Tutorial 04 - Section A

① $f(x) = \begin{cases} \frac{A}{x^4} & ; x \geq 1 \\ 0 & ; x < 1 \end{cases}$

(a) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\underbrace{\int_{-\infty}^{1} 0\,dx}_{=0} + \int_{1}^{\infty} \frac{A}{x^4}\,dx = 1$$

$$\left(-\frac{A}{3}\right) \int_{1}^{\infty} (-3)x^{-4}\,dx = 1$$

$$\left(-\frac{A}{3}\right) \left[x^{-3}\right]_{1}^{\infty} = 1$$

$$-\frac{A}{3} \left[\frac{1}{x^3}\right]_{1}^{\infty} = 1$$

$$-\frac{A}{3} \left[0 - 1\right] = 1$$

$$A = 3$$

(b) $P(X > 5) = 1 - P(X \leq 5)$

$$= 1 - \int_{1}^{5} \frac{3}{x^4}\,dx$$

$$= 1 + \int_{1}^{5} (-3)x^{-4}\,dx$$

$$= 1 + \left[x^{-3}\right]_{1}^{5}$$

$$= 1 + \frac{1}{5^3} - 1$$

$$= 1/125 = 0.008$$

$$P(X < n+1 \mid X > n) = \frac{P(n < X < n+1)}{P(X > n)}$$

$$= \frac{P(n < X < n+1)}{1 - P(X \leq n)}$$

$$= \frac{\int_n^{n+1} \frac{3}{x^4}\, dx}{1 - \int_1^n \frac{3}{x^4}\, dx}$$

$$= \frac{-\int_n^{n+1} (-3)\, x^{-4}\, dx}{1 + \int_1^n (-3)\, x^{-4}\, dx}$$

$$= \frac{-\left[x^{-3}\right]_n^{n+1}}{1 + \left[x^{-3}\right]_1^n}$$

$$= \frac{-\frac{1}{(n+1)^3} + \frac{1}{n^3}}{1 + \frac{1}{n^3} - 1}$$

$$= 1 - \left(\frac{n}{n+1}\right)^3$$

(2) $Y = \{1, 2, 3, \dots\}$

$$F(n) = \sum_{y=1}^{n} p(y) \quad (\text{def}^n)$$

$$F(1) = p(1)$$

$$F(2) = p(1) + p(2) = F(1) + p(2)$$

$$\Rightarrow p(2) = F(2) - F(1)$$

$$F(3) = \underbrace{p(1) + p(2)}_{= F(2)} + p(3)$$

$$\Rightarrow p(3) = F(3) - F(2)$$

$$\vdots$$

$$F(n) = \underbrace{p(1) + p(2) + \dots + p(n-1)}_{F(n-1)} + p(n)$$

$$\Rightarrow p(n) = F(n) - F(n-1) \text{ for } n = 2, 3, \dots$$

Thus,
$$p(y) = \begin{cases} F(1) \; ; & y = 1 \\ F(y) - F(y-1) \; ; & y = 2, 3, \dots \end{cases}$$

(3) $F(x) = \begin{cases} 0 & ; x<0 \\ mx^n & ; 0 \le x \le 2 \\ 1 & ; x>2 \end{cases}$

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx = 2/3$$

For $x<0$, $f(x) = \frac{d\, F(x)}{dx} = \frac{d\,(0)}{dx} = 0$

For $0 \le x \le 2$, $f(x) = \frac{d\, F(x)}{dx} = \frac{d\,(mx^n)}{dx} = mnx^{n-1}$

For $x>2$, $f(x) = \frac{d\, F(x)}{dx} = \frac{d\,(1)}{dx} = 0$

pdf

$$f(x) = \begin{cases} mnx^{n-1} & ; 0 \le x \le 2 \\ 0 & ; \text{otherwise} \end{cases}$$

$$E(x) = \int_0^2 x \cdot mnx^{n-1}\, dx = 2/3$$

$$mn \int_0^2 x^n\, dx = \frac{2}{3}$$

$$\frac{mn}{(n+1)} \left[x^{n+1} \right]_0^2 = \frac{2}{3}$$

$$\frac{mn}{(n+1)} \cdot 2^{n+1} = \frac{2}{3}$$

$$\frac{mn \cdot 2^n}{n+1} = \frac{1}{3} \longleftarrow (1)$$

From cdf, $F(2) = m \cdot 2^n$

we've $P(X \leq 2) = 1 \Rightarrow F(2) = 1$

$\therefore \; m \cdot 2^n = 1$

① $\Rightarrow \dfrac{n}{n+1} = \dfrac{1}{3}$ $\qquad m = \dfrac{1}{2^n} = \dfrac{1}{2^{1/2}}$

