Tutorial 04

Section B

① Let $f(x,y,z) = x^2 z - y^3 z^5 + xy = 9$

Then the surface is a level surface of f.

Therefore, the gradient of f at point P = (1,2,-1)

is normal to the surface.

vector $\Rightarrow$ $\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$

$= \langle 2xz + y, -3y^2 z^5 + x, x^2 - 5z^4 y^3 \rangle$

~~$\nabla f(p)$~~.

$\nabla f(1,2,-1) = \langle 0, 13, -39 \rangle$

$= 13 \langle 0, 1, -3 \rangle$

The tangent plane at P $\Rightarrow$

$\nabla f(1,2,-1) \cdot \langle x-1, y-2, z+1 \rangle = 0$

$13\langle 0,1,-3 \rangle \langle x-1, y-2, z+1 \rangle = 0$

$13 [0 + y-2 - 3(z+1)] = 0$

$0 + y - 2 - 3z - 3 = 0$

$y - 3z - 5 = 0$ //

Laplace operator $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$.

when $u = u(x, y)$ & $x = r\cos\theta$, $y = r\sin\theta$.

Consider partial derivatives,

$$\frac{\partial x}{\partial r} = \cos\theta \quad (1), \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta \quad (2)$$

$$\frac{\partial y}{\partial r} = \sin\theta \quad (3), \qquad \frac{\partial y}{\partial \theta} = r\cos\theta \quad (4)$$

u si~~s~~ a f^n of both x & y. ~~&~~ x & y are the f^n of ~~x &~~ bot r & θ.

∴ By chain Rule

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}.$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\cdot\sin\theta. \qquad \text{(by (1) \& (3))}$$

$$\frac{\partial u}{\partial r} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}. \quad (5)$$

Now, calculate for $\frac{\partial^2 u}{\partial r^2}$

$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial}{\partial r}\left[\frac{\partial u}{\partial r}\right] = \frac{\partial}{\partial r}\left[\cos\theta\cdot\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}\right].$$

$$= \cos\theta\frac{\partial}{\partial r}\cdot\frac{\partial u}{\partial x} + \sin\theta\frac{\partial}{\partial r}\cdot\frac{\partial u}{\partial y}$$

Since $\frac{\partial u}{\partial x}$ & $\frac{\partial u}{\partial y}$ are both f^ns of x and y

apply the chain Rule,

(2)

$$\frac{\partial^2 u}{\partial r^2} = \cos\theta\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\frac{\partial y}{\partial r}\right]$$

$$+\sin\theta\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\cdot\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\frac{\partial y}{\partial r}\right].$$

Substiting for the partial derivaties of x & y,

$$\frac{\partial^2 u}{\partial r^2} = \cos\theta\left[\frac{\partial}{\partial x}\cdot\frac{\partial u}{\partial x}\cdot\cos\theta + \frac{\partial}{\partial y}\cdot\frac{\partial u}{\partial x}\cdot\sin\theta\right]$$

$$+\sin\theta\left[\frac{\partial}{\partial x}\cdot\frac{\partial u}{\partial y}\cdot\cos\theta + \frac{\partial}{\partial y}\cdot\frac{\partial u}{\partial y}\cdot\sin\theta\right].$$

(by 1 & 3)

$$\frac{\partial^2 u}{\partial r^2} = \cos^2\theta\,\frac{\partial^2 u}{\partial x^2} + 2\sin\theta\cos\theta\,\frac{\partial^2 u}{\partial x\partial y} + \sin^2\theta\,\frac{\partial^2 u}{\partial y^2}$$

Now calculate the, $\frac{\partial u}{\partial\theta}$ & $\frac{\partial^2 u}{\partial\theta^2}$.

$$\frac{\partial u}{\partial\theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial\theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial\theta}.$$

$$\frac{\partial u}{\partial\theta} = -r\sin\theta\,\frac{\partial u}{\partial x} + r\cos\theta\,\frac{\partial u}{\partial y}.$$

(by 2 & 4)

Consider

$$\frac{\partial u}{\partial \theta} = -r\sin\theta \frac{\partial u}{\partial x} + r\cos\theta \frac{\partial u}{\partial y} \qquad (3)$$

