t)^{n+1}} \, dt$$

$$|R_n(-0.98,0)| = \left| \int_0^{-0.98} \frac{(0.98+t)^n}{(1+t)^{n+1}} \, dt \right| < 10^{-6}$$

By finding such n, we've $n = 560$.

$$\therefore \; T_n(-0.98,0) = \sum_{k=1}^{560} \frac{(-1)^{k-1}}{k} \cdot (-0.98)^k$$

(b) Similarly, by considering the f^n $f(x) = \ln(1+x)$,

(i) $\ln(2) = \ln(1+1)$ where $x = 1 > 0$. Follow the same procedure.

(ii) $\ln(5) = \ln(4 \cdot 5/4)$

$$= \ln(4) + \ln(5/4)$$

$$= 2 \cdot \underbrace{\ln(2)}_{\checkmark} + \ln(1 + 1/4)$$

Consider $\ln(1 + 1/4)$. Let $x = 1/4 > 0$ and follow the same procedure. Then, we've $\ln(5)$.

(iii) $\ln(500) = \ln(5 \cdot 100)$

$$= \ln(5) + \ln(100)$$

$$= \ln(5) + \ln(5^2 \cdot 2^2)$$

$$= 3 \cdot \underbrace{\ln(5)}_{\checkmark} + 2 \cdot \underbrace{\ln(2)}_{\checkmark}$$

(iv) $\ln(0.2) = \ln(1/5)$

$$= \ln 1 - \ln 5$$

$$= -\underbrace{\ln 5}_{\checkmark}$$

(v) $\ln(0.02) = \ln(2/100)$

$$= \ln(2) - \ln(100)$$

$$= \ln(2) - \ln(5^2 \cdot 2^2)$$

$$= \ln(2) - 2 \ln(5) - 2 \cdot \ln(2)$$

$$= -2 \cdot \underbrace{\ln(5)}_{\checkmark} - \underbrace{\ln(2)}_{\checkmark}$$

(2)

(c) (i) Let $f(x) = e^x$

Then, $f^{(1)}(x) = e^x$

$f^{(2)}(x) = e^x$

$\vdots$

$f^{(n)}(x) = e^x$

$$f(x) = T_n(x,0) + R_n(x,0)$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} \cdot x^k + \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot x^{n+1}$$

where ξ between x and 0.

$$= f(0) + \sum_{k=1}^{n} \frac{f^{(k)}(0)}{k!} \cdot x^k + \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot x^{n+1}$$

$$= 1 + \sum_{k=1}^{n} \frac{x^k}{k!} + \frac{e^{\xi} \cdot x^{n+1}}{(n+1)!}$$

(d)

(i) $f(x) = e^{x}$

Shown that, $f(x) = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!}$

$$\Rightarrow f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Let $U_n = \frac{x^n}{n!}$. Then, $U_{n+1} = \frac{x^{n+1}}{(n+1)!}$

By using Ratio Test for the convergence of f,

$$\lim_{n\to\infty} \left| \frac{U_{n+1}}{U_n} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| < 1$$

$$\Rightarrow |x| \cdot \underbrace{\lim_{n\to\infty} \frac{1}{n+1}}_{=0} < 1$$

$$\Rightarrow \quad 0 < 1$$

This satisfies $\forall x \in \mathbb{R}$

Hence, the range of convergence of e^x is $\mathbb{R}$.

(e)

(i) $f(x) = e^{x}$

Shown that, $R_n(x,0) = \frac{e^{\xi} x^{n+1}}{(n+1)!}$ where ξ between x and 0.

