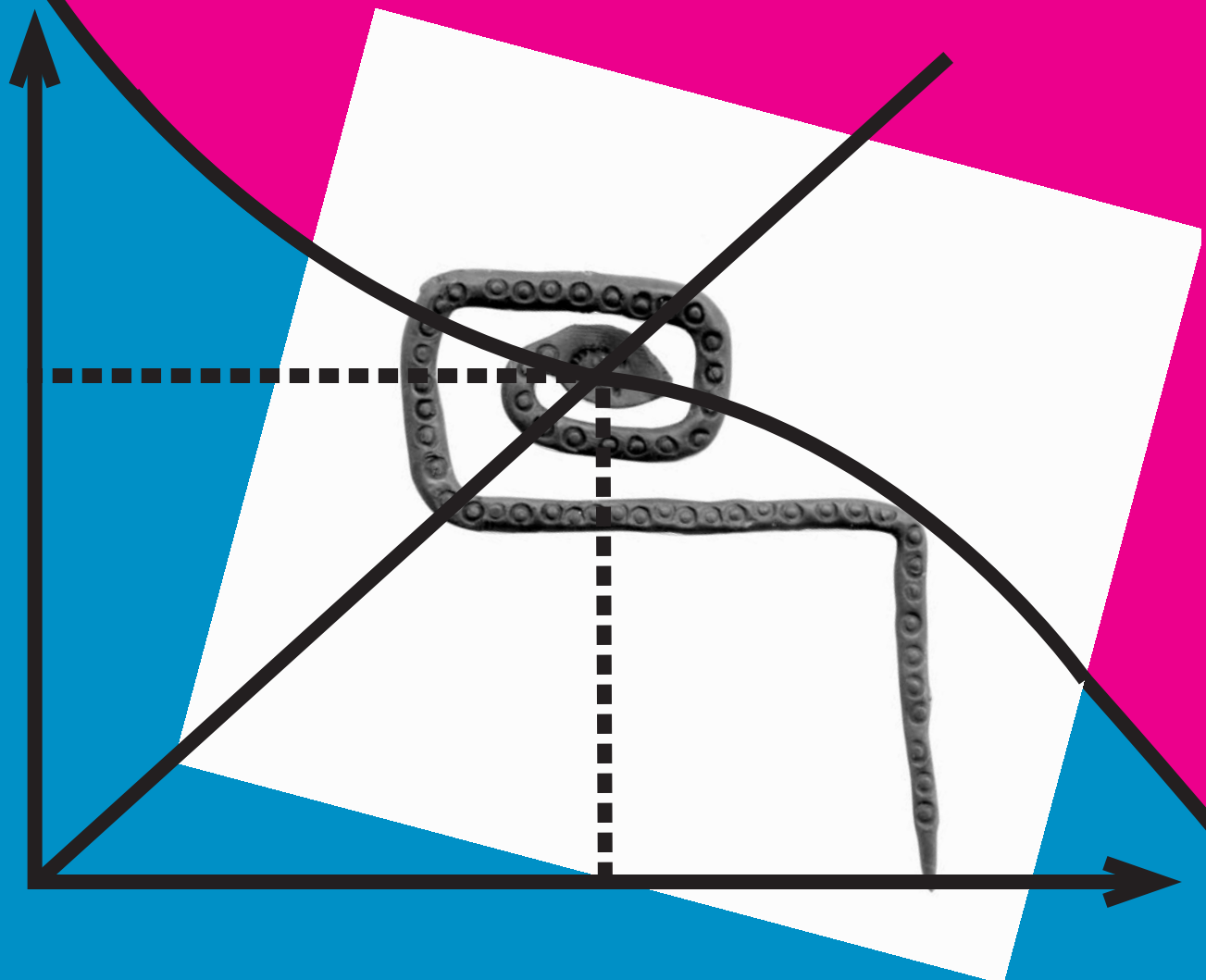


**NUMERICAL ANALYSIS
AND
NUMERICAL METHODS
WITH
MATHEMATICA
AND
CASIO CALCULATORS**



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To my students of o8 batch

With best wishes for a bright future

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Write down the n^{th} degree Taylor polynomial for the function $f(x) = x^{\frac{1}{10}}$, near $x = c > 0$.

$$f(x) = P_n(x, c) + R_n(c, x)$$

$$P_n(x, c) = \sum_{r=0}^n (x-c)^r \frac{f^{(r)}(c)}{r!}$$

$$R_n(x, c) = (x-c)^{(n+1)} \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

Here $\xi \in (x, c) \cup (c, x)$

i.e. ξ lies between x and c

i.e. $0 < c < \xi < x$ or $0 < x < \xi < c$

$$f(x) = x^{\frac{1}{10}}$$

$$f^{(r)}(x) = \frac{1}{10} \left(\frac{1}{10} - 1 \right) \left(\frac{1}{10} - 2 \right) \dots \left(\frac{1}{10} - (r-1) \right) x^{\frac{1}{10}-r}$$

$$P_n(x, c) = f(c) + \sum_{r=1}^n \frac{(x-c)^r}{r!} \frac{1}{10} \left(\frac{1}{10} - 1 \right) \left(\frac{1}{10} - 2 \right) \dots \left(\frac{1}{10} - (r-1) \right) c^{\frac{1}{10}-r}$$

Show that $|R_n| < \begin{cases} \frac{x^{\frac{1}{10}}}{10(n+1)} \left(\frac{x-c}{c} \right)^{n+1} & \text{if } x > c \\ \frac{c^{\frac{1}{10}}}{10(n+1)} \left(\frac{c-x}{x} \right)^{n+1} & \text{if } x < c \end{cases}$

$$R_n(x, c) = (x-c)^{(n+1)} \frac{f^{(n+1)}(\xi)}{(n+1)!} = \frac{(x-c)^{(n+1)}}{(n+1)!} \frac{1}{10} \left(\frac{1}{10} - 1 \right) \left(\frac{1}{10} - 2 \right) \dots \left(\frac{1}{10} - n \right) \xi^{\frac{1}{10}-(n+1)}$$

$$|R_n(x, c)|$$

$$= \left| \frac{(x-c)^{(n+1)}}{(n+1)!} \frac{1}{10} \left(\frac{1}{10} - 1 \right) \left(\frac{1}{10} - 2 \right) \dots \left(\frac{1}{10} - n \right) \xi^{\frac{1}{10}-(n+1)} \right|$$

$$< \frac{|x-c|^{n+1}}{(n+1)!} \frac{1}{10} |(-1)(-2)\dots(-n)| |\xi|^{\frac{1}{10}-(n+1)} = \frac{|x-c|^{n+1} |\xi|^{\frac{1}{10}-(n+1)}}{10(n+1)}$$

When $0 < c < \xi < x$

$$|R_n(x, c)| < \frac{|x-c|^{n+1} |\xi|^{\frac{1}{10}-(n+1)}}{10(n+1)} = \frac{(x-c)^{n+1}}{10(n+1)} \frac{\xi^{\frac{1}{10}}}{\xi^{n+1}} < \frac{(x-c)^{n+1}}{10(n+1)} \frac{x^{\frac{1}{10}}}{c^{n+1}} = \frac{x^{\frac{1}{10}}}{10(n+1)} \left(\frac{x-c}{c} \right)^{n+1}$$

When $0 < x < \xi < c$

$$|R_n(x, c)| < \frac{|x-c|^{n+1} |\xi|^{\frac{1}{10}-(n+1)}}{10(n+1)} = \frac{(c-x)^{n+1}}{10(n+1)} \frac{\xi^{\frac{1}{10}}}{\xi^{n+1}} < \frac{(c-x)^{n+1}}{10(n+1)} \frac{c^{\frac{1}{10}}}{x^{n+1}} = \frac{c^{\frac{1}{10}}}{10(n+1)} \left(\frac{c-x}{x} \right)^{n+1}$$

$$\text{Therefore } |R_n(x, c)| < \begin{cases} \frac{x^{10}}{10(n+1)} \left(\frac{x-c}{c}\right)^{n+1} & \text{if } x > c > 0 \\ \frac{c^{10}}{10(n+1)} \left(\frac{c-x}{x}\right)^{n+1} & \text{if } 0 < x < c \end{cases}$$

Let $\bar{P}_n(x, c)$ and $\bar{u}_r(c, x)$ denote the corresponding rounded quantities

$$\begin{aligned} & |f(x) - \bar{P}_n(x, c)| \\ &= |f(x) - P_n(x, c) + P_n(x, c) - \bar{P}_n(x, c)| \\ &< |f(x) - P_n(x, c)| + |P_n(x, c) - \bar{P}_n(x, c)| \\ &= |R_n(x, c)| + \left| \sum_{r=0}^n (u_r(x, c) - \bar{u}_r(c, x)) \right| \\ &= |R_n(x, c)| + \left| \sum_{r=1}^n (u_r(x, c) - \bar{u}_r(c, x)) \right| \\ &< |R_n(x, c)| + \sum_{r=1}^n |u_r(x, c) - \bar{u}_r(c, x)| \\ &< |R_n(x, c)| + \sum_{r=1}^n \frac{1}{2} 10^{-p} \\ &= |R_n(x, c)| + \frac{n}{2} 10^{-p} \end{aligned}$$

$$\text{Therefore } |f(x) - \bar{P}_n(x, c)| < |R_n(x, c)| + \frac{n}{2} 10^{-p}$$

$$|f(x) - \bar{P}_n(x, c)| < \frac{1}{2} 10^{-m}$$

$$\Leftrightarrow |f(x) - \bar{P}_n(x, c)| < |R_n(x, c)| + \frac{n}{2} 10^{-p} < \frac{1}{2} 10^{-m}$$

$$\Leftrightarrow |f(x) - \bar{P}_n(x, c)| < |R_n(x, c)| + \frac{n}{2} 10^{-p} < \frac{1}{2} 10^{-(m+1)} + \frac{n}{2} 10^{-p} < \frac{1}{2} 10^{-m}$$

This means that we can make $|f(x) - \bar{P}_n(x, c)| < \frac{1}{2} 10^{-m}$ by selecting n and p so that

$$|R_n(x, c)| < \frac{1}{2} 10^{-(m+1)} \quad \text{and} \quad \frac{1}{2} 10^{-(m+1)} + \frac{n}{2} 10^{-p} < \frac{1}{2} 10^{-m}$$

$$\text{But } \frac{1}{2} 10^{-(m+1)} + \frac{n}{2} 10^{-p} < \frac{1}{2} 10^{-m} \Leftrightarrow n 10^{-p} < 10^{-(m+1)} (10-1) \Leftrightarrow 10^{p-(m+1)} > \frac{n}{9} \Leftrightarrow p > m+1 + \log_{10} \left(\frac{n}{9} \right)$$

So when m is given

$$\text{find } n \text{ so that it satisfies } |R_n(x, c)| < \frac{1}{2} 10^{-(m+1)} \quad \text{and then find } p \text{ so that } p > m+1 + \log_{10} \left(\frac{n}{9} \right)$$

Deduce that the value of $1000^{\frac{1}{10}}$ correct to three decimal places is 1.995.

Here $m = 3$ and $x = 1000$. We also realize that $c = 1024$ is a good choice since $c^{\frac{1}{10}} = 1024^{\frac{1}{10}} = (2^{10})^{\frac{1}{10}} = 2$.

Also in this situation $0 < x < \xi < c$ so we have $|R_n(x, c)| < \frac{c^{\frac{1}{10}}}{10(n+1)} \left(\frac{c-x}{x} \right)^{n+1}$

Therefore

$$|R_n(x, c)| < \frac{1}{2} 10^{-(m+1)}$$

$$\Leftrightarrow |R_n(x, c)| < \frac{c^{\frac{1}{10}}}{10(n+1)} \left(\frac{c-x}{x} \right)^{n+1} < \frac{1}{2} 10^{-(m+1)}$$

$$\Leftrightarrow \frac{1024^{\frac{1}{10}}}{10(n+1)} \left(\frac{1024-1000}{1000} \right)^{n+1} < \frac{1}{2} 10^{-(3+1)}$$

$$\Leftrightarrow \frac{2}{10(n+1)} \left(\frac{24}{1000} \right)^{n+1} < \frac{1}{2} 10^{-4}$$

$$\Leftrightarrow (n+1) \left(\frac{10^3}{24} \right)^{n+1} > 4 \cdot 10^3$$

$$\Leftrightarrow (n+1) 10^{3n} > 96.24^n$$

$$\Leftrightarrow (2+1) 10^6 = 3000000 > 55296 = 96.24^2$$

$$\Leftrightarrow n = 2$$

$$p > m+1 + \log_{10} \left(\frac{n}{9} \right) = 3+1 + \log_{10} \left(\frac{2}{9} \right) = 4 - \log_{10} \left(\frac{9}{2} \right) \Leftrightarrow p = 4$$

So to find $f(x)$ to an accuracy to the 3rd decimal place ($m = 3$), we have to take 3 terms ($n+1 = 2+1 = 3$) of the Taylor Polynomial and round each term to the 4th decimal place ($p = 4$).

$$P_n(x, c) = f(c) + \sum_{r=1}^n \frac{(x-c)^r}{r!} \frac{1}{10} \left(\frac{1}{10} - 1 \right) \left(\frac{1}{10} - 2 \right) \dots \left(\frac{1}{10} - (r-1) \right) c^{\frac{1}{10}-r}$$

$$P_2(1000, 1024)$$

$$= f(1024) + \sum_{r=1}^2 \frac{(1000-1024)^r}{r!} \frac{1}{10} \left(\frac{1}{10} - 1 \right) \left(\frac{1}{10} - 2 \right) \dots \left(\frac{1}{10} - (r-1) \right) 1024^{\frac{1}{10}-r}$$

$$= (2^{10})^{\frac{1}{10}} + \frac{(-24)}{1} \frac{1}{10} (2^{10})^{-\frac{9}{10}} + \frac{(-24)^2}{2} \frac{1}{10} \left(-\frac{9}{10} \right) (2^{10})^{-\frac{19}{10}}$$

$$= 2 - \frac{24}{10} \frac{1}{2^9} - \frac{9 \cdot 24^2}{200} \frac{1}{2^{19}}$$

$$= 2 - 0.00468 \dots - 0.00004 \dots$$

$$\bar{P}_2(1000, 1024)$$

$$= 2.0000 - 0.0047 - 0.0000 = 1.9953$$

So to the 3rd decimal place accuracy

$$1000^{\frac{1}{10}} = 1.995$$

Determine the value of $1025^{\frac{1}{10}}$ correct to ten decimal places.

