MODEL QUESTIONS WITH SOLUTIONS

MA(101)

VECTORS AND 3-D GEOMETRY

Q 1. Determine the lengths $|\overrightarrow{OP}|$ of the vectors $\underline{OP}$ given that 0 is the origin and the points P are:
(a) (1,3,4) (b) (2,4,5) (c) (4,0,2)

Solution: . (a) $\overrightarrow{OP} = \underline{i} + 3\underline{j} + 4\underline{k}$ $\left|\overrightarrow{OP}\right| = \sqrt{(\underline{i} + 3\underline{j} + 4\underline{k})\cdot(\underline{i} + 3\underline{j} + 4\underline{k})} = \sqrt{(1^2 + 3^2 + 4^2)} = \sqrt{26}$

(b) $\overrightarrow{OP} = 2\underline{i} + 4\underline{j} + 5\underline{k}$ $\left|\overrightarrow{OP}\right| = \sqrt{(2\underline{i} + 4\underline{j} + 5\underline{k})\cdot(2\underline{i} + 4\underline{j} + 5\underline{k})} = \sqrt{(2^2 + 4^2 + 5^2)} = \sqrt{45}$

(c) $\overrightarrow{OP} = 4\underline{i} + 0\underline{j} + 2\underline{k}$ $\left|\overrightarrow{OP}\right| = \sqrt{(4\underline{i} + 0\underline{j} + 2\underline{k})\cdot(4\underline{i} + 0\underline{j} + 2\underline{k})} = \sqrt{(4^2 + 0^2 + 2^2)} = \sqrt{20}$

Q2. Find the lengths $|\overrightarrow{OP}|$, the direction cosines and the angles $(\theta_1, \theta_2, \theta_3)$ of the vectors $\underline{OP}$, where the points P are:
(a) (2,-I,-1); (b) (4,0,2); (c) (- 1,2,1).

Solution: (a) $\overrightarrow{OP} = 2\underline{i} - \underline{j} - \underline{k}$ $\left|\overrightarrow{OP}\right| = \sqrt{(2^2 + 1^2 + 1^2)} = \sqrt{6}$, $\dfrac{\overrightarrow{OP}}{|OP|} = \dfrac{2\underline{i} - \underline{j} - \underline{k}}{\sqrt{6}}$, then

$\cos\theta_1 = \dfrac{2}{\sqrt{6}}, \cos\theta_2 = \dfrac{-1}{\sqrt{6}}$ and $\cos\theta_3 = \dfrac{-1}{\sqrt{6}}$.

(b) $\overrightarrow{OP} = 4\underline{i} + 0\underline{j} + 2\underline{k}$ $\left|\overrightarrow{OP}\right| = \sqrt{(4^2 + 0^2 + 2^2)} = \sqrt{20}$, $\dfrac{\overrightarrow{OP}}{|OP|} = \dfrac{4\underline{i} + 0\underline{j} + 2\underline{k}}{\sqrt{20}}$, then $\cos\theta_1 = \dfrac{4}{\sqrt{20}}, \cos\theta_2 = 0$

and $\cos\theta_3 = \dfrac{2}{20}$

(c) $\overrightarrow{OP} = -\underline{i} + 2\underline{j} + 1\underline{k}$ $\left|\overrightarrow{OP}\right| = \sqrt{(1^2 + 2^2 + 1^2)} = \sqrt{6}$, $\dfrac{\overrightarrow{OP}}{|OP|} = -\dfrac{-\underline{i} + 2\underline{j} + \underline{k}}{\sqrt{6}}$, then $\cos\theta_1 = \dfrac{-1}{\sqrt{6}}, \cos\theta_2 = \dfrac{2}{\sqrt{6}}$

and $\cos\theta_3 = \dfrac{1}{\sqrt{6}}$

Q3. Find the direction ratios, the direction cosines and the angles $(\theta_1, \theta_2, \theta_3)$ of the vectors $\underline{OP}$ where the points P are:
(a) (1,1,1); (b) (-1,1,1); (c) (21,-1).