$n = \dfrac{1}{2}$

④ $f(x) = \begin{cases} x^2(2x + 3/2) & ; \; 0 < x \leq 1 \\ 0 & ; \; \text{otherwise} \end{cases}$

$Y = \dfrac{2}{X} + 3$

$E(Y) = E\left(\dfrac{2}{X} + 3\right) = E\left(\dfrac{2}{X}\right) + E(3) = 2 \cdot E\left(\dfrac{1}{X}\right) + 3$

$$E\left(\frac{1}{X}\right) = \int_{-\infty}^{\infty} \frac{1}{x} \cdot f(x)\, dx$$

$$= \int_0^1 \frac{1}{x} \cdot x^2(2x + 3/2)\, dx$$

$$= \int_0^1 \left(2x^2 + \frac{3}{2}x\right) dx$$

$$= \left[\frac{2}{3}x^3 + \frac{3}{4}x^2\right]_0^1$$

$$= \frac{2}{3} + \frac{3}{4}$$

$$= \frac{17}{12}$$

$E(Y) = 2 \cdot \dfrac{17}{12} + 3 = \dfrac{35}{6} = 5.83$

$$\mathrm{Var}\left(\frac{1}{X}\right) = E\left[\left(\frac{1}{X} - E\left(\frac{1}{X}\right)\right)^2\right] \quad (\mathrm{def}^n)$$

$$= E\left(\frac{1}{X^2}\right) - \left(E\left(\frac{1}{X}\right)\right)^2$$

$$E\left(\frac{1}{X^2}\right) = \int_{-\infty}^{\infty} \frac{1}{x^2} \cdot f(x)\, dx = 5/2$$

$$\mathrm{Var}\left(\frac{1}{X}\right) = \frac{5}{2} - \left(\frac{17}{12}\right)^2 = \frac{71}{144}$$

$$\mathrm{Var}(Y) = 4 \cdot \frac{71}{144} = \frac{71}{36} = \underline{\underline{1.97}}$$

Question 05

(a) $E(X) = \sum_{r=1}^{5} r \cdot P(X=r)$

$= 1\times 0 + 2\times\frac{1}{4} + 3\times\frac{1}{3} + 4\times\frac{1}{3} + 5\times\frac{1}{12}$

$= \frac{13}{4} = 3.25$

$E(Y) = \sum_{r=1}^{5} r \cdot P(Y=r)$

$= 1\times\frac{1}{5} + 2\times 0 + 3\times\frac{2}{5} + 4\times 0 + 5\times\frac{2}{5}$

$= \frac{17}{5} = 3.4$

$Var(X) = E\left[(X - E(X))^2\right] = E(X^2) - [E(X)]^2$

$E(X^2) = \sum_{r=1}^{5} r^2 \cdot P(X=r)$

$= 1^2\times 0 + 2^2\times\frac{1}{4} + 3^2\times\frac{1}{3} + 4^2\times\frac{1}{3} + 5^2\times\frac{1}{12}$

$= \frac{137}{12}$

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$Var(X) = \frac{137}{12} - \left(\frac{13}{4}\right)^2 = \frac{41}{48} = 0.854$

$Var(Y) = E\left[(Y - E(Y))^2\right] = E(Y^2) - [E(Y)]^2$

$E(Y^2) = \sum_{r=1}^{5} r^2 \cdot P(Y=r)$

$= 1^2\times\frac{1}{5} + 2^2\times 0 + 3^2\times\frac{2}{5} + 4^2\times 0 + 5^2\times\frac{2}{5}$

$= \frac{69}{5}$

$Var(Y) = \frac{69}{5} - \left(\frac{17}{5}\right)^2 = \frac{56}{25} = 2.24$

(b) $Z = X - Y$

$X = \{1, 2, 3, 4, 5\}$ and $Y = \{1, 2, 3, 4, 5\}$

Thus, $Z = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$

$$P(Z=-4) = P(X=1) \times P(Y=5)$$
$$= 0 \times 2/5$$
$$= 0$$

$$P(Z=-3) = P(X=1) \times P(Y=4) + P(X=2) \times P(Y=5)$$
$$= 0 \times 0 + \frac{1}{4} \times \frac{2}{5}$$
$$= \frac{1}{10}$$