Here $m = 10$ and $x = 1025$. We also realize that $c = 1024$ is a good choice since $c^{\frac{1}{10}} = 1024^{\frac{1}{10}} = (2^{10})^{\frac{1}{10}} = 2$.

Also in this situation $0 < c < \xi < x$ so we have $|R_n(x, c)| < \frac{x^{\frac{1}{10}}}{10(n+1)} \left(\frac{x-c}{c}\right)^{n+1}$

Therefore

$$\begin{aligned} |R_n(x, c)| &< \frac{1}{2} 10^{-(m+1)} \\ \Leftrightarrow |R_n(x, c)| &< \frac{x^{\frac{1}{10}}}{10(n+1)} \left(\frac{x-c}{c}\right)^{n+1} < \frac{1}{2} 10^{-(m+1)} \\ \Leftrightarrow \frac{1025^{\frac{1}{10}}}{10(n+1)} \left(\frac{1025-1024}{1024}\right)^{n+1} &< \frac{3}{10(n+1)} \left(\frac{1}{1024}\right)^{n+1} < \frac{1}{2} 10^{-(10+1)} \\ \Leftrightarrow \frac{3}{10(n+1)} \frac{1}{1024^{n+1}} &< \frac{1}{2} 10^{-11} \\ \Leftrightarrow (n+1)1024^{n+1} &> 60.1000^3 \\ \Leftrightarrow (3+1)1024^{3+1} = 4.1024.1024^3 &> 60.1000^3 \\ \Leftrightarrow n &= 3 \end{aligned}$$

$$p > m + 1 + \log_{10} \left(\frac{n}{9}\right) = 10 + 1 + \log_{10} \left(\frac{3}{9}\right) = 11 - \log_{10}(3) \Leftrightarrow p = 11$$

So to find $f(x)$ to an accuracy to the 10th decimal place ($m = 10$), we have to take 4 terms ($n + 1 = 3 + 1 = 4$) of the Taylor Polynomial and round each term to the 11th decimal place ($p = 11$).

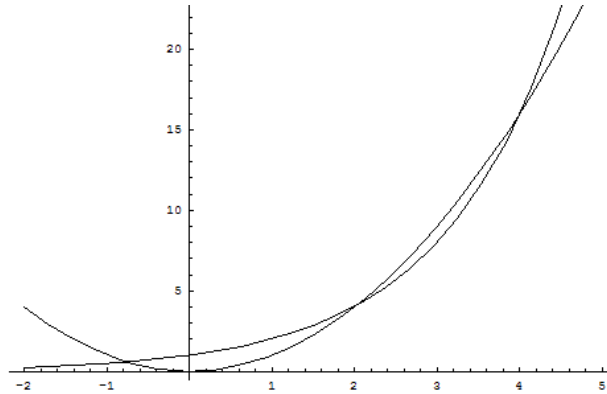
$$\begin{aligned} P_n(x, c) &= f(c) + \sum_{r=1}^n \frac{(x-c)^r}{r!} \frac{1}{10} \left(\frac{1}{10} - 1\right) \left(\frac{1}{10} - 2\right) \dots \left(\frac{1}{10} - (r-1)\right) c^{\frac{1}{10}-r} \\ P_3(1025, 1024) &= f(1024) + \sum_{r=1}^3 \frac{(1025-1024)^r}{r!} \frac{1}{10} \left(\frac{1}{10} - 1\right) \left(\frac{1}{10} - 2\right) \dots \left(\frac{1}{10} - (r-1)\right) 1024^{\frac{1}{10}-r} \\ &= (2^{10})^{\frac{1}{10}} + \frac{(1)}{1} \frac{1}{10} (2^{10})^{-\frac{9}{10}} + \frac{(1)^2}{2} \frac{1}{10} \left(-\frac{9}{10}\right) (2^{10})^{-\frac{19}{10}} + \frac{(1)^2}{6} \frac{1}{10} \left(-\frac{9}{10}\right) \left(-\frac{19}{10}\right) (2^{10})^{-\frac{29}{10}} \\ &= 2 + \frac{1}{10} \frac{1}{2^9} - \frac{9}{200} \frac{1}{2^{19}} + \frac{57}{2000} \frac{1}{2^{29}} \\ &= 2 + 0.0001953125 - 0.000000085830 \dots + 0.00000000053 \dots \end{aligned}$$

$$\begin{aligned} \overline{P}_3(1025, 1024) &= 2.00000000000 + 0.00019531250 - 0.00000008583 + 0.00000000005 \\ &= 2.00019531255 - 0.00000008583 \\ &= 2.00019522672 \end{aligned}$$

So to the 10th decimal place accuracy $1025^{\frac{1}{10}} = 2.0001952267$

Consider the equation $x^2 = 2^x$.

1. Graphically obtain real intervals which contain real roots.
-



We draw the graphs of $y_1(x) = x^2$ and $y_2(x) = 2^x$ on the same plane. Then the solutions are given by the x Coordinates of the points of intersection.

The graph show that there are 3 real roots in $(-1,0), (0,3), (3,5)$

Let $f(x) = 2^x - x^2$

$$\left. \begin{array}{l} f(-1) = 2^{-1} - (-1)^2 = \frac{1}{2} - 1 = -\frac{1}{2} < 0 \\ f(0) = 2^0 - 0^2 = 1 - 0 = 1 > 0 \end{array} \right\} \Rightarrow x \in (-1, 0)$$

$$\left. \begin{array}{l} f(1) = 2^1 - 1^2 = 2 - 1 = 1 > 0 \\ f(3) = 2^3 - 3^2 = 8 - 9 = -1 < 0 \end{array} \right\} \Rightarrow x \in (1, 3)$$

$$\left. \begin{array}{l} f(3) = 2^3 - 3^2 = 8 - 9 = -1 < 0 \\ f(5) = 2^5 - 5^2 = 32 - 25 = 7 > 0 \end{array} \right\} \Rightarrow x \in (3, 5)$$

We further see that

Let $f(x) = 2^x - x^2$

$$\left. \begin{array}{l} f(-0.8) = 2^{-0.8} - (-0.8)^2 \approx -0.0656508225 \\ f(-0.7) = 2^{-0.7} - (-0.7)^2 \approx 0.1255722067 \end{array} \right\} \Rightarrow x \in (-0.8, -0.7)$$

$$\left. \begin{array}{l} f(1.9) = 2^{1.9} - 1.9^2 \approx 0.1221319661 > 0 \\ f(2.1) = 2^{2.1} - 2.1^2 \approx -0.1229061499 < 0 \end{array} \right\} \Rightarrow x \in (1.9, 2.1)$$

$$\left. \begin{array}{l} f(3.9) = 2^{3.9} - 3.9^2 \approx -0.2814721354 > 0 \\ f(4.1) = 2^{4.1} - 4.1^2 \approx 0.3383754006 < 0 \end{array} \right\} \Rightarrow x \in (3.9, 4.1)$$

2. Find the number of iterations required to find the answers accurate to the 2nd decimal place by the Bisection method and find the roots to this accuracy.

We know that $|b_n - a_n| = \left(\frac{1}{2}\right)^n |b_0 - a_0|$

Also

$$a_n < \alpha < b_n$$

$$\Rightarrow a_n - x_n < \alpha < b_n - x_n$$

$$\Leftrightarrow a_n - \frac{a_n + b_n}{2} < \alpha - x_n < b_n - \frac{a_n + b_n}{2}$$

$$\Leftrightarrow -\left(\frac{b_n - a_n}{2}\right) < \alpha - x_n < \left(\frac{b_n - a_n}{2}\right)$$

$$\Leftrightarrow |\alpha - x_n| < \frac{1}{2}|a_n - b_n|$$

$$b_n < \alpha < a_n$$

$$\Rightarrow b_n - x_n < \alpha < a_n - x_n$$

$$\Leftrightarrow b_n - \frac{a_n + b_n}{2} < \alpha - x_n < a_n - \frac{a_n + b_n}{2}$$

$$\Leftrightarrow -\left(\frac{a_n - b_n}{2}\right) < \alpha - x_n < \left(\frac{a_n - b_n}{2}\right)$$

$$\Leftrightarrow |\alpha - x_n| < \frac{1}{2}|b_n - a_n|$$

In any situation

$$|\alpha - x_n| < \frac{1}{2}|b_n - a_n| = \left(\frac{1}{2}\right)^{n+1} |b_0 - a_0|$$

For a m decimal place accuracy

$$|\alpha - x_n| < \frac{1}{2}10^{-m} \Leftrightarrow |\alpha - x_n| < \frac{1}{2}|b_n - a_n| = \left(\frac{1}{2}\right)^{n+1} |b_0 - a_0| < \frac{1}{2}10^{-m} \Leftrightarrow \left(\frac{1}{2}\right)^{n+1} |b_0 - a_0| < \frac{1}{2}10^{-m}$$

$$\Leftrightarrow 2^n > |b_0 - a_0|10^m \Leftrightarrow n > \frac{\ln|b_0 - a_0| + m \ln 10}{\ln 2}$$

In our problem $m = 2$

For the root in $(-0.8, -0.7)$

$$n > \frac{\ln|b_0 - a_0| + m \ln 10}{\ln 2} = \frac{\ln|(-0.7) - (-0.8)| + 2 \ln 10}{\ln 2} = \frac{\ln 10}{\ln 2} = 3.321928095$$

$$\Leftrightarrow n = 4$$

We use CASIO fx-991ES

n	a_n	b_n	$x_n = \frac{a_n + b_n}{2}$	$f(x_n)$
	A	B	X	Formula =AB-AB+2^X-X^2
0	-0.8	-0.7	A/2+B/2	0.0321035575 > 0
1	=	X =-0.75	A/2+B/2	-0.01623637572 < 0
2	X =-0.775	=	A/2+B/2	0.00806771461 > 0
3	=	X =-0.7625	A/2+B/2	-0.00405077568 < 0
4	X =-0.77875	=	A/2+B/2	0.0020168552 > 0
5			-0.765625	

So the root accurate to 2 decimal places is $\alpha = -0.77$. Other roots are found in a similar manner

3. Write the equation in two different ways to be solved by the Simple-Iterative Method.

$$x^2 = 2^x \Leftrightarrow x = 2^{\frac{x}{2}} = g_1(x)$$

$$x^2 = 2^x \Leftrightarrow 2 \ln x = x \ln 2 \Leftrightarrow x = \frac{2 \ln x}{\ln 2} = g_2(x)$$

4. Comment on the possibility of finding real roots by these two ways.

$g_1(x) = 2^{\frac{x}{2}} = x > 0$. Therefore it is not possible to find the root $x \in (-0.8, -0.7)$ by this method

$$g_1'(x) = \frac{d}{dx} \left(2^{\frac{x}{2}} \right) = \frac{d}{dx} \left(\exp \left(\ln 2^{\frac{x}{2}} \right) \right) = \frac{d}{dx} \left(\exp \left(\frac{x}{2} \ln 2 \right) \right) = \exp \left(\frac{x}{2} \ln 2 \right) \frac{\ln 2}{2} = \frac{\ln 2}{2} 2^{\frac{x}{2}}$$

$$|g_1'(x)| < 1$$

$$\Leftrightarrow \left| \frac{\ln 2}{2} 2^{\frac{x}{2}} \right| < 1 \Leftrightarrow \frac{\ln 2}{2} 2^{\frac{x}{2}} < 1 \Leftrightarrow 2^{\frac{x}{2}} < \frac{2}{\ln 2} \Leftrightarrow \frac{x}{2} \ln 2 < \ln \left(\frac{2}{\ln 2} \right) \Leftrightarrow x < \frac{2}{\ln 2} \ln \left(\frac{2}{\ln 2} \right) \approx 3.057532746 \Rightarrow x < 3.9$$

Therefore the root in (3.9,4.1) cannot be found by this method

$g_2(x) = \frac{2 \ln x}{\ln 2} \Rightarrow x > 0$. Therefore it is not possible to find the root $x \in (-0.8, -0.7)$ by this method

$$g_2''(x) = \frac{2}{x \ln 2}$$

$$|g_2'(x)| < 1 \Leftrightarrow \left| \frac{2}{x \ln 2} \right| < 1 \Leftrightarrow \frac{2}{x \ln 2} < 1 \Leftrightarrow x > \frac{2}{\ln 2} \approx 2.885390082 \Rightarrow 2.1 < x$$

Therefore the root in (1.9,2.1) cannot be found by this method

1. Use the Secant method to find the real roots.

Secant(Chord) Method by CASIO fx-991ES

n	x_{n-1}	x_n	$x_{n+1} = \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$
	Y	X	Formula $= \frac{(Y(2^X - X^2) - X(2^Y - Y^2))}{(2^X - X^2) - (2^Y - Y^2)}$
0	-0.8	-0.7	-0.7656679309
1	X =-0.7	Ans =-0.7656679309	-0.7666949268
2	X =-0.7656679309	Ans =-0.7666949268	-0.7666646826

Other roots can be found in a similar manner

Error calculations with Secant(Chord) method and the Regular-Falsi(False Position) methods will not be discussed.