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left[-r\sin\theta \frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial \theta}\left[r\cos\theta \frac{\partial u}{\partial y}\right].$$

$$= -r\sin\theta \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial u}{\partial x}\frac{\partial}{\partial \theta}(-r\sin\theta)$$

$$+ \cancel{\frac{\partial}{\partial \theta} r\cos}\cdot r\cos\theta \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial u}{\partial y}\frac{\partial}{\partial \theta}(r\cos\theta)$$

$$= \left(-\frac{\partial u}{\partial x} r\cos\theta - \frac{\partial u}{\partial y} r\sin\theta\right) - r\sin\theta \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right)$$

$$+ r\cos\theta \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial y}\right)$$

Apply chain rule.

$$\frac{\partial^2 u}{\partial \theta^2} = \left(-\frac{\partial u}{\partial x} r\cos\theta - \frac{\partial u}{\partial y} r\sin\theta\right) - r\sin\theta\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\frac{\partial u}{\partial x}\frac{\partial y}{\partial \theta}\right]$$

$$+ r\cos\theta\left[\frac{\partial}{\partial x}\frac{\partial u}{\partial y}\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}\right]$$

$$= \left(-\frac{\partial u}{\partial x} r\cos\theta - \frac{\partial u}{\partial y} r\sin\theta\right) + r^2\sin^2\theta \frac{\partial^2 u}{\partial x^2}$$

$$- 2r^2\sin\theta\cos\theta \frac{\partial^2 u}{\partial y \partial x} + r^2\cos^2\theta \frac{\partial^2 u}{\partial y^2}$$

(by (2) & (4)).

Now consider

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$= \cos^2\theta \frac{\partial^2 u}{\partial x^2} \cancel{+ 2\sin\theta\cos\theta \frac{\partial^2 u}{\partial x \partial y}} + \sin^2\theta \frac{\partial^2 u}{\partial y^2}$$

$$+ \frac{1}{r}\left[\cancel{\cos\theta \frac{\partial u}{\partial x}} + \cancel{\sin\theta \frac{\partial u}{\partial y}}\right]$$

$$+ \frac{1}{r^{\cancel{2}}}\left[-\cancel{r}\cancel{\cos\theta \frac{\partial u}{\partial x}} - \cancel{r}\cancel{\sin\theta \frac{\partial u}{\partial y}}\right] + \sin^2\theta \frac{\partial^2 u}{\partial x^2}$$

$$\cancel{- 2\sin\theta\cos\theta \frac{\partial u}{\partial y \partial x}} + \cos^2\theta \frac{\partial^2 u}{\partial y^2}$$

$$= \frac{\partial^2 u}{\partial x^2}\Big[\underbrace{\sin^2\theta + \cos^2\theta}_{1}\Big] + \frac{\partial^2 u}{\partial y^2}\Big[\underbrace{\sin^2\theta + \cos^2\theta}_{1}\Big]$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad //$$

④ (a) $f(x,y) = x^4 + y^4 - x^2 - y^2 + 1$

For critical points, $f_x = 0$ and $f_y = 0$

$f_x = 4x^3 - 2x$ and $f_y = 4y^3 - 2y$

$f_x = 0 \Rightarrow 2x(2x^2 - 1) = 0$

$x = 0$ or $x = \pm 1/\sqrt{2}$

$f_y = 0 \Rightarrow 2y(2y^2 - 1) = 0$

$y = 0$ or $y = \pm 1/\sqrt{2}$

critical points are

$(0,0)$, $(0, 1/\sqrt{2})$, $(0, -1/\sqrt{2})$, $(1/\sqrt{2}, 0)$, $(-1/\sqrt{2}, 0)$,
$(1/\sqrt{2}, 1/\sqrt{2})$, $(1/\sqrt{2}, -1/\sqrt{2})$, $(-1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$

