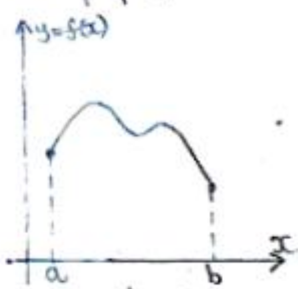
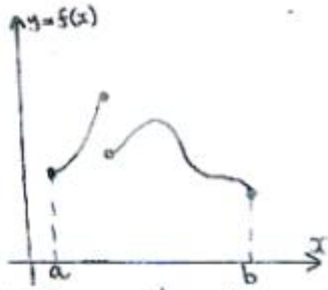
<u>Continuity</u>

In English, the word continuous would mean unbroken. The same notion will be maintained when we try to define a continuous function. Before giving a rigorous definition regarding the continuity of a function, we will explain this notion using a graph.

Let f be a function and suppose a and b are two points in the domain of f such that $a < b$. f is said to be continuous in $[a, b]$ if the graph of $y = f(x)$ can be drawn from $x = a$ to $x = b$ without lifting the pen from the paper.



The graph of a continuous function.



The graph of a function which is not continuous.

Now, we are ready for the rigorous definition which we have promised. We will first define the continuity of a function at a point. Later on, we will extend this definition to discuss the continuity of a function in an interval.

Definition 1

Let f be a real valued function defined on $\mathbb{R}$. f is said to be continuous at $x = a$ if and only if

i) $\lim_{x \to a} f(x)$ exists

ii) $\lim_{x \to a} f(x) = f(a)$

Example 1

Check to see whether the following function is continuous at $x = 1$.

$$f(x) = \begin{cases} x+3 & , x \leq 1 \\ 2-x & , x > 1 \end{cases}$$

Let $\varepsilon > 0$. Then

$$\begin{aligned} 1-\varepsilon < x < 1 &\Rightarrow -\varepsilon < x-1 < 0 \\ &\Rightarrow -\varepsilon < (x+3)-4 < 0 \\ &\Rightarrow |f(x)-4| < \varepsilon \end{aligned}$$

$$\therefore \lim_{x \to 1^-} f(x) = 4$$

$$1 < x < 1+\varepsilon \Rightarrow 0 < x-1 < \varepsilon$$
$$\Rightarrow 0 < -[(2-x)-1] < \varepsilon$$
$$\Rightarrow 0 < -[f(x)-1] < \varepsilon$$
$$\Rightarrow |f(x)-1| < \varepsilon$$

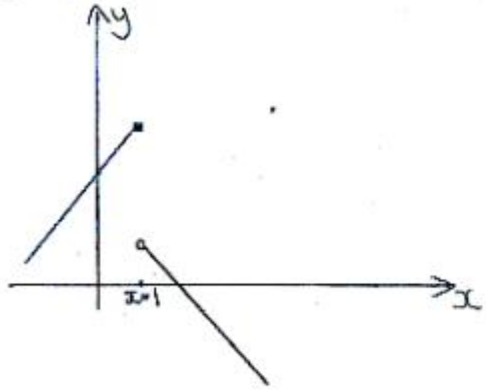
$$\therefore \lim_{x \to 1^+} f(x) = 1$$

So, $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$

$\therefore$ By theorem 10 in chapter 4, $\lim_{x \to 1} f(x)$ doesn't exist.

$\therefore$ f is not continuous at $x = 1$.

Now, let us see the graph of $y = f(x)$.



While drawing the graph,
we had to lift the pen at $x=1$.
$\therefore$ We can't expect the function to be continuous at $x=1$.

When a function is not continuous at a particular point, we say that it is discontinuous there. So, the above function is discontinuous at $x=1$.

For the rest of this chapter, whenever a function f is mentioned, it is assumed to be a real valued function defined on $\mathbb{R}$.

<u>Theorem 1</u>

A function f is continuous at $x=a$ if and only if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that
$|x-a| < \delta \Longrightarrow |f(x)-f(a)| < \varepsilon$

<u>Proof</u>: Assume f is continuous at $x=a$.
Let $\varepsilon > 0$. Because $\lim_{x \to a} f(x) = f(a)$,
$\exists \delta > 0$ such that

$$0 < |x-a| < \delta \Longrightarrow |f(x)-f(a)| < \varepsilon$$

$$\begin{aligned} |x-a| < \delta &\Longrightarrow x=a \text{ or } 0<|x-a|<\delta \\ &\Longrightarrow f(x)=f(a) \text{ or } |f(x)-f(a)|<\varepsilon \\ &\Longrightarrow |f(x)-f(a)|<\varepsilon \end{aligned}$$

Now, assume this claim.
Let $\varepsilon > 0$. From the claim, $\exists \delta > 0$ such that $|x-a| < \delta \Rightarrow |f(x)-f(a)| < \varepsilon$

$$0 < |x-a| < \delta \Rightarrow |x-a| < \delta$$
$$\Rightarrow |f(x) - f(a)| < \varepsilon$$

$\therefore \lim_{x \to a} f(x)$ exists, and
$$\lim_{x \to a} f(x) = f(a)$$

$\therefore f$ is continuous at $x = a$.

<u>Definition 2</u> Let f be a function.

i) If f is continuous at x $\forall x \in [a,b]$, then f is said to be continuous in the interval $[a,b]$.

ii) If f is continuous at x $\forall x \in R$, then f is said to be continuous in R.