2. Find the number of iterations required to find the answers accurate to the 4th decimal place by the Newton-Rapson method and find the roots to this accuracy.
-

$$f(x) = 2^x - x^2 = 0 \Rightarrow f'(x) = 2^x \ln 2 - 2x$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{2^{x_n} - x_n^2}{2^{x_n} \ln 2 - 2x_n} = \frac{2^{x_n}(x_n \ln 2 - 1) - x_n^2}{2^{x_n} \ln 2 - 2x_n} = g(x_n)$$

$$g(x) = x - \frac{2^x - x^2}{2^x \ln 2 - 2x} = \frac{2^x(x \ln 2 - 1) - x^2}{2^x \ln 2 - 2x}$$

Assume that $f'(x) = 2^x \ln 2 - 2x = 0$ at a root of $f(x)$

Then

$$f'(x) = 2^x \ln 2 - 2x = x(x \ln 2 - 2) = 0 \Leftrightarrow x = 0, x = \frac{2}{\ln 2} \approx 2.885390082 \Rightarrow x \text{ is not a root of } f(x)$$

Hence $f'(x) \neq 0$ at a roots of $f(x)$

Therefore the Newton-Raphson method can be used to find all the roots.

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{2^x - x^2}{2^x \ln 2 - 2x} = \frac{2^x(x \ln 2 - 1) - x^2}{2^x \ln 2 - 2x}$$

$$h(x) = g'(x) = 1 - \frac{f'(x)}{f'(x)} - f(x) \cdot \frac{-f''(x)}{(f'(x))^2} = \frac{f(x)f''(x)}{(f'(x))^2} = \frac{(2^x - x^2)(2^x(\ln 2)^2 - 2)}{(2^x \ln 2 - 2x)^2}$$

$$f'(x) = 2^x \ln 2 - 2x$$

$$f''(x) = 2^x(\ln 2)^2 - 2 = 0 \Leftrightarrow 2^x = \frac{2}{(\ln 2)^2} \Leftrightarrow x = \frac{\ln 2 - 2 \ln(\ln 2)}{\ln 2} = \beta \approx 2.057532746$$

Also $x \rightarrow \pm\infty$, $f'(x) \rightarrow \infty$, therefore $x = \beta$ is a minimum of $f'(x)$

$$\text{Also } f'(\beta) \approx -1.22967541 < 0$$

and

$$f'(-0.7) = 1.826682139$$

$$f'(1.9) = -1.21308325$$

$$f'(2.1) = -1.228412985$$

$$f'(3.9) = 2.547666999$$

Therefore $f'(x) = 0$ only once in each of $(-0.7, 1.9)$ and $(1.9, 3.9)$.

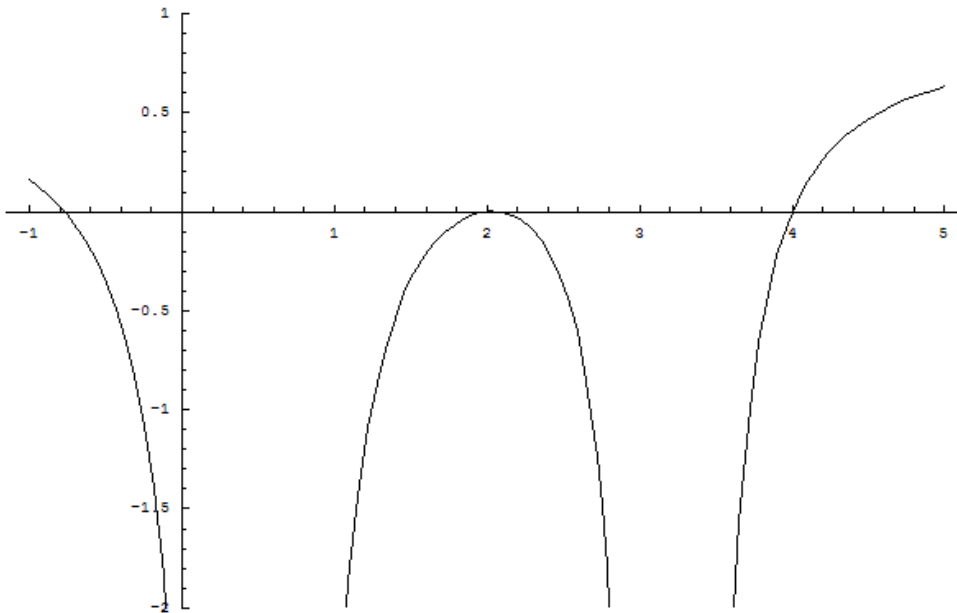
Also $f(x) \neq 0$ and $f''(x) \neq 0$ on these intervals.

Therefore $g'(x) \rightarrow \pm\infty$ once in each of $(-0.7, 1.9)$ and $(1.9, 3.9)$.

Considering these and other conditions one can draw a rough sketch of $g'(x)$ as follows

```
In[8]:= h[x_] := (2^x - x^2) (2^x (Log[2])^2 - 2) / (2^x Log[2] - 2 x)^2
```

```
In[10]:= Plot[h[x], {x, -1, 5}, PlotRange -> {1, -2}]
```



```
Out[10]= - Graphics -
```

From the graph we see

$g'(x)$ is decreasing on $(-0.8, -0.7)$

$$\Leftrightarrow g'(-0.7) < g'(x) < g'(-0.8)$$

$$\Leftrightarrow -0.06413571879 < g'(x) < 0.02834995957$$

$$\Rightarrow 0 < |g'(x)| < |g'(-0.7)| \approx 0.06413571879$$

On $(-0.8, -0.7)$; $|g'(x)| < |g'(-0.7)| \approx 0.06413571879 < 0.065 = r$

$g'(x)$ is increasing on $(1.9, 2)$

$$\Leftrightarrow g'(1.9) < g'(x) < g'(2)$$

$$\Leftrightarrow -0.01717034634 < g'(x) < 0$$

$$\Rightarrow 0 < |g'(x)| < |g'(1.9)| \approx 0.01717034634$$

$g'(x)$ is decreasing on $(2, 2.1)$

$$\Leftrightarrow g'(2.1) < g'(x) < g'(2)$$

$$\Leftrightarrow -0.00486633601 < g'(x) < 0$$

$$\Rightarrow 0 < |g'(x)| < |g'(2.1)| \approx 0.00486633601$$

On $(1.9, 2.1)$; $|g'(x)| < |g'(1.9)| \approx 0.01717034634 < 0.018 = r$

$g'(x)$ is increasing on (3.9,4.1)

$$\Leftrightarrow g'(3.9) < g'(x) < g'(4.1)$$

$$\Leftrightarrow -0.2243091074 < g'(x) < 0.1553532351$$

$$\Rightarrow 0 < |g'(x)| < |g'(3.9)| \approx 0.2243091074$$

On (3.9,4.1); $|g'(x)| < |g'(3.9)| \approx 0.2243091074 < 0.225 = r$

$$\begin{aligned} |\alpha - x_n| < \frac{1}{2}10^{-m} &\Leftrightarrow |\alpha - x_n| < r^n |\alpha - x_0| < \frac{1}{2}10^{-m} \Leftrightarrow r^n |\alpha - x_0| < r^n |b - a| < \frac{1}{2}10^{-m} \Leftrightarrow r^n |b - a| < \frac{1}{2}10^{-m} \\ &\Leftrightarrow r^n < \frac{10^{-m}}{2|b - a|} \Leftrightarrow \left(\frac{1}{r}\right)^n > 2|b - a|10^m \Leftrightarrow n \ln \frac{1}{r} > \ln 2|b - a| + m \ln 10 \Leftrightarrow n > \frac{\ln 2|b - a| + m \ln 10}{\ln \frac{1}{r}} \end{aligned}$$

On (-0.8,-0.7); $|g'(x)| < 0.065 = r$

$$n > \frac{\ln 2|(-0.7) - (-0.8)| + 4 \ln 10}{\ln \frac{1}{0.065}} = \frac{\ln 2 - \ln 10 + 4 \ln 10}{3 \ln 10 - \ln 65} = \frac{\ln 2 + 3 \ln 10}{2 \ln 10 - \ln 65} \approx 6.455183322 \Leftrightarrow n = 7$$

Lets do the iterations starting from $x_0 = -0.75$

In[13]: `f[x_] := 2x - x2`

In[14]: `x = -75 / 100;`

`For[i = 1, i ≤ 7, i++, {x = x - $\frac{f[x]}{f'[x]}$, Print[N[x, 10]]}]`

-0.7667892659

-0.7666647028

-0.7666646960

-0.7666646960

-0.7666646960

-0.7666646960

-0.7666646960

So the root accurate to 4 decimal places is $\alpha = -0.7667$

On (1.9,2.1); $|g'(x)| < 0.018 = r$

$$n > \frac{\ln 2|(2.1) - (1.9)| + 4 \ln 10}{\ln \frac{1}{0.018}} \approx 2.064540166 \Leftrightarrow n = 3$$

Lets do the iterations starting from $x_0 = 1.95$

2 and 4 are the only positive integers satisfying $x^y = y^x$

The only negative integers are -2 and -4.

3. Use a suitable method to find at least one complex root.

We use the Newton Raphson method with $x_0 = 10 + 10i$

In[9]:= **x = 10 + 10 I ;**

For [i = 1, i ≤ 20, i ++, {x = N[x - $\frac{f[x]}{f'[x]}$, 1000], Print[N[x, 20]]}]

8.6738654364325811263 + 10.2263204857772237504 i

7.4965533927967540307 + 10.7808280530536215699 i

7.0358613025925046864 + 11.9716071723298178340 i

7.824368383801721431 + 11.894737587472193836 i

7.662795530825642645 + 11.945101462068614175 i

7.654495508970462616 + 11.953625240066227678 i

7.654506496006766085 + 11.953678912892782106 i

7.654506496977927274 + 11.953678912263155851 i

7.654506496977927274 + 11.953678912263155851 i

7.654506496977927274 + 11.953678912263155851 i

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7.654506496977927274 + 11.953678912263155851 i

7.654506496977927274 + 11.953678912263155851 i

So $\alpha \approx 7.654506496977927274 + 11.953678912263155851i$

Trying with a built in functions of MATHEMATICA

In[14]:= **Clear [x] ;**

In[15]:= FindRoot[f[x] == 0, {x, 10 + 10 I}, AccuracyGoal → 20, WorkingPrecision → 34, MaxIterations → 1000]

Out[15]= {x → 7.654506496977927273823937027283793 + 11.95367891226315585118232997224737 i}

Note: Rounding error at each iteration is not considered in any of the discussions. However the author has

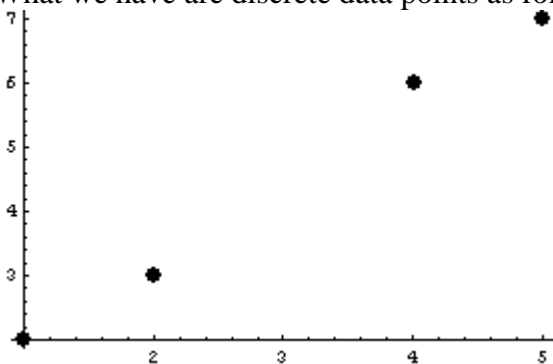
shown that $\left| \alpha - \bar{x}_n \right| < r^n \left| \alpha - x_0 \right| + \frac{1 - r^{n+1}}{1 - r} e$

Where $\bar{x}_n = g(\bar{x}_{n-1})$ and $\left| \bar{x}_n - x_n \right| < e$

Consider the following table

x	y
1	2
2	3
4	6
5	7

What we have are discrete data points as follows



1. What is the Least Degree of the Polynomial that passes through all these points?.

Here we have 4 points $(1,2), (2,3), (4,6), (5,7)$. In other words we have 4 information. As these data are all different, we can find 4 unknown coefficients uniquely by using these information. If $f(x), g(x), h(x), k(x)$ are 4 different known functions we can combine them linearly using 4 unknowns a, b, c, d to produce $y(x)$ as $y(x) = af(x) + bg(x) + ch(x) + dk(x)$.

The simplest of functions are $f(x) = x^3, g(x) = x^2, h(x) = x, k(x) = 1$ which gives

$y(x) = ax^3 + bx^2 + cx + d$ which is a cubic(third degree) polynomial. It has 4 coefficients and is the least degree having 4 coefficients. These polynomials which exactly pass through all the points are called Lagrange Polynomials.