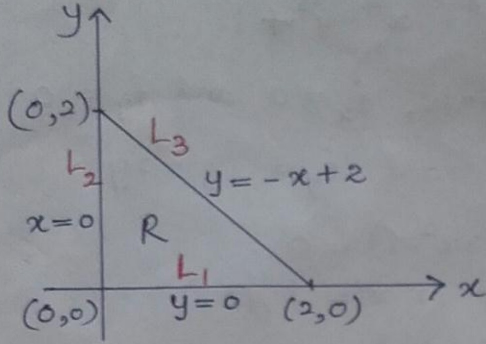
(b) Use the second derivative test

$D = f_{xx}(a,b) \cdot f_{yy}(a,b) - [f_{xy}(a,b)]^2$

$f_{xx} = 12x^2 - 2$, $f_{yy} = 12y^2 - 2$ and $f_{xy} = 0$

point	D	$f_{xx}(a,b)$	$f(a,b)$
$(0,0)$	$(-2)(-2) > 0$	$-2 < 0$	a local maximum
$(0, 1/\sqrt{2})$	$(-2)(4) < 0$		a saddle point
$(0, -1/\sqrt{2})$	$(-2)(4) < 0$		a saddle point
$(1/\sqrt{2}, 0)$	$(4)(-2) < 0$		a saddle point
$(-1/\sqrt{2}, 0)$	$(4)(-2) < 0$		a saddle point
$(1/\sqrt{2}, 1/\sqrt{2})$	$(4)(4) > 0$	$4 > 0$	a local minimum
$(1/\sqrt{2}, -1/\sqrt{2})$	$(4)(4) > 0$	$4 > 0$	a local minimum
$(-1/\sqrt{2}, 1/\sqrt{2})$	$(4)(4) > 0$	$4 > 0$	a local minimum
$(-1/\sqrt{2}, -1/\sqrt{2})$	$(4)(4) > 0$	$4 > 0$	a local minimum

⑦ $f(x,y) = x^2+y^2-x-y+1$



$y-2 = \left(\frac{0-2}{2-0}\right)(x-0)$

$y-2 = (-1)x$

$y = -x+2$

f is a polynomial. So, it is continuous on the given closed, bounded set in $\mathbb{R}^2$.

Use the Extreme Value Th$^{\underline{m}}$.

* For critical points, $f_x = 0$ and $f_y = 0$

$f_x = 2x-1$ and $f_y = 2y-1$

$f_x = 0 \Rightarrow x = 1/2$ and $f_y = 0 \Rightarrow y = 1/2$

$\therefore (1/2, 1/2)$ is the only critical point of f in the interior of R.

* The boundary lines of R are L_1, L_2 and L_3.

<u>On line L_1,</u> $y=0$ and $0 \leq x \leq 2$

$f(x,0) = x^2-x+1$

Let $g(x) = x^2-x+1$

$\Rightarrow g'(x) = 2x-1$

$g'(x) = 0 \Rightarrow x = 1/2$

$\therefore (1/2, 0)$ is an extreme point of f on L_1.

And also, (0,0) and (2,0) are extreme points of f on L_1.

On line L_2, $x=0$ and $0 \le y \le 2$.

$f(0,y) = y^2 - y + 1$

Let $h(y) = y^2 - y + 1$

$\Rightarrow h'(y) = 2y - 1$

$h'(y) = 0 \Rightarrow y = 1/2$

$\therefore (0, 1/2)$ is an extreme point of f on L_2.
And also, $(0,0)$ and $(0,2)$ are extreme points of f on L_2.

On line L_3, $y = -x + 2$

$f(x,y) = x^2 + y^2 - x - y + 1$

$f(x, -x+2) = x^2 + (-x+2)^2 - x - (-x+2) + 1$

$= 2x^2 - 4x + 3$

Let $l(x) = 2x^2 - 4x + 3$

$\Rightarrow l'(x) = 4x - 4$

$l'(x) = 0 \Rightarrow x = 1$ and then $y = 1$

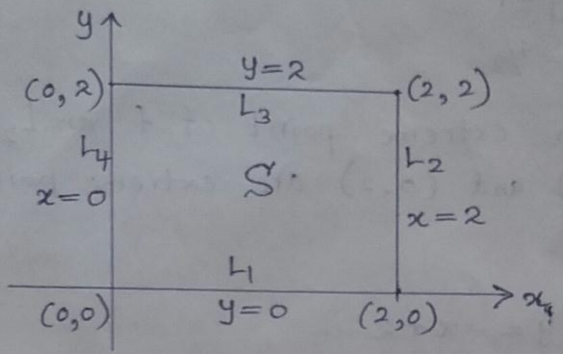
$\therefore (1,1)$ is an extreme point of f on L_3.
And also, $(0,2)$ and $(2,0)$ are extreme points of f on L_3.