Solution :

(a) $\overrightarrow{OP} = \underline{i} + \underline{j} + \underline{k}$ then direction ratios line OP is 1:1:1

$\left|\overrightarrow{OP}\right| = \sqrt{(1^2 + 1^2 + 1^2)} = \sqrt{3}$, $\dfrac{\overrightarrow{OP}}{|OP|} = -\dfrac{\underline{i} + \underline{j} + \underline{k}}{\sqrt{3}}$, then direction cosine of line OP :

$\cos\theta_1 = \dfrac{1}{\sqrt{3}}, \cos\theta_2 = \dfrac{1}{\sqrt{3}}$ and $\cos\theta_3 = \dfrac{1}{\sqrt{3}}$

(b) $\overrightarrow{OP} = -\underline{i} + \underline{j} + \underline{k}$ then direction ratios line OP is $-1:1:1$

$\left|\overrightarrow{OP}\right| = \sqrt{(1^2 + 1^2 + 1^2)} = \sqrt{3}$, $\dfrac{\overrightarrow{OP}}{|OP|} = -\dfrac{-\underline{i} + \underline{j} + \underline{k}}{\sqrt{3}}$, then direction cosine of line OP :

$\cos\theta_1 = \dfrac{-1}{\sqrt{3}}, \cos\theta_2 = \dfrac{1}{\sqrt{3}}$ and $\cos\theta_3 = \dfrac{1}{\sqrt{3}}$

(c) $\overrightarrow{OP} = 2\underline{i} + \underline{j} - \underline{k}$ then direction ratios line OP is $2:1:-1$

$\left|\overrightarrow{OP}\right| = \sqrt{(2^2 + 1^2 + 1^2)} = \sqrt{6}$, $\dfrac{\overrightarrow{OP}}{|OP|} = \dfrac{2\underline{i} + \underline{j} - \underline{k}}{\sqrt{6}}$, then direction cosine of line OP : $\cos\theta_1 = \dfrac{2}{6}, \cos\theta_2 = \dfrac{1}{\sqrt{6}}$

and $\cos\theta_3 = \dfrac{-1}{\sqrt{6}}$

Q4. Determine the angles $(\theta_1, \theta_2, \theta_3)$ for the vectors with the direction cosines:

(a) $\left\{\dfrac{\sqrt{3}}{2}, 0, \dfrac{1}{2}\right\}$, (b) $\left\{\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right\}$, $\left\{\dfrac{1}{3}, -\dfrac{1}{3}, \dfrac{\sqrt{7}}{3}\right\}$

Solution: (a) $\left\{\dfrac{\sqrt{3}}{2}, 0, \dfrac{1}{2}\right\}, \Rightarrow \quad \cos\alpha = \dfrac{\sqrt{3}}{2}, \cos\beta = 0, \quad \cos\gamma = \dfrac{1}{2}$

$\alpha = \pi/6, \beta = \pi/2$ and $\gamma = \pi/3$

(b) $\left\{\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right\} \Rightarrow \alpha = \beta = \gamma = \cos^{-1}\dfrac{1}{\sqrt{3}}$

(c) $\left\{\dfrac{1}{3}, -\dfrac{1}{3}, \dfrac{\sqrt{7}}{3}\right\} \Rightarrow \quad \alpha = \cos^{-1}\dfrac{1}{3}, \quad \beta = \cos^{-1} -\dfrac{1}{3}, \quad \gamma = \cos^{-1}\dfrac{\sqrt{7}}{3}$

Q5. Determine the lengths $|\overrightarrow{AB}|$ of the vectors $\underline{AB}$, given that the end points A and B. Use your results to determine the direction cosines for each of these vectors.

(a) A=(1,1,1), B=(2,O,6)
(b) A=(2,-1,1), B=(-2,2,2)
(c) A=(-1,3,1), B=(-2,-1,0).

Use your results to determine the direction cosines for each of these vectors.

Solution:

$\overrightarrow{AB} = \underline{i} - \underline{j} + 5\underline{k}$ and $\left|\overrightarrow{AB}\right| = \sqrt{1^2 + 1^2 + 5^2} = \sqrt{27}$.

$\cos\alpha = \dfrac{1}{\sqrt{27}}, \cos\beta = \dfrac{-1}{\sqrt{27}}, \cos\gamma = \dfrac{5}{\sqrt{27}},$

(b) A=(2,-1,1), B=(-2,2,2)

$\overrightarrow{AB} = -4\underline{i} + \underline{j} + \underline{k}$ and $\left|\overrightarrow{AB}\right| = \sqrt{4^2 + 1^2 + 1^2} = \sqrt{18}$.

$\cos\alpha = \dfrac{-4}{\sqrt{18}}, \cos\beta = \dfrac{1}{\sqrt{18}}, \cos\gamma = \dfrac{1}{\sqrt{18}},$

(c) A=(-1,3,1), B= (-2,-1,0).

$\overrightarrow{AB} = -\underline{i} - 4\underline{j} - 1\underline{k}$ and $\left|\overrightarrow{AB}\right| = \sqrt{1^2 + 4^2 + 1^2} = \sqrt{18}$.