Then,

$$\lim_{n\to\infty} |R_n(x,0)| = \lim_{n\to\infty} \left| \frac{e^{\xi} x^{n+1}}{(n+1)!} \right|$$

$$= |e^{\xi}| \cdot \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!}$$

Choose $m \in \mathbb{Z}^+$ s.t. $|x| < m \leq n$

• Then, $0 \leq \frac{|x|^n}{n!} = \frac{|x|^n}{(m-1)!\, m(m+1) \cdots (n-2)(n-1)n}$

$$\leq \frac{|x|^n}{(m-1)!\, \underbrace{m \cdot m \cdots m \cdot m \cdot m}_{(n-m+1)\text{ times}}}$$

$$= \frac{|x|^n}{m^{n-m+1}(m-1)!} = \frac{|x|^n}{m^{n+1}} \cdot \frac{m^m}{(m-1)!}$$

$$< \left(\frac{|x|}{m}\right)^n \cdot \frac{m^m}{(m-1)!} \quad \Rightarrow 0 \leq \frac{|x|^n}{n!} \leq \frac{m^m}{(m-1)!} \cdot \left(\frac{|x|}{m}\right)^n$$

Since $0 \leq |x| < m$, $0 \leq \frac{|x|}{m} < 1$.

Then, $\left\{\left(\frac{|x|}{m}\right)^n\right\}_{n=0}^{\infty}$ is a geometric sequence and converges to 0 as $n \to \infty$.

Since $\lim_{n\to\infty} 0 = 0$ and $\frac{m^m}{(m-1)!} \cdot \lim_{n\to\infty} \left(\frac{|x|}{m}\right)^n = 0$

by sandwitch theorem,

$$\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$$

$\therefore \lim_{n\to\infty} |R_n(x,0)| = |e^{\xi}| \cdot \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$

i.e. $|R_n(x,0)| \longrightarrow 0$ as $n \to \infty$.

(f) (i) $f(x) = e^x$ (5)

Shown that $R_n(x,0) = \dfrac{e^{\xi} x^{n+1}}{(n+1)!}$ where ξ between x and 0

Thus, for $x=1$,

$$R_n(1,0) = \frac{e^{\xi}}{(n+1)!} \quad \text{where } \xi \in (0,1)$$

For $0 < \xi < 1$, $1 < e^{\xi} < e$.

$$\Rightarrow \frac{1}{(n+1)!} < \frac{e^{\xi}}{(n+1)!} < \frac{e}{(n+1)!}$$

Thus, for $|R_n(1,0)| < 10^{-10}$ i.e. $\left|\dfrac{e^{\xi}}{(n+1)!}\right| < 10^{-10}$

$$\Rightarrow \frac{1}{(n+1)!} < 10^{-10}$$

$n=10 \Rightarrow \dfrac{1}{(10+1)!} = 0.25052 \times 10^{-7} > 10^{-10}$

$n=11 \Rightarrow \dfrac{1}{(11+1)!} = 0.20876 \times 10^{-8} > 10^{-10}$

$n=12 \Rightarrow \dfrac{1}{(12+1)!} = 0.1659 \times 10^{-9} > 10^{-10}$

$n=13 \Rightarrow \dfrac{1}{(13+1)!} = 0.114707 \times 10^{-10} < 10^{-10}$

$\therefore n = 13$

Shown that $e^x = 1 + \displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k!}$.

Hence, $e = 1 + \displaystyle\sum_{k=1}^{13} \frac{1}{k!} = 2.718281828$

(9) (i) For $x=4$,

$$R_n(4,0) = \frac{e^{\xi}\cdot 4^{n+1}}{(n+1)!} \quad \text{where } \xi \in (0,4)$$

For $0<\xi<4$, $1<e^{\xi}<e^{4}$

$$\Rightarrow \frac{4^{n+1}}{(n+1)!} < \frac{e^{\xi}\cdot 4^{n+1}}{(n+1)!} < \frac{e^{4}\cdot 4^{n+1}}{(n+1)!}$$