*

	points	values of f	
critical point →	$(1/2, 1/2)$	$\frac{1}{4} + \frac{1}{4} - \frac{1}{2} - \frac{1}{2} + 1 = \frac{1}{2}$	← absolute minimum value
vertices of R	$(0,0)$	1	
	$(2,0)$	$4 - 2 + 1 = 3$	← absolute maximum value
	$(0,2)$	$4 - 2 + 1 = 3$	
	$(1/2, 0)$	$\frac{1}{4} - \frac{1}{2} + 1 = \frac{3}{4}$	
	$(0, 1/2)$	$\frac{1}{4} - \frac{1}{2} + 1 = \frac{3}{4}$	
	$(1,1)$	$1 + 1 - 1 - 1 + 1 = 1$	

⑥ $f(x,y) = x^4 + y^4 - 4xy + 2$

$S = \{ (x,y) \mid 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 2 \}$



f is a polynomial. So, it is continuous on the given closed, bounded set in $\mathbb{R}^2$.

* For critical points, $f_x = 0$ and $f_y = 0$

$f_x = 4x^3 - 4y$ and $f_y = 4y^3 - 4x$

$f_x = 0 \Rightarrow x^3 = y$ and $f_y = 0 \Rightarrow y^3 = x$

$\Rightarrow (x^3)^3 = x$

$x^9 - x = 0$

$x(x^8 - 1) = 0$

$x(x^4 - 1)(x^4 + 1) = 0$

$x(x^2 - 1)(x^2 + 1)(x^4 + 1) = 0$

$x(x-1)(x+1)(x^2+1)(x^4+1) = 0$

$x = 0$ or $x = \pm 1$

$x = 0 \Rightarrow y = 0$, $x = 1 \Rightarrow y = 1$, $x = -1 \Rightarrow y = -1$

critical points are $(0,0)$, $(1,1)$, $(-1,-1)$

But $(-1,-1)$ is not in the interior of S.

$\therefore$ critical points are $(0,0)$ and $(1,1)$.

* The boundary lines of S are L_1, L_2, L_3 and L_4. 3

On line L_1, $y=0$ and $0 \le x \le 2$

$f(x,0) = x^4 + 2$

$f'(x,0) = 4x^3$

$f'(x,0) = 0 \Rightarrow x = 0$

$\therefore$ $(0,0)$ is an extreme point of f on L_1

On line L_2, $x=2$ and $0 \le y \le 2$

$f(2,y) = 16 + y^4 - 8y + 2 = y^4 - 8y + 18$

$f'(2,y) = 4y^3 - 8$

$f'(2,y) = 0 \Rightarrow y^3 = 2$

$y = 2^{1/3}$

$\therefore$ $(2, 2^{1/3})$ is an extreme point of f on L_2.

On line L_3, $y=2$ and $0 \le x \le 2$

$f(x,2) = x^4 + 16 - 8x + 2 = x^4 - 8x + 18$

$f'(x,2) = 4x^3 - 8$

$f'(x,2) = 0 \Rightarrow x^3 = 2$

$x = 2^{1/3}$

$\therefore$ $(2^{1/3}, 2)$ is an extreme point of f on L_3.

On line L_4, $x=0$ and $0 \le y \le 2$

$f(0,y) = y^4 + 2$

$f'(0,y) = 4y^3$

$f'(0,y) = 0 \Rightarrow y = 0$

$\therefore$ $(0,0)$ is an extreme point of f on L_4.

*

point	values of f
$(0,0)$	2
$(1,1)$	$1+1-4+2=0$
$(2,0)$	$16+2=18$
$(2,2)$	$16+16-16+2=18$
$(0,2)$	$16+2=18$
$(2,2^{1/3})$	$16+2^{4/3}-8.2^{1/3}+2=10.44$
$(2^{1/3},2)$	$2^{4/3}+16-8.2^{1/3}+2=10.44$

$\therefore$ the absolute maximum value is 18
and the absolute minimum value is 0