Answer: 3

2. Find this polynomial by Matrix inversion.

To find coefficients we let x_i, y_i agree with those given in the table and get 4 simultaneous equations of a, b, c, d .

$$y(1) = a + b + c + d = 2$$

$$y(2) = 8a + 4b + 2c + d = 3$$

$$y(4) = 64a + 16b + 4c + d = 6$$

$$y(5) = 125a + 25b + 5c + d = 7$$

Which give the following matrix equations

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 64 & 16 & 4 & 1 \\ 125 & 25 & 5 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 6 \\ 7 \end{pmatrix}$$

$$XA = Y$$

Clearly the matrix X is non singular. X^{-1} can be found in lot of ways. The most efficient method is the Gauss Elimination. So the final result is

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = A = X^{-1}Y = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 64 & 16 & 4 & 1 \\ 125 & 25 & 5 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 3 \\ 6 \\ 7 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} -1 & 2 & -2 & 1 \\ 11 & -20 & 16 & -7 \\ -38 & 58 & -34 & 14 \\ 40 & -40 & 20 & -8 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 6 \\ 7 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} -1 \\ 9 \\ -8 \\ 24 \end{pmatrix}$$

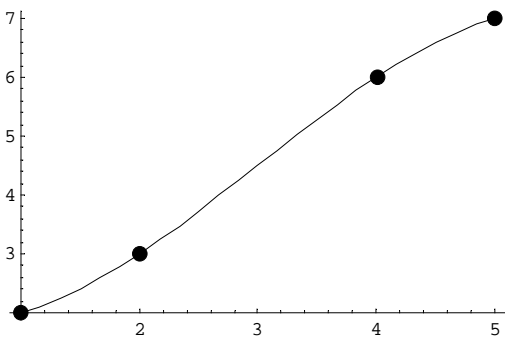
$$a = -\frac{1}{12}, b = \frac{3}{4}, c = -\frac{2}{3}, d = 2.$$

Therefore the polynomial is

$$y(x) = -\frac{1}{12}x^3 + \frac{3}{4}x^2 - \frac{2}{3}x + 2$$

It is guaranteed that

$y(1) = 2, y(2) = 3, y(4) = 6, y(5) = 7$ so that the curve will exactly goes through these 4 points



3. Find the polynomial by Lagrange Method.

It is very difficult to find the above polynomial by matrix inversion. So we assume the polynomial in the Lagrange form as follows

$$y(x) = a_0(x-x_1)(x-x_2)(x-x_3) + a_1(x-x_0)(x-x_2)(x-x_3) \\ + a_2(x-x_0)(x-x_1)(x-x_3) + a_3(x-x_0)(x-x_1)(x-x_2)$$

Clearly this is a 3ed degree polynomial and has 4 coefficients a_0, a_1, a_2, a_3 . The x terms with a_0 is not having $(x-x_0)$ and the x terms with a_1 is not having $(x-x_1)$ and so on. As a result for example if we put $x = x_0$ all the other terms except for the first term with a_0 will cancel out when we put $x = x_0$ giving us a very easy method of finding a_0

$$y(x_0) = a_0(x_0-x_1)(x_0-x_2)(x_0-x_3) = y_0 \Rightarrow a_0 = \frac{y_0}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}$$

This method is called the Lagrange Method of finding coefficients.

For the problem given

x	y
$x_0 = 1$	$y_0 = 2$
$x_1 = 2$	$y_1 = 3$
$x_2 = 4$	$y_2 = 6$
$x_3 = 5$	$y_3 = 7$

Therefore

$$a_0 = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} = \frac{2}{(1-2)(1-4)(1-5)} = \frac{2}{(-1)(-3)(-4)} = -\frac{1}{6}$$

$$a_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} = \frac{3}{(2-1)(2-4)(2-5)} = \frac{3}{(1)(-2)(-3)} = \frac{1}{2}$$

$$a_2 = \frac{y_2}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} = \frac{6}{(4-1)(4-2)(4-5)} = \frac{6}{(3)(2)(-1)} = -1$$

$$a_3 = \frac{y_3}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} = \frac{7}{(5-1)(5-2)(5-4)} = \frac{7}{(4)(3)(1)} = \frac{7}{12}$$

So the polynomial is

$$y(x) = -\frac{1}{6}(x-2)(x-4)(x-5) + \frac{1}{2}(x-1)(x-4)(x-5) - (x-1)(x-2)(x-5) + \frac{7}{12}(x-1)(x-2)(x-4)$$

Although the process of finding coefficients is much easier than matrix inversion, the polynomial is complex looking. However by expanding

$y(x)$

$$\begin{aligned} &= \left[-\frac{1}{6} + \frac{1}{2} - 1 + \frac{7}{12} \right] x^3 \\ &+ \left[-\frac{1}{6}(-2-4-5) + \frac{1}{2}(-1-4-5) - (-1-2-5) + \frac{7}{12}(-1-2-4) \right] x^2 \\ &+ \left[-\frac{1}{6}((-2)(-4) + (-4)(-5) + (-5)(-2)) + \frac{1}{2}((-1)(-4) + (-4)(-5) + (-5)(-1)) \right. \\ &\quad \left. - ((-1)(-2) + ((-2)(-5) + (-5)(-1))) + \frac{7}{12}((-1)(-2) + (-2)(-4) + (-4)(-1)) \right] x \\ &+ \left[-\frac{1}{6}(-2)(-4)(-5) + \frac{1}{2}(-1)(-4)(-5) - (-1)(-2)(-5) + \frac{7}{12}(-1)(-2)(-4) \right] \\ &= \left(-\frac{1}{6} + \frac{1}{2} - 1 + \frac{7}{12} \right) x^3 + \left(\frac{11}{6} - 5 + 8 - \frac{49}{12} \right) + \left(-\frac{19}{3} + \frac{29}{2} - 17 + \frac{49}{6} \right) x + \left(\frac{20}{3} - 10 + 10 - \frac{14}{3} \right) \\ &= \left(\frac{-2+6-12+7}{12} \right) x^3 + \left(\frac{22+36-49}{12} \right) x^2 + \left(\frac{-38+87-102+49}{6} \right) x + \frac{6}{3} \\ &= -\frac{1}{12} x^3 + \frac{9}{12} x^2 - \frac{4}{6} x + 2 \\ &= -\frac{1}{12} x^3 + \frac{3}{4} x^2 - \frac{2}{3} x + 2 \end{aligned}$$

It is also clear that finding the polynomial by Lagrange manner is not so good, especially when the number of points is high. Although the curve will exactly pass through all the points, the curve will try to do it "somehow" making the intermediate values "unreliable".

The solution is break the graph into several pieces and to have a Piecewise Functions.

4. Fit the data by piecewise polynomials of degree 1.

Here the data are grouped as follows

x	y
$x_0 = 1$	$y_0 = 2$
$x_1 = 2$	$y_1 = 3$

We can find the polynomial by the Lagrange method

$$y(x) = \frac{y_0}{(x_0 - x_1)}(x - x_1) + \frac{y_1}{(x_1 - x_0)}(x - x_0) = \frac{2}{(1-2)}(x-2) + \frac{3}{(2-1)}(x-1) = -2(x-2) + 3(x-1) = x+1$$

x	y
$x_0 = 2$	$y_0 = 3$
$x_1 = 4$	$y_1 = 6$

We can find the polynomial by the Lagrange method

$$y(x) = \frac{y_0}{(x_0 - x_1)}(x - x_1) + \frac{y_1}{(x_1 - x_0)}(x - x_0) = \frac{3}{(2-4)}(x-4) + \frac{6}{(4-2)}(x-2) = -\frac{3}{2}(x-4) + 3(x-2) = \frac{3}{2}x$$

x	y
$x_0 = 4$	$y_0 = 6$
$x_1 = 5$	$y_1 = 7$

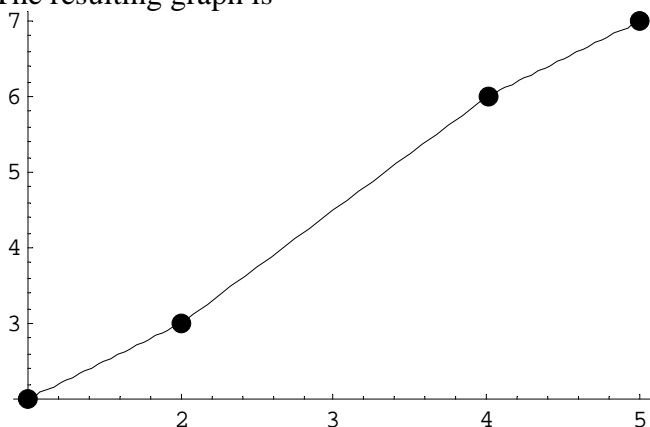
We can find the polynomial by the Lagrange method

$$y(x) = \frac{y_0}{(x_0 - x_1)}(x - x_1) + \frac{y_1}{(x_1 - x_0)}(x - x_0) = \frac{6}{(4-5)}(x-5) + \frac{7}{(5-4)}(x-4) = -6(x-5) + 7(x-4) = x+2$$

Therefore the functions is

$$y(x) = \begin{cases} x+1 & ; 1 \leq x \leq 2 \\ \frac{3}{2}x & ; 2 \leq x \leq 4 \\ x+2 & ; 4 \leq x \leq 5 \end{cases}$$

The resulting graph is



Note that the function is not differentiable at $x = 2$ and $x = 4$. In fact $y'(x) = \begin{cases} 1 & ; 1 \leq x \leq 2 \\ \frac{3}{2} & ; 2 \leq x \leq 4 \\ 1 & ; 4 \leq x \leq 5 \end{cases}$

This data can also be fitted piecewise by one quadratic polynomial and one linear one.

Here the data are grouped as follows

x	y
$x_0 = 1$	$y_0 = 2$
$x_1 = 2$	$y_1 = 3$
$x_2 = 4$	$y_2 = 6$

We can find the polynomial by the Lagrange method

$$\begin{aligned}
 y(x) &= \frac{y_0}{(x_0 - x_1)(x_0 - x_2)}(x - x_1)(x - x_2) + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)}(x - x_0)(x - x_2) + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}(x - x_0)(x - x_1) \\
 &= \frac{2}{(1-2)(1-4)}(x-2)(x-4) + \frac{3}{(2-1)(2-4)}(x-1)(x-4) + \frac{6}{(4-1)(4-2)}(x-1)(x-2) \\
 &= \frac{2}{(-1)(-3)}(x^2 - 6x + 8) + \frac{3}{(1)(-2)}(x^2 - 5x + 4) + \frac{6}{(3)(2)}(x^2 - 3x + 2) \\
 &= \frac{2}{3}(x^2 - 6x + 8) - \frac{3}{2}(x^2 - 5x + 4) + (x^2 - 3x + 2) \\
 &= \frac{1}{6}((4-9+6)x^2 + (-24+45-18)x + (32-36+12)) = \frac{1}{6}(x^2 + 3x + 8) = \frac{1}{6}x^2 + \frac{1}{2}x + \frac{4}{3}
 \end{aligned}$$

Last section is identical to the above linear piecewise polynomial and hence the polynomial with

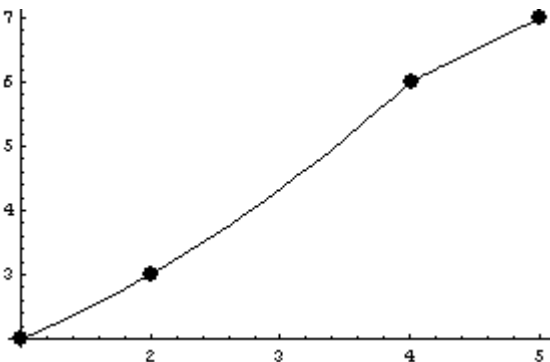
x	y
$x_0 = 4$	$y_0 = 6$
$x_1 = 5$	$y_1 = 7$

$$y(x) = x + 2$$

Therefore the functions is

$$y(x) = \begin{cases} \frac{1}{6}x^2 + \frac{1}{2}x + \frac{4}{3} & ; 1 \leq x \leq 4 \\ x + 2 & ; 4 \leq x \leq 5 \end{cases}$$

And the graph is



Note also that $y'(x) = \begin{cases} \frac{1}{3}x + \frac{1}{2} & ; 1 \leq x \leq 4 \\ 1 & ; 4 \leq x \leq 5 \end{cases}$ and $y'(4) = \begin{cases} \frac{1}{3}(4) + \frac{1}{2} = \frac{11}{6} \\ 1 \end{cases}$ so the function is not

differentiable at $x = 4$.

If in addition if $y'(x)$ values are also given we can increase the degree of the polynomial to meet these requirements. The resulting polynomial is called the Hermit Polynomial.

For example assume that the following table is given

x	y	y'
1	2	0
2	3	1
4	6	2
5	7	0

Then we have 8 data and we can assume a 7th degree polynomial

$$y(x) = a_7x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

And

$$y'(x) = 7a_7x^6 + 6a_6x^5 + 5a_5x^4 + 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$$

So the conditions are

$$y(1) = a_7 + a_6 + a_5 + a_4 + a_3 + a_2 + a_1 + a_0 = 1$$

$$y'(1) = 7a_7 + 6a_6 + 5a_5 + 4a_4 + 3a_3 + 2a_2 + a_1 = 0$$

$$y(2) = 128a_7 + 64a_6 + 32a_5 + 16a_4 + 8a_3 + 4a_2 + 2a_1 + a_0 = 3$$

$$y'(2) = 448a_7 + 192a_6 + 80a_5 + 32a_4 + 12a_3 + 2a_2 + a_1 = 1$$

$$y(4) = 16384a_7 + 4096a_6 + 1024a_5 + 256a_4 + 64a_3 + 16a_2 + 4a_1 + a_0 = 6$$

$$y'(4) = 28672a_7 + 6144a_6 + 1280a_5 + 256a_4 + 48a_3 + 8a_2 + a_1 = 2$$

$$y(5) = 78125a_7 + 15625a_6 + 3125a_5 + 625a_4 + 125a_3 + 25a_2 + 5a_1 + a_0 = 7$$