$\cos\alpha = \frac{-1}{\sqrt{18}}$, $\cos\beta = \frac{4}{\sqrt{18}}$, $\cos\gamma = \frac{-1}{\sqrt{18}}$,

Q(6). Write down the position vectors OP in terms of the unit vectors i, j, k given that 0 is the origin and the points P are: (a) (1,1,1) (b) (2,3,4) (c) (1,2,3)

Solution:

(a) (1,1,1) $\overrightarrow{OP} = \underline{i} + \underline{j} + \underline{k}$, (b) (2,3,4) $\overrightarrow{OP} = 2\underline{i} + 3\underline{j} + 4\underline{k}$, (c) (1,2,3) $\overrightarrow{OP} = \underline{i} + 2\underline{j} + 3\underline{k}$,

Q(7). Determine the values of α, β and γ in order to that:

$$(1-\alpha)\underline{i} + \beta(1-\alpha^2)\underline{j} + (\gamma-2)\underline{k} = \frac{1}{2}\underline{i} + 3\underline{j} + 2\underline{k}$$

Solution:

Equating i, j and k components

$$(1-\alpha) = \frac{1}{2} \Rightarrow \alpha = \frac{1}{2} \quad \beta(1-\alpha^2) = 3 \Rightarrow \beta = 4 \text{ and } (\gamma-2) = 2, \Rightarrow \gamma = 4$$

Q(8). Form the sum $\underline{a} + \underline{b}$ and difference $\underline{a} - \underline{b}$ of the vectors:

(a) $\underline{a} = 3\underline{i} + 3\underline{j} + 2\underline{k}$ $\quad \underline{b} = \underline{i} + \underline{j} + 2\underline{k}$

(b) $\underline{a} = -\underline{i} + 2\underline{j} - 2\underline{k}$ $\quad \underline{b} = \underline{i} - \underline{j} + 2\underline{k}$

(c) $\underline{a} = 3\underline{i} + \underline{j} + 2\underline{k}$ $\quad \underline{b} = \underline{i} + \underline{3j} + 2\underline{k}$

Solution:

(a) $\underline{a} = 3\underline{i} + 3\underline{j} + 2\underline{k}$ $\quad \underline{b} = \underline{i} + \underline{j} + 2\underline{k}$

$\Rightarrow \underline{a} + \underline{b} = 4\underline{i} + 4\underline{j} + 4\underline{k} \Rightarrow \underline{a} - \underline{b} = 2\underline{i} + 2\underline{j}$

(b) $\underline{a} = -\underline{i} + 2\underline{j} - 2\underline{k}$ $\quad \underline{b} = \underline{i} - \underline{j} + 2\underline{k}$

$\Rightarrow \underline{a} + \underline{b} = \underline{j} \Rightarrow \underline{a} - \underline{b} = -2\underline{i} + 3\underline{j} - 4\underline{k}$

(c) $\underline{a} = 3\underline{i} + \underline{j} + 2\underline{k}$ $\quad \underline{b} = \underline{i} + \underline{3j} + 2\underline{k}$

$\Rightarrow \underline{a} + \underline{b} = 4\underline{i} + 4\underline{j} + 4\underline{k} \Rightarrow \underline{a} - \underline{b} = 2\underline{i} + 2\underline{j}$

Q(9). State which of the following pairs of vectors a and b are parallel and which are anti-parallel:

(a) $\underline{a} = \underline{i} - 3\underline{j} + \underline{k}$ $\quad \underline{b} = -4\underline{i} + 12\underline{j} - 4\underline{k}$

(b) $\underline{a} = -2\underline{i} + 3\underline{j} - \underline{k}$ $\quad \underline{b} = 2\underline{i} - 3\underline{j} + \underline{k}$

(c) $\underline{a} = 4\underline{i} - \underline{j} - 3\underline{k}$ $\quad \underline{b} = 3\underline{i} + 2\underline{j} + 3\underline{k}$

Solution:

(a) $\underline{a} = \underline{i} - 3\underline{j} + \underline{k}$ $\quad \underline{b} = -4\underline{i} + 12\underline{j} - 4\underline{k} \Rightarrow \underline{b} = -4\underline{a}$, $\underline{b} // \underline{a}$,

(b) $\underline{a} = -2\underline{i} + 3\underline{j} - \underline{k}$ $\quad \underline{b} = 2\underline{i} - 3\underline{j} + \underline{k} \Rightarrow \underline{b} = -\underline{a}$, $\underline{b} // \underline{a}$,

(c) $\underline{a} = 4\underline{i} - \underline{j} - 3\underline{k}$ $\quad \underline{b} = 3\underline{i} + 2\underline{j} + 3\underline{k} \Rightarrow \underline{b}$ and $\underline{a}$, are anti parallel.