Thus, for $|R_n(4,0)|<10^{-6}$,

$$\frac{4^{n+1}}{(n+1)!} < 10^{-6}$$

$$n=16 \Rightarrow \frac{4^{16+1}}{(16+1)!} = 0.483\times 10^{-4} > 10^{-6}$$

$$n=17 \Rightarrow \frac{4^{17+1}}{(17+1)!} = 0.107\times 10^{-4} > 10^{-6}$$

$$n=18 \Rightarrow \frac{4^{18+1}}{(18+1)!} = 0.22596\times 10^{-5} > 10^{-6}$$

$$n=19 \Rightarrow \frac{4^{19+1}}{(19+1)!} = 0.45193\times 10^{-6} < 10^{-6}$$

$\therefore n=19$

Hence, $e^{4} = 1+\sum_{k=1}^{19}\frac{4^k}{k!} = 54.59814948$

(h) Let $f(x) = x^4$ is continuous on $x \in \mathbb{R}$ (6)

Let E be the true value for the approximation.

Then, $|e - E| < r = 10^{-10}$.

Let, $\dfrac{f(e) - f(E)}{e - E} = f'(c)$ for some c between e and E.

Then, $\left|\dfrac{f(e) - f(E)}{e - E}\right| = |f'(c)| = |4c^3|$

$$\Rightarrow \left|\frac{e^4 - E^4}{e - E}\right| = 4c^3 \quad (\because c > 0)$$

Since $e < c < E$, $e^3 < c^3 < E^3$

$4e^3 < 4c^3 < 4E^3$

So, $4e^3 < \left|\dfrac{e^4 - E^4}{e - E}\right| < 4E^3$

$$4e^3 |e - E| < |e^4 - E^4| < 4E^3 \underbrace{|e - E|}_{< 10^{-10}}$$

$$< 80.34215 \times 10^{-10}$$

$$= 0.8 \times 10^{-10} < 10^{-6}$$

Mean Value Th$^{\underline{m}}$

If f is differentiable on (a, b) and cts on $[a, b]$ then

$$\frac{f(b) - f(a)}{b - a} = f'(c) \text{ for some } c \in (a, b).$$

(i) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$ ⑦

$\Rightarrow e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$

Assume that **e** is rational.

Then, $\exists\, a, b \in \mathbb{Z}$, $b \neq 0$ s.t. $e = \frac{a}{b}$

Since $a = be$ is an integer $b! \cdot e$ is an integer.

So, $b! \cdot e = b! \left[1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{b!} \right] +$

$b! \left[\frac{1}{(b+1)!} + \frac{1}{(b+2)!} + \cdots \right]$

Let $N = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{b!}$.

Then, $b! \cdot e = b! \cdot N + \left[\frac{1}{(b+1)} + \frac{1}{(b+1)(b+2)} + \cdots \right]$

$$= b! \cdot N + \frac{1}{(b+1)} \left[1 + \frac{1}{(b+2)} + \frac{1}{(b+2)(b+3)} + \cdots \right]$$

$$< b! \cdot N + \frac{1}{(b+1)} \left[1 + \frac{1}{(b+1)} + \frac{1}{(b+1)^2} + \cdots \right]$$

$$= b! \cdot N + \frac{1}{(b+1)} \left[\frac{1}{1 - \frac{1}{(b+1)}} \right] \quad \left(\because \left| \frac{1}{b+1} \right| < 1 \right)$$

$$= b! \cdot N + \frac{1}{(b+1)} \cdot \frac{(b+1)}{b}$$

$$= b! \cdot N + \frac{1}{b}$$

Then, $b! \cdot e < b! \cdot N + \frac{1}{b} \Rightarrow b! (e - N) < \frac{1}{b}$

Since $e > N$, $e - N > 0$

Thus, $\boxed{0 < b!(e-N) < \frac{1}{b} < 1}$

But $a = b! \cdot e$ is an integer and

$b! \cdot N = b! \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(b-1)!} + \frac{1}{b!} \right)$

$= 1 + b + b(b-1) + \cdots + b!$ is an integer.

Hence, $b! \cdot e - b! \cdot N = b!(e - N)$ should be an integer ※

$\therefore$ e is irrational