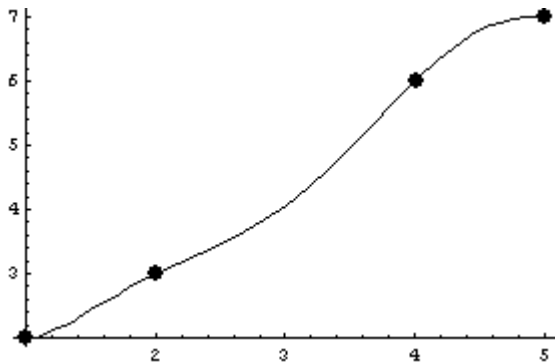
$$y'(5) = 109375a_7 + 18750a_6 + 3125a_5 + 500a_4 + 75a_3 + 10a_2 + a_1 = 0$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ 128 & 64 & 32 & 16 & 8 & 4 & 2 & 1 \\ 448 & 192 & 80 & 32 & 12 & 4 & 1 & 0 \\ 16384 & 4096 & 1024 & 256 & 64 & 16 & 4 & 1 \\ 28672 & 6144 & 1280 & 256 & 48 & 8 & 1 & 0 \\ 78125 & 15625 & 3125 & 625 & 125 & 25 & 5 & 1 \\ 109375 & 18750 & 3125 & 500 & 75 & 10 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_7 \\ a_6 \\ a_5 \\ a_4 \\ a_3 \\ a_2 \\ a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \\ 1 \\ 6 \\ 2 \\ 7 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a_7 \\ a_6 \\ a_5 \\ a_4 \\ a_3 \\ a_2 \\ a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} \frac{19}{864} & \frac{1}{144} & -\frac{1}{108} & \frac{1}{36} & \frac{1}{108} & \frac{1}{36} & -\frac{19}{864} & \frac{1}{144} \\ -\frac{431}{864} & -\frac{23}{144} & \frac{25}{108} & -\frac{11}{18} & -\frac{17}{108} & -\frac{5}{9} & \frac{367}{864} & -\frac{19}{144} \\ \frac{1343}{288} & \frac{73}{48} & -\frac{43}{18} & \frac{11}{2} & \frac{19}{18} & \frac{9}{2} & -\frac{959}{288} & \frac{49}{48} \\ -\frac{6655}{288} & -\frac{371}{48} & \frac{235}{18} & -26 & -\frac{65}{18} & -19 & \frac{3935}{288} & -\frac{199}{48} \\ \frac{584}{9} & \frac{45}{2} & -\frac{1447}{36} & \frac{827}{12} & \frac{247}{36} & \frac{539}{12} & -\frac{284}{9} & \frac{19}{2} \\ -\frac{7331}{72} & -\frac{149}{4} & \frac{2455}{36} & -\frac{607}{6} & -\frac{263}{36} & -\frac{178}{3} & \frac{2947}{72} & -\frac{49}{4} \\ \frac{2185}{27} & \frac{290}{9} & -\frac{1550}{27} & \frac{680}{9} & \frac{110}{27} & \frac{365}{9} & -\frac{745}{27} & \frac{74}{9} \\ -\frac{650}{27} & -\frac{100}{9} & \frac{500}{27} & -\frac{200}{9} & -\frac{25}{27} & -\frac{100}{9} & \frac{202}{27} & -\frac{20}{9} \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 3 \\ 1 \\ 6 \\ 2 \\ 7 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{864}{1} \\ \frac{1}{288} \\ -\frac{91}{288} \\ \frac{281}{96} \\ -\frac{106}{9} \\ \frac{571}{24} \\ -\frac{605}{27} \\ \frac{88}{9} \end{pmatrix}$$

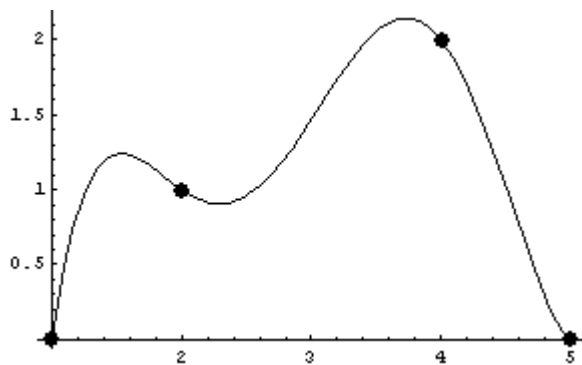
So the $y(x)$ and its graph are

$$Y(x) = \frac{88}{9} - \frac{605x}{27} + \frac{571x^2}{24} - \frac{106x^3}{9} + \frac{281x^4}{96} - \frac{91x^5}{288} + \frac{x^6}{288} + \frac{x^7}{864}$$



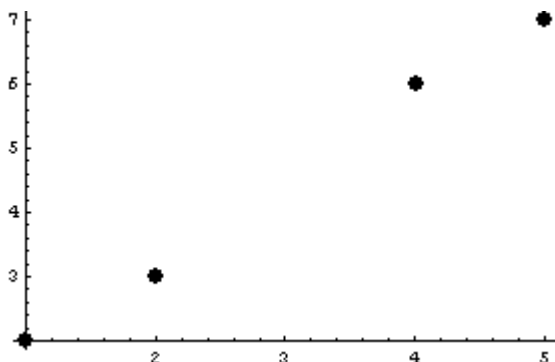
And $y'(x)$ and its graph are

$$Y'(x) = -\frac{605}{27} + \frac{571x}{12} - \frac{106x^2}{3} + \frac{281x^3}{24} - \frac{455x^4}{288} + \frac{x^5}{48} + \frac{7x^6}{864}$$



5. Fit the data by spline polynomials of degree 2.

When $y'(x)$ is not given, we can assume a higher degree polynomial and use the condition for continuity as an additional data to make $y(x)$ also continuous at $x = 2, 4$.



If we assume second degree ie $y(x) = ax^2 + bx + c$. Since there are 4 points and 3 intervals we have to assume 3 such equations making the variables to be found equals to $3 \times 3 = 9$.

By the given data points we can make $2 \times (4 - 2) + 2 = 6$ equations. Differentiability at the two internal points give us 2 equations. Altogether we have 8 equations. So we need only one more equation. Some possibilities to make the last equation is $y'(1) = 0$, $y'(5) = 0$, $y'(1) = y'(5)$, $y''(2)$ exists, $y''(4)$ exists

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Note also that we can't make $y(x)$ double differentiable at both $x = 2$ and $x = 4$. So we will make $y(x)$ Double differentiable only at $x = 2$.

x	y
1	2
2	3
4	6
5	7

$$y_1(x) = a_2x^2 + a_1x + a_0; y_1'(x) = 2a_2x + a_1; y_1''(x) = 2a_2$$

$$y_2(x) = b_2x^2 + b_1x + b_0; y_2'(x) = 2b_2x + b_1; y_2''(x) = 2b_2$$

$$y_3(x) = c_2x^2 + c_1x + c_0; y_3'(x) = 2c_2x + c_1$$

$$y_1(1) = a_2 + a_1 + a_0 = 2$$

$$y_1(2) = 4a_2 + 2a_1 + a_0 = 3$$

$$y_1'(2) = 4a_2 + a_1 = y_2'(2) = 4b_2 + b_1$$

$$y_1''(2) = 2a_2 = y_2''(2) = 2b_2$$

$$y_2(2) = 4b_2 + 2b_1 + b_0 = 3$$

$$y_2(4) = 16b_2 + 4b_1 + b_0 = 6$$

$$y_2'(4) = 8b_2 + b_1 = y_3'(4) = 8c_2 + c_1$$

$$y_3(4) = 16c_2 + 4c_1 + c_0 = 6$$

$$y_3(5) = 25c_2 + 5c_1 + c_0 = 7$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & -4 & -1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 1 & 0 & -8 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 16 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 25 & 5 & 1 \end{pmatrix} \begin{pmatrix} a_2 \\ a_1 \\ a_0 \\ b_2 \\ b_1 \\ b_0 \\ c_2 \\ c_1 \\ c_0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \\ 3 \\ 6 \\ 0 \\ 6 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} a_2 \\ a_1 \\ a_0 \\ b_2 \\ b_1 \\ b_0 \\ c_2 \\ c_1 \\ c_0 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 \\ -2 & 2 & -1 & -1 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ \frac{8}{3} & -\frac{5}{3} & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 \\ -2 & 2 & -2 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ \frac{8}{3} & -\frac{8}{3} & \frac{8}{3} & -\frac{4}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \\ -\frac{2}{3} & \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{5}{6} & -\frac{5}{6} & 1 & -1 & 1 \\ 6 & -6 & 6 & -3 & -\frac{15}{2} & \frac{15}{2} & -9 & 8 & -8 \\ -\frac{40}{3} & \frac{40}{3} & -\frac{40}{3} & \frac{20}{3} & \frac{50}{3} & -\frac{50}{3} & 20 & -15 & 16 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \\ 3 \\ 6 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} \\ \frac{1}{2} \\ \frac{4}{3} \\ \frac{1}{6} \\ \frac{1}{2} \\ \frac{4}{3} \\ -\frac{5}{6} \\ \frac{17}{2} \\ -\frac{44}{3} \end{pmatrix}$$

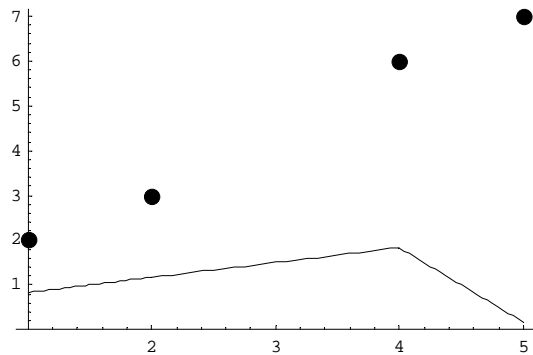
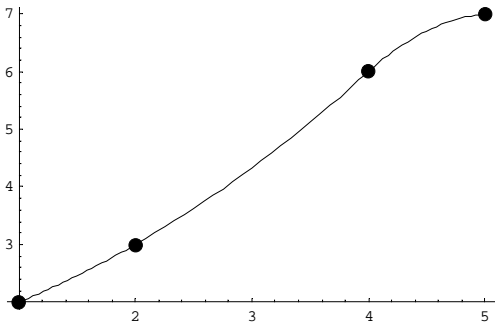
$$y_1(x) = \frac{1}{6}x^2 + \frac{1}{2}x + \frac{4}{3}$$

$$y_2(x) = \frac{1}{6}x^2 + \frac{1}{2}x + \frac{4}{3}$$

$$y_3(x) = -\frac{5}{6}x^2 + \frac{17}{2}x - \frac{44}{3}$$

$$y(x) = \begin{cases} \frac{1}{6}x^2 + \frac{1}{2}x + \frac{4}{3} & ; 1 \leq x \leq 4 \\ -\frac{5}{6}x^2 + \frac{17}{2}x - \frac{44}{3} & ; 4 \leq x \leq 5 \end{cases} ; y'(x) = \begin{cases} \frac{1}{3}x + \frac{1}{2} & ; 1 \leq x \leq 4 \\ -\frac{5}{3}x + \frac{17}{2} & ; 4 \leq x \leq 5 \end{cases}$$

Graphs are as follows



Suppose in general that we have n number of points. So there are $n - 2$ internal points. At each internal point we can write 2 equations. So we get $2(n - 2) = 2n - 4$ equations. At the 2 external points we can write 2 more equations making the total number $2n - 4 + 2 = 2n - 2$ of equations due to $y(x)$.

At each internal point we can make the function differentiable hence we get $n - 2$ more equations. Also we can let $y'(x) = 0$ at the two end points. So the total number of equations due to $y'(x)$ is $n - 2 + 2 = n$.

At each internal point we can also make the function twice differentiable hence we get $n - 1$ more equations. So the number of equations due to $y''(x)$ is $n - 2$.

Therefore the total number of equation we can write using $y(x), y'(x), y''(x)$ is

$$2n - 2 + n + n - 2 = 4n - 4 = 4(n - 1) .$$

Note that there are $n - 1$ intervals; therefore we can find 4 unknowns for each interval.

It is clear from this discussion that a cub polynomial can be fitted for each interval with these properties.

To do this for the given data

x	y
1	2
2	3
4	6
5	7

Assume

$$y_1(x) = a_3x^3 + a_2x^2 + a_1x + a_0; y_1'(x) = 3a_3x^2 + 2a_2x + a_1; y_1''(x) = 6a_3x + 2a_2$$

$$y_2(x) = b_3x^3 + b_2x^2 + b_1x + b_0; y_2'(x) = 3b_3x^2 + 2b_2x + b_1; y_2''(x) = 6b_3x + 2b_2$$

$$y_3(x) = c_3x^3 + c_2x^2 + c_1x + c_0; y_3'(x) = 3c_3x^2 + 2c_2x + c_1; y_3''(x) = 6c_3x + 2c_2$$