Q(10). Express the following vectors $\underline{a}$ as the product of a scalar and a unit vector:

(a) $\underline{a} = 2\underline{i} - 3\underline{j} + \underline{k}$ (b) $\underline{a} = -2\underline{i} + 3\underline{j} - \underline{k}$ (c) $\underline{a} = 4\underline{i} - \underline{j} - 3\underline{k}$

Solution:

(a) $\underline{a} = 2\underline{i} - 3\underline{j} + \underline{k} \Rightarrow \frac{\underline{a}}{|\underline{a}|} = \frac{2\underline{i} - 3\underline{j} + \underline{k}}{\sqrt{14}}$, $\quad \underline{a} = \sqrt{14}$ times unit vector

(b) $\underline{a} = -2\underline{i} + 3\underline{j} - \underline{k} \Rightarrow \frac{\underline{a}}{|\underline{a}|} = \frac{-2\underline{i} - 3\underline{j} + \underline{k}}{\sqrt{14}}$, $\underline{a} = \sqrt{14}$ times unit vector

(c) $\underline{a} = 4\underline{i} - \underline{j} - 3\underline{k}$ $\Rightarrow \dfrac{\underline{a}}{|\underline{a}|} = \dfrac{4\underline{i} - \underline{j} - 3\underline{k}}{\sqrt{26}}$, $\underline{a} = \sqrt{26}$ times unit vector

Q(11). Find the vectors $\underline{AB}$, and their direction cosines given that A and B have position vectors a and b, respectively, where

(a) $\underline{a} = \underline{i} - 3\underline{j} + 2\underline{k}$ $\underline{b} = -\underline{i} + \underline{j} - 4\underline{k}$

(b) $\underline{a} = -2\underline{i} + 3\underline{j} - \underline{k}$ $\underline{b} = 2\underline{i} - \underline{j} + \underline{k}$

(c) $\underline{a} = 4\underline{i} - \underline{j} + \underline{k}$ $\underline{b} = 3\underline{i} + 2\underline{j} + 3\underline{k}$

Solution:

(a) $\underline{a} = \underline{i} - 3\underline{j} + 2\underline{k}$ $\underline{b} = -\underline{i} + \underline{j} - 4\underline{k}$

$\overrightarrow{AB} = \underline{b} - \underline{a} = -2\underline{i} + 4\underline{j} - 6\underline{k}$ direction cosines of AB is $-2:4:-6$

(b) $\underline{a} = -2\underline{i} + 3\underline{j} - \underline{k}$ $\underline{b} = 2\underline{i} - \underline{j} + \underline{k}$

$\overrightarrow{AB} = \underline{b} - \underline{a} = 4\underline{i} - 4\underline{j} + 2\underline{k}$ direction cosines of AB is $4:-4:2$

(c) $\underline{a} = 4\underline{i} - \underline{j} + \underline{k}$ $\underline{b} = 3\underline{i} + 2\underline{j} + 3\underline{k}$

$\overrightarrow{AB} = \underline{b} - \underline{a} = -\underline{i} + 3\underline{j} + 2\underline{k}$ direction cosines of AB is $-1:3:2$

Q(12). Find the scalar products $\underline{a} \cdot \underline{b}$ and hence find the angle between the vectors $\underline{a}$ and $\underline{b}$ given that:

(a) $\underline{a} = \underline{i} - 3\underline{j} + \underline{k}$ $\underline{b} = -4\underline{i} + 12\underline{j} - 4\underline{k}$

(b) $\underline{a} = -2\underline{i} + 3\underline{j} - \underline{k}$ $\underline{b} = 2\underline{i} - 3\underline{j} + \underline{k}$

(c) $\underline{a} = 4\underline{i} - \underline{j} - 3\underline{k}$ $\underline{b} = 3\underline{i} + 2\underline{j} + 3\underline{k}$

Solution:

(a) $\underline{a} = \underline{i} - 3\underline{j} + \underline{k}$ $\underline{b} = -4\underline{i} + 12\underline{j} - 4\underline{k} \Rightarrow \underline{a} \cdot \underline{b} = -4 - 36 - 4 = -44$

(b) $\underline{a} = -2\underline{i} + 3\underline{j} - \underline{k}$ $\underline{b} = 2\underline{i} - 3\underline{j} + \underline{k} \Rightarrow \underline{a} \cdot \underline{b} = -4 - 9 - 4 = -44$

(c) $\underline{a} = 4\underline{i} - \underline{j} - 3\underline{k}$ $\underline{b} = 3\underline{i} + 2\underline{j} + 3\underline{k} \Rightarrow \underline{a} \cdot \underline{b} = 12 - 2 - 9 = 1$