$$P(Z=-2) = P(X=1) \times P(Y=3) + P(X=2) \times P(Y=4)$$
$$+ P(X=3) \times P(Y=5)$$
$$= 0 \times 2/5 + \frac{1}{4} \times 0 + \frac{1}{3} \times \frac{2}{5}$$
$$= \frac{2}{15}$$

$$P(Z=-1) = P(X=1) \times P(Y=2) + P(X=2) \times P(Y=3)$$
$$+ P(X=3) \times P(Y=4) + P(X=4) \times P(Y=5)$$
$$= 0 \times 0 + \frac{1}{4} \times \frac{2}{5} + \frac{1}{3} \times 0 + \frac{1}{3} \times \frac{2}{5}$$
$$= \frac{7}{30}$$

$$P(Z=0) = P(X=1) \times P(Y=1) + P(X=2) \times P(Y=2)$$
$$+ P(X=3) \times P(Y=3) + P(X=4) \times P(Y=4)$$
$$+ P(X=5) \times P(Y=5)$$
$$= 0 \times 1/5 + 1/4 \times 0 + \frac{1}{3} \times \frac{2}{5} + 1/3 \times 0 + \frac{1}{12} \times \frac{2}{5}$$
$$= \frac{1}{6}$$

$$P(Z=1) = P(X=2)\times P(Y=1) + P(X=3)\times P(Y=2)$$ (6)

$$+ P(X=4)\times P(Y=3) + P(X=5)\times P(Y=4)$$

$$= \frac{1}{4}\times\frac{1}{5} + \frac{1}{3}\times 0 + \frac{1}{3}\times\frac{2}{5} + \frac{1}{12}\times 0$$

$$= \frac{11}{60}$$

$$P(Z=2) = P(X=3)\times P(Y=1) + P(X=4)\times P(Y=2)$$

$$+ P(X=5)\times P(Y=3)$$

$$= \frac{1}{3}\times\frac{1}{5} + \frac{1}{3}\times 0 + \frac{1}{12}\times\frac{2}{5}$$

$$= \frac{1}{10}$$

$$P(Z=3) = P(X=4)\times P(Y=1) + P(X=5)\times P(Y=2)$$

$$= \frac{1}{3}\times\frac{1}{5} + \frac{1}{12}\times 0$$

$$= \frac{1}{15}$$

$$P(Z=4) = P(X=5)\times P(Y=1)$$

$$= \frac{1}{12}\times\frac{1}{5}$$

$$= \frac{1}{60}$$

r	-4	-3	-2	-1	0	1	2	3	4
$P(Z=r)$	0	$\frac{1}{10}$	$\frac{2}{15}$	$\frac{7}{30}$	$\frac{1}{6}$	$\frac{11}{60}$	$\frac{1}{10}$	$\frac{1}{15}$	$\frac{1}{60}$

$$E(Z) = \sum_{r=-4}^{4} r.\,P(Z=r)$$

$$= (-4)\times 0 + (-3)\times\frac{1}{10} + (-2)\times\frac{2}{15} + (-1)\times\frac{7}{30} + 0\times\frac{1}{6}$$

$$+ 1\times\frac{11}{60} + 2\times\frac{1}{10} + 3\times\frac{1}{15} + 4\times\frac{1}{60} = \underline{\underline{-0.15}}$$

OR $E(Z) = E(X-Y) = E(X) - E(Y) = \frac{13}{4} - \frac{17}{5} = \underline{\underline{-0.15}}$

$$Var(Z) = E\left[\left(Z - E(Z)\right)^2\right] = E(Z^2) - \left[E(Z)\right]^2$$

$$E(Z^2) = \sum_{r=-4}^{4} r^2 . P(Z=r)$$

$$= (-4)^2 \times 0 + (-3)^2 \times \frac{1}{10} + (-2)^2 \times \frac{2}{15} + (-1)^2 \times \frac{7}{30} + 0 \times \frac{1}{6}$$

$$+ 1^2 \times \frac{11}{60} + 2^2 \times \frac{1}{10} + 3^2 \times \frac{1}{15} + 4^2 \times \frac{1}{60}$$

$$= 3.1166$$

$$Var(Z) = 3.1166 - (-0.15)^2 = 3.094$$

$$Var(Z) = Var(X - Y)$$

$$= Var(X) + Var(Y) \quad (\because X, Y \text{ independent})$$

$$= \frac{41}{48} + \frac{56}{25}$$

$$= \underline{\underline{3.094}}$$