$$y_1'(1) = 3a_3 + 2a_2 + a_1 = 0$$

$$y_1(1) = a_3 + a_2 + a_1 + a_0 = 2$$

$$y_1(2) = 8a_3 + 4a_2 + 2a_1 + a_0 = 3$$

$$y_1'(2) = 12a_3 + 4a_2 + a_1 = y_2'(2) = 12b_3 + 4b_2 + b_1$$

$$y_1''(2) = 12a_3 + 2a_2 = y_2''(2) = 12b_3 + 2b_2$$

$$y_2(2) = 8b_3 + 4b_2 + 2b_1 + b_0 = 3$$

$$y_2(4) = 64b_3 + 16b_2 + 4b_1 + b_0 = 6$$

$$y_2'(4) = 48b_3 + 8b_2 + b_1 = y_3'(4) = 48c_3 + 8c_2 + c_1$$

$$y_2''(4) = 24b_3 + 2b_2 = y_3''(4) = 24c_3 + 2c_2$$

$$y_3(4) = 64c_3 + 16c_2 + 4c_1 + c_0 = 6$$

$$y_3(5) = 125c_3 + 25c_2 + 5c_1 + c_0 = 7$$

$$y_3'(5) = 75c_3 + 10c_2 + c_1 = 0$$

$$\begin{pmatrix} 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 4 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 4 & 1 & 0 & -12 & -4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 2 & 0 & 0 & -12 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 4 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 64 & 16 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 48 & 8 & 1 & 0 & -48 & -8 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 24 & 2 & 0 & 0 & -24 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 64 & 16 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 125 & 25 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 75 & 10 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \\ b_3 \\ b_2 \\ b_1 \\ b_0 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 3 \\ 0 \\ 0 \\ 3 \\ 6 \\ 0 \\ 0 \\ 6 \\ 7 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \\ b_3 \\ b_2 \\ b_1 \\ b_0 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{pmatrix} = \begin{pmatrix} \frac{23}{35} & \frac{34}{35} & -\frac{34}{35} & \frac{11}{35} & \frac{6}{35} & -\frac{3}{14} & \frac{3}{14} & -\frac{4}{35} & -\frac{1}{35} & \frac{6}{35} & -\frac{6}{35} & \frac{2}{35} \\ -\frac{127}{35} & -\frac{171}{35} & \frac{171}{35} & -\frac{44}{35} & -\frac{24}{35} & \frac{6}{7} & -\frac{6}{7} & \frac{16}{35} & \frac{4}{35} & -\frac{24}{35} & \frac{24}{35} & -\frac{8}{35} \\ \frac{44}{7} & \frac{48}{7} & -\frac{48}{7} & \frac{11}{7} & \frac{6}{7} & -\frac{15}{14} & \frac{15}{14} & -\frac{4}{7} & -\frac{1}{7} & \frac{6}{7} & -\frac{6}{7} & \frac{2}{7} \\ -\frac{116}{35} & -\frac{68}{35} & \frac{103}{35} & -\frac{22}{35} & -\frac{12}{35} & \frac{3}{7} & -\frac{3}{7} & \frac{8}{35} & \frac{2}{35} & -\frac{12}{35} & \frac{12}{35} & -\frac{4}{35} \\ -\frac{1}{14} & -\frac{3}{14} & \frac{3}{14} & -\frac{1}{7} & \frac{1}{28} & \frac{1}{7} & -\frac{1}{7} & \frac{1}{7} & \frac{1}{28} & -\frac{3}{14} & \frac{3}{14} & -\frac{1}{14} \\ \frac{26}{35} & \frac{78}{35} & -\frac{78}{35} & \frac{52}{35} & -\frac{13}{35} & -\frac{9}{7} & \frac{9}{7} & -\frac{38}{35} & -\frac{19}{70} & \frac{57}{35} & -\frac{57}{35} & \frac{19}{35} \\ -\frac{86}{35} & -\frac{258}{35} & \frac{258}{35} & -\frac{172}{35} & \frac{43}{35} & \frac{45}{14} & -\frac{45}{14} & \frac{88}{35} & \frac{22}{35} & -\frac{132}{35} & \frac{132}{35} & -\frac{44}{35} \\ \frac{88}{35} & \frac{264}{35} & -\frac{264}{35} & \frac{176}{35} & -\frac{44}{35} & -\frac{10}{7} & \frac{17}{7} & -\frac{64}{35} & -\frac{16}{35} & \frac{96}{35} & -\frac{96}{35} & \frac{32}{35} \\ \frac{2}{35} & \frac{6}{35} & -\frac{6}{35} & \frac{4}{35} & -\frac{1}{35} & -\frac{3}{14} & \frac{3}{14} & -\frac{11}{35} & \frac{6}{35} & \frac{34}{35} & -\frac{34}{35} & \frac{23}{35} \\ -\frac{4}{5} & -\frac{12}{5} & \frac{12}{5} & -\frac{8}{5} & \frac{2}{5} & 3 & -3 & \frac{22}{5} & -\frac{12}{5} & -\frac{63}{5} & \frac{63}{5} & -\frac{41}{5} \\ \frac{26}{7} & \frac{78}{7} & -\frac{78}{7} & \frac{52}{7} & -\frac{13}{7} & -\frac{195}{14} & \frac{195}{14} & -\frac{143}{7} & \frac{78}{7} & \frac{372}{7} & -\frac{372}{7} & \frac{236}{7} \\ -\frac{40}{7} & -\frac{120}{7} & \frac{120}{7} & -\frac{80}{7} & \frac{20}{7} & \frac{150}{7} & -\frac{150}{7} & \frac{220}{7} & -\frac{120}{7} & -\frac{505}{7} & \frac{512}{7} & -\frac{320}{7} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \\ 3 \\ 0 \\ 0 \\ 3 \\ 6 \\ 0 \\ 0 \\ 6 \\ 7 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 3 \\ -\frac{9}{2} \\ 4 \\ 0 \\ 0 \\ \frac{3}{2} \\ 0 \\ -\frac{1}{2} \\ 6 \\ -\frac{45}{2} \\ 32 \end{pmatrix}$$

$$y_1(x) = -\frac{1}{2}x^3 + 3x^2 - \frac{9}{2}x + 4$$

$$y_2(x) = \frac{3}{2}x$$

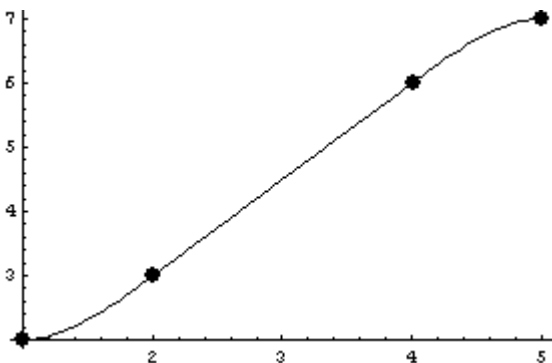
$$y_3(x) = -\frac{1}{2}x^3 + 6x^2 - \frac{45}{2}x + 32$$

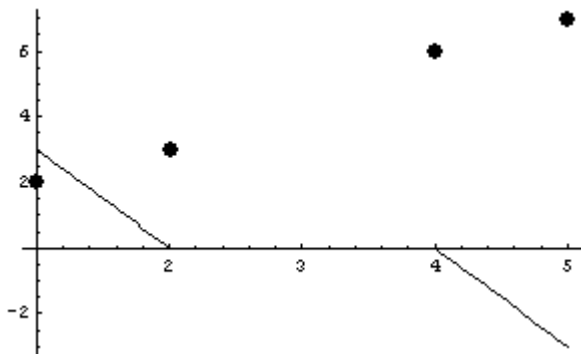
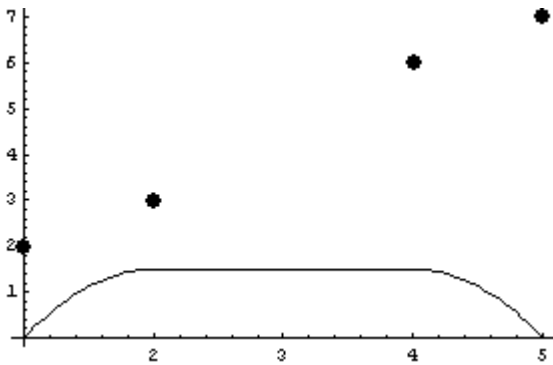
$$y(x) = \begin{cases} -\frac{1}{2}x^3 + 3x^2 - \frac{9}{2}x + 4 & ; 1 \leq x \leq 2 \\ \frac{3}{2}x & ; 2 \leq x \leq 4 \\ -\frac{1}{2}x^3 + 6x^2 - \frac{45}{2}x + 32 & ; 4 \leq x \leq 5 \end{cases}$$

$$y'(x) = \begin{cases} -\frac{3}{2}x^2 + 6x - \frac{9}{2} & ; 1 \leq x \leq 2 \\ \frac{3}{2} & ; 2 \leq x \leq 4 \\ -\frac{3}{2}x^2 + 12x - \frac{45}{2} & ; 4 \leq x \leq 5 \end{cases} \quad y''(x) = \begin{cases} -3x + 6 & ; 1 \leq x \leq 2 \\ 0 & ; 2 \leq x \leq 4 \\ -3x + 12 & ; 4 \leq x \leq 5 \end{cases}$$

So $y(x)$ is both differentiable and double differentiable at points $x = 2$ and $x = 4$.

The graphs of $y(x), y'(x), y''(x)$ are given below.





6. Fit the data by least square polynomial of degree 1, 2 and 3.

All the methods discussed previously are exact in the sense that they assume that the data points are exactly correct. But normally know is that there are errors. Also we may know the behavior in advance and want to find a physical quantity like gravitational acceleration by doing this experiment.

Suppose that we know that the data are linearly related. This means that you are going to predict $y(x)$ by $\hat{y}(x) = ax + b$

But how to find a, b ? If we have n number of points, the error at the x_i is

$E_i = y(x_i) - \hat{y}(x_i) = y_i - \hat{y}_i = y_i - (ax_i + b)$. So the sum of square error is

$\sum_{i=1}^n E_i^2 = \sum_{i=1}^n (y_i - (ax_i + b))^2$. Note that this value changes when we change a, b . That is to sat that this value

is a function of the two variables a, b . This means

$$E(a, b) = \sum_{i=1}^n (y_i - (ax_i + b))^2$$

Now we can find a, b so that $E(a, b)$ is minimized. Then at such points

$$\frac{\partial E}{\partial a} = 2 \sum_{i=1}^n (y_i - (ax_i + b))(-x_i) = 2 \left(- \sum_{i=1}^n x_i y_i + a \sum_{i=1}^n x_i^2 + b \sum_{i=1}^n x_i \right) = 0$$

$$\frac{\partial E}{\partial b} = 2 \sum_{i=1}^n (y_i - (ax_i + b))(-1) = 2 \left(- \sum_{i=1}^n y_i + a \sum_{i=1}^n x_i + b \sum_{i=1}^n 1 \right) = 2 \left(- \sum_{i=1}^n y_i + a \sum_{i=1}^n x_i + bn \right) = 0$$

So we have the following system of equations to solve

$$a \sum_{i=1}^n x_i^2 + b \sum_{i=1}^n x_i = \sum_{i=1}^n x_i y_i$$

$$a \sum_{i=1}^n x_i + bn = \sum_{i=1}^n y_i$$

$$\begin{pmatrix} \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & n \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n x_i y_i \\ \sum_{i=1}^n y_i \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & n \end{pmatrix}^{-1} \begin{pmatrix} \sum_{i=1}^n x_i y_i \\ \sum_{i=1}^n y_i \end{pmatrix}$$

It can be shown that these corresponds to a minimum

For the given data set we do the computation as follows

					sum
x_i	1	2	4	5	12
y_i	2	3	6	7	18
x_i^2	1	4	16	25	46
$x_i y_i$	2	6	24	35	67

Here $n = 4$

Now the matrix is

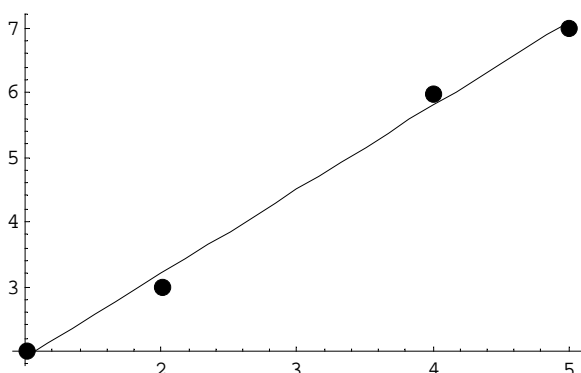
$$\begin{pmatrix} 46 & 12 \\ 12 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 67 \\ 18 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 46 & 12 \\ 12 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 67 \\ 18 \end{pmatrix} = \frac{1}{46 \times 4 - 12 \times 12} \begin{pmatrix} 4 & -12 \\ -12 & 46 \end{pmatrix} \begin{pmatrix} 67 \\ 18 \end{pmatrix} = \frac{2}{8(23-18)} \begin{pmatrix} 2 & -6 \\ -6 & 23 \end{pmatrix} \begin{pmatrix} 67 \\ 18 \end{pmatrix} = \frac{1}{20} \begin{pmatrix} 26 \\ 12 \end{pmatrix} = \begin{pmatrix} \frac{13}{10} \\ \frac{3}{5} \end{pmatrix}$$

So the least square line is

$$\hat{y}(x) = \frac{13}{10}x + \frac{3}{5} = 1.3x + 0.6$$

Following show the graph



One can do the same procedure by assuming a second degree polynomial $\hat{y}(x) = ax^2 + bx + c$. So the final matrix will be

$$\begin{pmatrix} \sum_{i=1}^n x_i^4 & \sum_{i=1}^n x_i^3 & \sum_{i=1}^n x_i^2 \\ \sum_{i=1}^n x_i^3 & \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i & n \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n x_i^2 y_i \\ \sum_{i=1}^n x_i y_i \\ \sum_{i=1}^n y_i \end{pmatrix}$$

					sum
x_i	1	2	4	5	12
y_i	2	3	6	7	18
x_i^2	1	4	16	25	46
x_i^3	1	8	64	125	198
x_i^4	1	16	256	625	898
$x_i y_i$	2	6	24	35	67
$x_i^2 y_i$	2	12	96	175	285

Hare $n = 4$

$$\begin{pmatrix} \sum_{i=1}^n x_i^4 & \sum_{i=1}^n x_i^3 & \sum_{i=1}^n x_i^2 \\ \sum_{i=1}^n x_i^3 & \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i & n \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n x_i^2 y_i \\ \sum_{i=1}^n x_i y_i \\ \sum_{i=1}^n y_i \end{pmatrix} \Rightarrow \begin{pmatrix} 898 & 198 & 46 \\ 198 & 46 & 12 \\ 46 & 12 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 285 \\ 67 \\ 18 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \frac{1}{9} & -\frac{2}{3} & \frac{13}{18} \\ -\frac{2}{3} & \frac{41}{10} & -\frac{139}{30} \\ \frac{13}{18} & -\frac{139}{30} & \frac{263}{45} \end{pmatrix} \cdot \begin{pmatrix} 285 \\ 67 \\ 18 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{13}{10} \\ \frac{3}{5} \end{pmatrix}$$

So the 2nd degree least square polynomial is

$$\hat{y}(x) = 0x^2 + \frac{13}{10}x + \frac{3}{5} = 1.3x + 0.6$$

Note that it is actually the 1st degree least square polynomial. This can happen, but not in general

Using the same arguments to the earlier one can do the same procedure by assuming a third degree polynomial $\hat{y}(x) = ax^3 + bx^2 + cx + d$. So the final matrix will be

$$\begin{pmatrix} \sum_{i=1}^n x_i^6 & \sum_{i=1}^n x_i^5 & \sum_{i=1}^n x_i^4 & \sum_{i=1}^n x_i^3 \\ \sum_{i=1}^n x_i^5 & \sum_{i=1}^n x_i^4 & \sum_{i=1}^n x_i^3 & \sum_{i=1}^n x_i^2 \\ \sum_{i=1}^n x_i^4 & \sum_{i=1}^n x_i^3 & \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i^3 & \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i & n \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n x_i^3 y_i \\ \sum_{i=1}^n x_i^2 y_i \\ \sum_{i=1}^n x_i y_i \\ \sum_{i=1}^n y_i \end{pmatrix}$$

When $n = 4$ one can notice that this is same as

$$\begin{pmatrix} x_1^3 & x_2^3 & x_3^3 & x_4^3 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1 & x_2 & x_3 & x_4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1^3 & x_1^2 & x_1 & 1 \\ x_2^3 & x_2^2 & x_2 & 1 \\ x_3^3 & x_3^2 & x_3 & 1 \\ x_4^3 & x_4^2 & x_4 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} x_1^3 & x_2^3 & x_3^3 & x_4^3 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1 & x_2 & x_3 & x_4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

$$X^T X A = X^T Y \Rightarrow (X^{-1})^T X^T X A = (X^{-1})^T X^T Y \Rightarrow X A = Y \Rightarrow A = X^{-1} Y$$

In other words this reduces to

$$\begin{pmatrix} x_1^3 & x_1^2 & x_1 & 1 \\ x_2^3 & x_2^2 & x_2 & 1 \\ x_3^3 & x_3^2 & x_3 & 1 \\ x_4^3 & x_4^2 & x_4 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}.$$

This is actually leading to the Lagrange Polynomial that we found earlier.