<u>Theorem 2</u>

A polynomial is continuous in R.

<u>Proof</u> Let f be a polynomial and let $a \in R$ be an arbitrary real number.
By theorem 8 in chapter 4,
$$\lim_{x \to a} f(x) = f(a)$$

$\therefore f$ is continuous at $x = a$ $\forall a \in R$.

$\therefore f$ is continuous in R.

Theorem 3

i) $f(x) = \sin x$ is continuous in R

ii) $f(x) = \cos x$ is continuous in R

Proof i) Let $a \in R$.

The fact $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ can

stated as $\lim_{x \to a} \frac{\sin\frac{x-a}{2}}{\frac{x-a}{2}} = 1$

$\therefore \exists \delta_1 > 0$ such that

$$0 < |x-a| < \delta_1 \implies \left| \frac{\sin\frac{x-a}{2}}{\frac{x-a}{2}} - 1 \right|$$

$$\implies \left| \frac{\sin\frac{x-a}{2}}{\frac{x-a}{2}} \right| - 1 < \frac{1}{2}$$

$$\implies \left| \sin\frac{x-a}{2} \right| < \frac{3}{4}|x-a|$$

Now,

$$|\sin x - \sin a| = 2\left|\cos\frac{x+a}{2}\right|\left|\sin\frac{x-a}{2}\right|$$
$$\leq 2\left|\sin\frac{x-a}{2}\right|$$

Let $\varepsilon > 0$.

Define $\delta = \operatorname{Min}\{\delta_1, \frac{2\varepsilon}{3}\}$

$$0 < |x-a| < \delta \implies$$
$$2\left|\sin\frac{x-a}{2}\right| < \frac{6}{4}|x-a| < \frac{6}{4}\left(\frac{2\varepsilon}{3}\right)$$
$$\implies |\sin x - \sin a| < \varepsilon$$

$\therefore \lim_{x \to a} \sin x = \sin a$

$\therefore f(x) = \sin x$ is continuous at $x = a$ where a is any real number.

$\therefore f(x)$ is continuous in R.

The proof of ii) is very similar.

Theorem 4

Let f and g be functions which are continuous at $x = a$. Then

i) $f+g$ is continuous at $x = a$.

ii) $f-g$ is continuous at $x = a$.

iii) fg is continuous at $x = a$.

iv) $\frac{f}{g}$ is continuous at $x = a$ provided that $g(a) \neq 0$.

The proof is an exercise for the student.

Definition 3

The points at which a function f is not continuous are called the discontinuities of f.

Some of the discontinuities can be removed by redefining the function.

Example 2

$$f(x) = \frac{x^2-8x+15}{x^2-x-6} \quad \forall x \neq -2, 3$$

For f to be continuous at $x=a$, $f(a)$ has to be defined as a finite value. Hence, we can see that our f has discontinuities at $x=-2$ and $x=3$.

Now, $f(x) = \frac{(x-3)(x-5)}{(x-3)(x+2)} = \frac{x-5}{x+2} \quad \forall x \neq 3$

When taking the limit $x \to 3$, we are interested only in a deleted neighbourhood of the type $(3-\delta, 3+\delta) \setminus \{3\}$.

$$\therefore \lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x-5}{x+2} = \frac{-2}{5}$$

Now, define f in this manner

$$f(x) = \begin{cases} \frac{x^2-8x+15}{x^2-x-6} & \text{if } x \neq -2, 3 \\ \frac{-2}{5} & \text{if } x = 3 \end{cases}$$

Then $\lim_{x\to 3} f(x) = f(3)$

$\therefore$ f is continuous at $x=3$.

When $x<-2$, $(x-3)(x-5) > 0$

$(x-3)(x+2) > 0$

$\therefore f(x) > 0$

When $-2<x<0$, $(x-3)(x-5) > 0$

$(x-3)(x+2) < 0$

$\therefore f(x) < 0$

$\therefore \lim_{x\to -2} f(x)$ doesn't exist.

$\therefore$ There is no way to redefine f to remove the discontinuity at -2.

Suppose a function f is continuous at $x=a$. According to the definition I,

$$\lim_{x\to a} f(x) = f(a) = f(\lim_{x\to a} x)$$

So, the operations of taking the function value and taking the limit are commutative.

Now, if a continuous function f is defined on a convergent sequence of points $\{x_n\}$, we would expect $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$

However, because we are talking about a different kind of limit here, that result needs to be proven. That will be our next theorem.

Theorem 5

Let $\{x_n\}$ be a sequence such that $x_n \to x_0$ and let f be a function which is continuous at x_0. Then $\{f(x_n)\}$ is a convergent sequence and $f(x_n) \to f(x_0)$.

Proof: Let $\varepsilon > 0$. $\exists \delta > 0$ such that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$$

Because $x_n \to x_0$, $\exists\, n_0 \in \mathbb{Z}_+$ such that

$$n > n_0 \implies |x_n - x_0| < \delta$$
$$\implies |f(x_n) - f(x_0)| < \varepsilon$$

$\therefore$ $\{f(x_n)\}$ is convergent and

$$\lim_{n\to\infty} f(x_n) = f(x_0)$$

Note that it was not necessary for f to be continuous at the sequence points $\{x_n\}$.