This is expected as we have 4 points and because cubic polynomial has 4 unknowns.

In other words Lagrange polynomial is actually a kind of least square polynomial where the least square error is 0.

Answer $y(x) = -\frac{1}{12}x^3 + \frac{3}{4}x^2 - \frac{2}{3}x + 2$

7. Find $y(1.6)$ and $y(6.1)$ by all these methods.

Method	Degree	$y(x)$	$y(1.6)$	$y(6.1)$
Lagrange	3-3-3	$y(x) = -\frac{1}{12}x^3 + \frac{3}{4}x^2 - \frac{2}{3}x + 2$	2.512	6.92575
Piecewise Lagrange	1-1-1	$y(x) = \begin{cases} x+1 & ; 1 \leq x \leq 2 \\ \frac{3}{2}x & ; 2 \leq x \leq 4 \\ x+2 & ; 4 \leq x \leq 5 \end{cases}$	2.6	
	2-1	$y(x) = \begin{cases} \frac{1}{6}x^2 + \frac{1}{2}x + \frac{4}{3} & ; 1 \leq x \leq 4 \\ x+2 & ; 4 \leq x \leq 5 \end{cases}$	2.56	
Piecewise Hermite (Spline)	2-2-2	$y(x) = \begin{cases} \frac{1}{6}x^2 + \frac{1}{2}x + \frac{4}{3} & ; 1 \leq x \leq 4 \\ -\frac{5}{6}x^2 + \frac{17}{2}x - \frac{44}{3} & ; 4 \leq x \leq 5 \end{cases}$	2.56	10.585

	3-3-3	$y(x) = \begin{cases} -\frac{1}{2}x^3 + 3x^2 - \frac{9}{2}x + 4 & ; 1 \leq x \leq 2 \\ \frac{3}{2}x & ; 2 \leq x \leq 4 \\ -\frac{1}{2}x^3 + 6x^2 - \frac{45}{2}x + 32 & ; 4 \leq x \leq 5 \end{cases}$	2.432	4.5195
Least Square	1	$\hat{y}(x) = \frac{13}{10}x + \frac{3}{5} = 1.3x + 0.6$	2.68	8.53
	2	$\hat{y}(x) = 0x^2 + \frac{13}{10}x + \frac{3}{5} = 1.3x + 0.6$	2.68	8.53
	3	$\hat{y}(x) = -\frac{1}{12}x^3 + \frac{3}{4}x^2 - \frac{2}{3}x + 2$	2.512	6.92575

We can find $y'(x), y''(x), \int y(x)dx$ also by doing the respective operations on these functions.

However $y'(x), y''(x)$ should exist to find them and hence we need at least Piecewise Hermit polynomials for this.

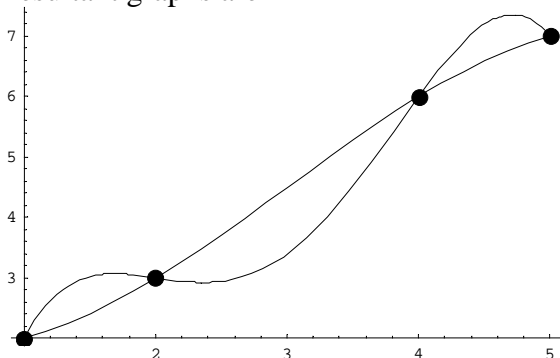
But integration does not require differentiability and hence Piecewise Lagrange polynomials are sufficient to find $\int y(x)dx$. This part of Numerical Methods is called Numerical Integration and will be discussed in detail in the next chapter.

It is very clear that there are infinitely many curves pass these 4 points. This is only one of them Another curve can be found by setting $z(x) = ae^x + b \sin x + c \ln x + dx$. That is

$f(x) = e^x, g(x) = \sin x, h(x) = \ln x, k(x) = x$. Than we get

$$y(x) : -0.0606624e^x + 4.93449 \cos[x] + 8.71869 \log[x] - 0.595648 \sin[x]$$

The resultant graphs are



Also such functions can be used to find Piecewise Lagrange, Hermit or even Least Square approximations.

Some of these calculations can be done using your calculators. See the catalogue for regression analysis.

You are not supposed to get inverses of large matrices manually. They are for illustration purposes only.

NUMERICAL ANALYSIS AND METHODS-Udaya Chinthaka Jayatilake
CHAPTER 3-Numerical Integration

Consider the CDF (Cumulative Distribution Function) of the Standard Normal Distribution

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{t^2}{2}} dt$$

Find the value of $F(2) = P(-\infty < x < 2)$ using

1. Normal Tables

$$0.4772$$

2. DISTR function in your calculator

$$Q(2)=0.47725$$

3. INTEGRATION function in your calculator

$$\int (e^{(-X^2 \div 2)} \div \sqrt{(2\pi)}, 0, 2) = 0.4772498681$$

See the catalogue of the calculator to find the methods that it uses to find the integral.

$$\int (e^{(-X^2 \div 2)} \div \sqrt{(2\pi)}, 0, 2, 10) = 0.4772498681$$

Normally the last number 100 means $n = 2^{10}$

Find the number of intervals needed to find the above integral accurate to 5 decimal places by each of the methods below.

Absolute error terms is given as follows for $\int_a^b f(x)dx$ with $\xi \in (a, b)$

1. Trapezoidal Rule $\frac{(b-a)h^2}{12} f^{(2)}(\xi)$

In the Trapezoidal rule points are found by sampling the given function $f(x)$ and piecewise Lagrange Polynomial $y(x)$ of degree 1 (straight line) is found and the area is found by integrating $y(x)$

We know that $F(2) = P(-\infty < x < 2)$

$$F(2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^2 e^{-\frac{t^2}{2}} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{-\frac{t^2}{2}} dt + \frac{1}{\sqrt{2\pi}} \int_0^2 e^{-\frac{t^2}{2}} dt = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^2 e^{-\frac{t^2}{2}} dt$$

Therefore what we actually have to find is the integral $\int_0^2 e^{-\frac{t^2}{2}} dt$

Let $f(x) = e^{-\frac{x^2}{2}}$, $a = 0$ and $b = 2$

$$f'(x) = -xe^{-\frac{x^2}{2}}$$

$$f''(x) = -x \left(-xe^{-\frac{x^2}{2}} \right) - e^{-\frac{x^2}{2}} = e^{-\frac{x^2}{2}} (x^2 - 1)$$

$$f'''(x) = e^{-\frac{x^2}{2}} (2x) + (x^2 - 1) \cdot -xe^{-\frac{x^2}{2}} = -e^{-\frac{x^2}{2}} x(x^2 - 3)$$

So the critical points of $f''(x)$ are given by

$$f'''(x) = e^{-\frac{x^2}{2}} x(x^2 - 3) = 0 \Leftrightarrow x = -\sqrt{3}, 0, \sqrt{3}$$

We divide the region into 4 as

$(-\infty, -\sqrt{3})$: $f'''(-2) > 0 \Rightarrow f''(x)$ is increasing on this interval. (actually we do not need this interval)

$(-\sqrt{3}, 0)$: $f'''(-1) < 0 \Rightarrow f''(x)$ is decreasing on this interval. (actually we do not need this interval)

$(0, \sqrt{3})$: $f'''(1) > 0 \Rightarrow f''(x)$ is increasing on this interval.

$(\sqrt{3}, \infty)$: $f'''(2) < 0 \Rightarrow f''(x)$ is decreasing on this interval. (actually we do not need $(2, \infty)$ interval)

Therefore

$f''(x)$ is increasing on $(0, \sqrt{3})$ and decreasing on $(\sqrt{3}, 2)$ (actually $f''(x)$ has a maximum at $\sqrt{3}$)

Therefore we check with $f''(x) = e^{-\frac{x^2}{2}} (x^2 - 1)$

$$f''(0) = -1$$

$$f''(\sqrt{3}) = 2e^{-\frac{3}{2}} = \frac{2}{\sqrt{e^3}} \approx 0.4462603203$$

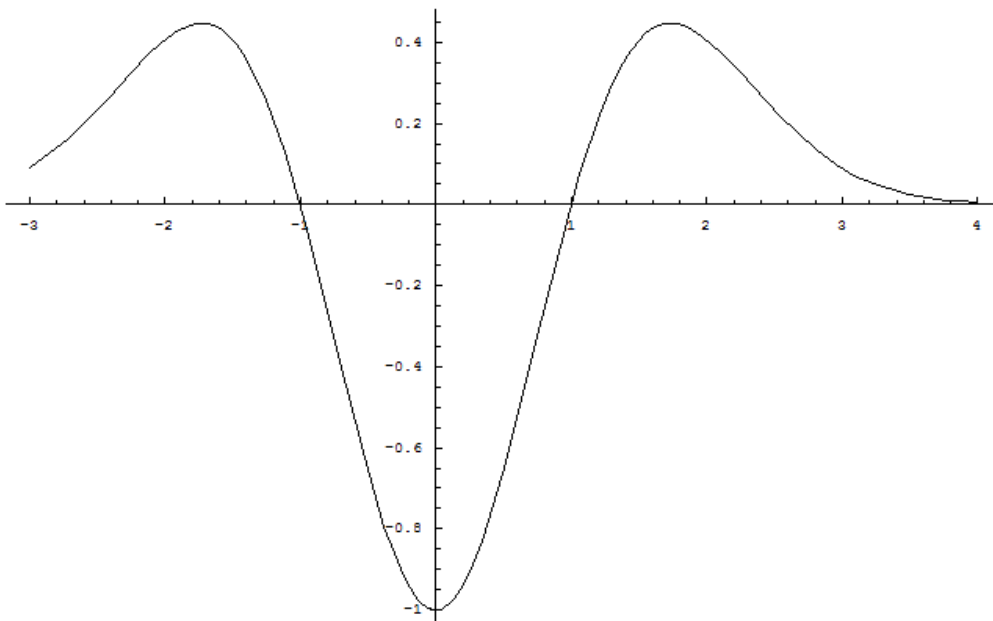
$$f''(2) = 3e^{-2} = \frac{3}{e^2} \approx -0.6766764162$$

Therefore on $(0, 2)$: $|f''(x)| \leq 1$

This is seen from the graph of $f''(x)$

```
In[5]:= f[x_] := Exp[-x^2/2]
```

```
In[8]:= Plot[f''[x], {x, -3, 4}]
```



```
Out[8]:= - Graphics -
```

$$\left| \int_a^b f(x) dx - \int_a^b p_1(x) dx \right| \leq \frac{(b-a)h^2}{12} \max |f''(x)| = \frac{(b-a)^3}{12n^2} \max |f''(x)| \leq \frac{(2-0)^3}{12n^2} 1 < \frac{1}{2} 10^{-5}$$

$$\Leftrightarrow n^2 > \frac{4}{3} 10^5$$

$$\Leftrightarrow n > 365.1483717$$

$$\Leftrightarrow n = 366$$

Also find the above integral accurate to 5 decimal places using any of the above two method.

However these n values are much larger than needed since the upper bound of the error is too high.

We will do the illustration with.

We sum using MATHEMATICA

In[12]:= `f[x_] := Exp[-x^2/2]`

$$\text{In[21]:= } N\left[\frac{1}{\sqrt{2\pi}} \frac{1}{2} \frac{2}{366} \left(f[0] + 2 \sum_{r=1}^{365} f\left[0 + \frac{2}{366} r\right] + f[2] \right), 10\right]$$

Out[21]= 0.4772495994

We will illustrate the procedure with Let $n = 10$

$$\text{So } h = \frac{b-a}{n} = \frac{2-0}{10} = 0.2$$

$$\text{The Trapezoidal Rule is } \int_a^b y(x) dx = \frac{h}{2} (y_0 + 2(y_1 + y_2 + \dots + y_{n-1}) + y_n)$$

r	x_r	$y_r = y(x_r) = f(x_r)$	
0	0	1	1
1	0.2		0.9801986733
2	0.4		0.9231163464
3	0.6		0.8352702114
4	0.8		0.7261490371
5	1.0		0.6065306597
6	1.2		0.4867522560
7	1.4		0.3753110989
8	1.6		0.2780373005
9	1.8		0.1978986991
10	2.0	0.1353352832	0.1353352832
			11.95386385

$$\int_0^2 e^{-\frac{x^2}{2}} dx \approx \frac{0.2}{2} (11.95386385) = 0.1195386385$$

$$\int_0^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \approx 0.4768901703$$

Find the number of intervals needed to find the above integral accurate to 5 decimal places by each of the methods below.

Absolute error terms is given as follows for $\int_a^b f(x)dx$ with $\xi \in (a, b)$

2. Simpson's Rule $\frac{(b-a)h^4}{180} f^{(4)}(\xi)$

In the Simpson's rule points are found by sampling the given function $f(x)$ and piecewise Lagrange Polynomial $y(x)$ of degree 2 (Quadratic) is found and the area is found by integrating $y(x)$.

Therefore the divisions n must be even

The calculation is similar to the Trapezoidal rule but more complex since we have to find an upper bound for $f^{(4)}(x)$ instead for $f^{(2)}(x)$. We will not do it here.

Also find the above integral accurate to 5 decimal places using any of the above two methods.

We will illustrate the procedure with Let $n = 10$ which is even

So $h = \frac{b-a}{n} = \frac{2-0}{10} = 0.2$

The Simpson's Rule is

$$\int_a^b y(x)dx = \frac{h}{3} (y_0 + 4(y_1 + y_3 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2}) + y_n)$$

r	x_r	$y_r = y(x_r) = f(x_r)$		
0	0	1		1
1	0.2		0.9801986733	3.920794693
2	0.4			0.9231163464
3	0.6		0.8352702114	3.341080846
4	0.8			0.7261490371
5	1.0		0.6065306597	2.426122639
6	1.2			0.4867522560
7	1.4		0.3753110989	1.501244395
8	1.6			0.2780373005
9	1.8		0.1978986991	0.7915947963
10	2.0	0.1353352832		0.1353352832
				17.94428253

$$\int_0^2 e^{-\frac{x^2}{2}} dx \approx \frac{0.2}{3} (17.94428253) = 1.196285502$$

$$\int_0^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \approx 0.4772488662$$

Simpson's rule is not always more accurate than the Trapezoidal rule.

Consider the following system of equations

$$5x + 3y + 2z = 17$$

$$3x + 7y - z = 14$$

$$2x + 5y + 3z = 21$$

1. Write the system in the matrix form $AX = B$.
-

$$\begin{pmatrix} 5 & 3 & 2 \\ 3 & 7 & -1 \\ 2 & 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 17 \\ 14 \\ 21 \end{pmatrix}$$

2. Find $|A|$ and comment on the nature of solution.
-

3. Solve the system by matrix inversion.
-

$$X = A^{-1}B = \begin{pmatrix} 5 & 3 & 2 \\ 3 & 7 & -1 \\ 2 & 5 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 17 \\ 14 \\ 21 \end{pmatrix} = \frac{1}{99} \begin{pmatrix} 26 & 1 & -17 \\ -11 & 11 & 11 \\ 1 & -19 & 26 \end{pmatrix} \begin{pmatrix} 17 \\ 14 \\ 21 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

4. Find reduced row echelon form of the augmented matrix $(A|B)$ and comment on the nature of solution.
-

5. Find the solution by back substitution.
-

6. Solve the system by the Gauss elimination method (same as 4. and 5.).
-

7. Find the solution by finding the reduced row echelon form of the augmented matrix $(A|B)$.
-

8. Solve the system by Gauss-Jordan elimination method (same as 7.).
-

9. Find the matrices H, G such that $X_{n+1} = HX_n + G$ to solve the system by the Gauss-Jacobi method.
-

$$\begin{array}{l} 5x + 3y + 2z = 17 \\ 3x + 7y - z = 14 \\ 2x + 5y + 3z = 21 \end{array} \Rightarrow \begin{array}{l} x = -\frac{3}{5}y - \frac{2}{5}z + \frac{17}{5} \\ y = -\frac{3}{7}x + \frac{1}{7}z + \frac{14}{7} \\ z = -\frac{2}{3}x - \frac{5}{3}y + \frac{21}{3} \end{array} \Rightarrow \begin{array}{l} x_{n+1} = -\frac{3}{5}y_n - \frac{2}{5}z_n + \frac{17}{5} \\ y_{n+1} = -\frac{3}{7}x_n + \frac{1}{7}z_n + \frac{14}{7} \\ z_{n+1} = -\frac{2}{3}x_n - \frac{5}{3}y_n + \frac{21}{3} \end{array}$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{3}{5} & -\frac{2}{5} \\ -\frac{3}{7} & 0 & \frac{1}{7} \\ -\frac{2}{3} & -\frac{5}{3} & 0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} + \begin{pmatrix} \frac{17}{5} \\ \frac{14}{7} \\ \frac{21}{3} \end{pmatrix} \Rightarrow X_{n+1} = HX_n + G$$

10. Find $\|H\|_1$ and $\|H\|_\infty$ and see whether it is possible to make a comment on the convergence. Otherwise find $\rho(H)$ and comment on the convergence.
-

Maximum absolute column sum norm

$$\begin{aligned}\|M\|_1 &= \max_j \left\{ \sum_i |m_{ij}| \right\} \\ &= \max \left\{ |0| + \left| -\frac{3}{7} \right| + \left| -\frac{2}{3} \right|, \left| -\frac{3}{5} \right| + |0| + \left| -\frac{5}{3} \right|, \left| -\frac{2}{5} \right| + \left| \frac{1}{7} \right| + |0| \right\} \\ &= \max \left\{ \frac{23}{21}, \frac{34}{15}, \frac{19}{35} \right\} = \max \left\{ \frac{115}{105}, \frac{238}{105}, \frac{19}{35} \right\} = \frac{34}{15} > 1\end{aligned}$$

Maximum absolute row sum norm

$$\begin{aligned}\|M\|_\infty &= \max_i \left\{ \sum_j |m_{ij}| \right\} \\ &= \max \left\{ |0| + \left| -\frac{3}{5} \right| + \left| -\frac{2}{5} \right|, \left| -\frac{3}{7} \right| + |0| + \left| \frac{1}{7} \right|, \left| -\frac{2}{3} \right| + \left| \frac{-5}{3} \right| + |0| \right\} \\ &= \max \left\{ 1, \frac{4}{7}, \frac{7}{3} \right\} = \frac{7}{3} > 1\end{aligned}$$

convergence $\Leftrightarrow \rho(H) < 1$ and $\rho(H) \leq \|H\|$ where $\|H\|$ is any norm .

This means

$$\|H\| < 1 \Rightarrow \rho(H) < 1 \Leftrightarrow \text{convergence.}$$

Here $\|H\| > 1$ hence we cannot make a comment about the convergence based on $\|H\|$

Therefore we have to find

$\rho(H) = \max_i \{|\lambda_i|\}$ the spectral radius of H where λ_i are the eigenvalues of H which are the roots of the characteristic polynomial $p(\lambda) = |\lambda I - H| = (-1)^n |H - \lambda I|$

$$\begin{aligned}& \left(\begin{array}{ccc} 0 & -\frac{3}{5} & -\frac{2}{5} \\ -\frac{3}{7} & 0 & \frac{1}{7} \\ -\frac{2}{3} & -\frac{5}{3} & 0 \end{array} - \lambda \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) = \begin{vmatrix} -\lambda & -\frac{3}{5} & -\frac{2}{5} \\ -\frac{3}{7} & -\lambda & \frac{1}{7} \\ -\frac{2}{3} & -\frac{5}{3} & -\lambda \end{vmatrix} \\ &= -\lambda \left(\lambda^2 + \frac{5}{21} \right) + \frac{3}{5} \left(\frac{3}{7} \lambda + \frac{2}{21} \right) - \frac{2}{5} \left(\frac{5}{7} - \frac{2}{3} \lambda \right) \\ &= -\lambda^3 + \left(-\frac{5}{21} + \frac{9}{35} + \frac{4}{15} \right) \lambda + \frac{2}{35} - \frac{2}{7} \\ &= -\lambda^3 + \frac{-25+27+28}{105} \lambda + \frac{2-10}{35} = -\lambda^3 + \frac{30}{105} \lambda + \frac{-8}{35} = -\lambda^3 + \frac{2}{7} \lambda - \frac{8}{35}\end{aligned}$$

$$p(\lambda) = |\lambda I - H| = (-1)^3 |H - \lambda I| = \lambda^3 - \frac{2}{7} \lambda + \frac{8}{35} = 0 \Leftrightarrow$$

$$\lambda \approx -0.764623, \quad 0.382311 + 0.39086i, \quad 0.382311 - 0.39086i$$

$$\Rightarrow |\lambda| \approx 0.764623, \quad 0.546748, \quad 0.546748$$

$$\rho(H) = \max_i \{|\lambda_i|\} \approx \max\{0.764623, 0.546748, 0.546748\} = 0.764623 < 1$$

Therefore the process converges.

-
11. If the process is not converging with the current H , modify the system until the convergence requirements are met.
-

When required this can be done by interchanging variables, interchanging or adding equations.

It is clear from the definitions of $\|H\|_1$ and $\|H\|_\infty$ that they become less (as a result $\rho(H)$ becomes less) if we get smaller numbers as the elements of H . This will happen if the diagonal of A having large numbers.

12. Find the solution by the Gauss-Jacobi method.
-

$$\text{In[31]:= } H = \begin{pmatrix} 0 & -3/5 & -2/5 \\ -3/7 & 0 & 1/7 \\ -2/3 & -5/3 & 0 \end{pmatrix}; G = \begin{pmatrix} 17/5 \\ 14/7 \\ 21/3 \end{pmatrix};$$

$$\text{In[32]:= } X = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; S = \{\};$$

For[i = 1, i ≤ 30, i++, {X = H.X + G, S = Append[S, MatrixForm[X]}]};

Print[N[S]]

$$\left\{ \begin{pmatrix} 2.4 \\ 1.71429 \\ 4.66667 \end{pmatrix}, \begin{pmatrix} 0.504762 \\ 1.6381 \\ 2.54286 \end{pmatrix}, \begin{pmatrix} 1.4 \\ 2.14694 \\ 3.93333 \end{pmatrix}, \begin{pmatrix} 0.538503 \\ 1.9619 \\ 2.48844 \end{pmatrix}, \begin{pmatrix} 1.22748 \\ 2.1247 \\ 3.37116 \end{pmatrix}, \begin{pmatrix} 0.776715 \\ 1.95553 \\ 2.64051 \end{pmatrix}, \right.$$

$$\begin{pmatrix} 1.17048 \\ 2.04434 \\ 3.22297 \end{pmatrix}, \begin{pmatrix} 0.884208 \\ 1.95879 \\ 2.81245 \end{pmatrix}, \begin{pmatrix} 1.09975 \\ 2.02283 \\ 3.14588 \end{pmatrix}, \begin{pmatrix} 0.92795 \\ 1.97809 \\ 2.89545 \end{pmatrix}, \begin{pmatrix} 1.05497 \\ 2.01594 \\ 3.08455 \end{pmatrix}, \begin{pmatrix} 0.956615 \\ 1.98852 \\ 2.93679 \end{pmatrix},$$

$$\begin{pmatrix} 1.03217 \\ 2.00956 \\ 3.04805 \end{pmatrix}, \begin{pmatrix} 0.975041 \\ 1.99308 \\ 2.96261 \end{pmatrix}, \begin{pmatrix} 1.01911 \\ 2.00536 \\ 3.02818 \end{pmatrix}, \begin{pmatrix} 0.985515 \\ 1.99584 \\ 2.97833 \end{pmatrix}, \begin{pmatrix} 1.01116 \\ 2.00311 \\ 3.0166 \end{pmatrix}, \begin{pmatrix} 0.991494 \\ 1.99759 \\ 2.98737 \end{pmatrix},$$

$$\begin{pmatrix} 1.0065 \\ 2.00184 \\ 3.00969 \end{pmatrix}, \begin{pmatrix} 0.995018 \\ 1.9986 \\ 2.9926 \end{pmatrix}, \begin{pmatrix} 1.0038 \\ 2.00108 \\ 3.00566 \end{pmatrix}, \begin{pmatrix} 0.997091 \\ 1.99918 \\ 2.99567 \end{pmatrix}, \begin{pmatrix} 1.00222 \\ 2.00063 \\ 3.00331 \end{pmatrix}, \begin{pmatrix} 0.9983 \\ 1.99952 \\ 2.99747 \end{pmatrix},$$

$$\left. \begin{pmatrix} 1.0013 \\ 2.00037 \\ 3.00194 \end{pmatrix}, \begin{pmatrix} 0.999006 \\ 1.99972 \\ 2.99852 \end{pmatrix}, \begin{pmatrix} 1.00076 \\ 2.00021 \\ 3.00113 \end{pmatrix}, \begin{pmatrix} 0.999419 \\ 1.99984 \\ 2.99914 \end{pmatrix}, \begin{pmatrix} 1.00044 \\ 2.00013 \\ 3.00066 \end{pmatrix}, \begin{pmatrix} 0.99966 \\ 1.9999 \\ 2.99949 \end{pmatrix} \right\}$$

13. Repeat the questions 9. through 12. with the Gauss-Seidel method.
-

$$\begin{aligned} 5x + 3y + 2z &= 17 & x &= -\frac{3}{5}y - \frac{2}{5}z + \frac{17}{5} & x_{n+1} &= -\frac{3}{5}y_n - \frac{2}{5}z_n + \frac{17}{5} \\ 3x + 7y - z &= 14 & \Rightarrow y &= -\frac{3}{7}x + \frac{1}{7}z + \frac{14}{7} & \Rightarrow y_{n+1} &= -\frac{3}{7}x_{n+1} + \frac{1}{7}z_n + \frac{14}{7} \\ 2x + 5y + 3z &= 21 & z &= -\frac{2}{3}x - \frac{5}{3}y + \frac{21}{3} & z_{n+1} &= -\frac{2}{3}x_{n+1} - \frac{5}{3}y_{n+1} + \frac{21}{3} \end{aligned}$$