Q13. Find unit vectors parallel to the vectors a where:

(a) $\underline{a} = \underline{i} - 3\underline{j} + \underline{k}$ (b) $\underline{a} = -2\underline{i} + 3\underline{j} - \underline{k}$ (c) $\underline{a} = 4\underline{i} - \underline{j} - 3\underline{k}$

Solution:

(a) $\underline{a} = \underline{i} - 3\underline{j} + \underline{k}$ Unit vector parallel to $\underline{a} = \underline{i} - 3\underline{j} + \underline{k}$ is $\Rightarrow \dfrac{\underline{a}}{|\underline{a}|} = \dfrac{\underline{i} - 3\underline{j} + \underline{k}}{\sqrt{11}}$,

(b) $\underline{a} = -2\underline{i} + 3\underline{j} - \underline{k}$ Unit vector parallel to $\underline{a} = -2\underline{i} + 3\underline{j} - \underline{k}$ is $\Rightarrow \dfrac{\underline{a}}{|\underline{a}|} = \dfrac{-2\underline{i} + 3\underline{j} - \underline{k}}{\sqrt{14}}$,

(c) $\underline{a} = 4\underline{i} - \underline{j} - 3\underline{k}$ Unit vector parallel to $\underline{a} = 4\underline{i} - \underline{j} - 3\underline{k}$ is $\Rightarrow \dfrac{\underline{a}}{|\underline{a}|} = \dfrac{4\underline{i} - \underline{j} - 3\underline{k}}{\sqrt{26}}$,

Q(14). Evaluate the vector products $\underline{b} x \underline{a}$ given that:

(a) $\underline{a} = -2\underline{i} + 3\underline{j} - \underline{k}$ $\underline{b} = 2\underline{i} - 3\underline{j} + \underline{k}$

(b) $\underline{a} = -\underline{i} + \underline{j} + \underline{k}$ $\underline{b} = 2\underline{i} + 4\underline{j} + 3\underline{k}$

(c) $\underline{a} = -\underline{i} - \underline{j} + \underline{k}$ $\underline{b} = 2\underline{i} + 2\underline{j} + 2\underline{k}$

Solution:

(a) $\underline{a} = -2\underline{i} + 3\underline{j} - \underline{k}$ $\quad \underline{b} = 2\underline{i} - 3\underline{j} + \underline{k} \Rightarrow \underline{b}x\underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -3 & 1 \\ -2 & 3 & -1 \end{vmatrix} = 0$

(b) $\underline{a} = -\underline{i} + \underline{j} + \underline{k}$ $\quad \underline{b} = 2\underline{i} + 4\underline{j} + 3\underline{k} \Rightarrow \underline{b}x\underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 4 & 3 \\ -1 & 1 & 1 \end{vmatrix} \Rightarrow \underline{b}x\underline{a} = \underline{i} - 5\underline{j} - 6\underline{k}$

(c) $\underline{a} = -\underline{i} - \underline{j} + \underline{k}$ $\quad \underline{b} = 2\underline{i} + 2\underline{j} + 2\underline{k} \Rightarrow \underline{b}x\underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 2 & 2 \\ -1 & -1 & 1 \end{vmatrix} \Rightarrow \underline{b}x\underline{a} = 4\underline{i} - 4\underline{j}$

Q(15). Evaluate the triple scalar products $\underline{a} \cdot (\underline{b}x\underline{c})$ and $\underline{b} \cdot (\underline{a}x\underline{c})$ given that:

(a) $\underline{a} = -2\underline{i} + 3\underline{j} - \underline{k}$ $\quad \underline{b} = 2\underline{i} - 3\underline{j} + \underline{k}$ and $\underline{c} = 4\underline{i} - \underline{j} - 3\underline{k}$

Solution:

$$\Rightarrow \underline{a} \cdot (\underline{b}x\underline{c}) = \begin{vmatrix} -2 & 3 & -1 \\ 2 & -3 & 1 \\ 4 & -1 & -3 \end{vmatrix} = 0 \Rightarrow \underline{b} \cdot (\underline{a}x\underline{c}) = \begin{vmatrix} 2 & -3 & 1 \\ -2 & 3 & -1 \\ 4 & -1 & -3 \end{vmatrix} = 0$$

(b) $\underline{a} = \underline{i} - 3\underline{j} + \underline{k}$ $\quad \underline{b} = -4\underline{i} + 12\underline{j} - 4\underline{k}$ and $\underline{c} = 2\underline{i} + 2\underline{j} + 2\underline{k}$

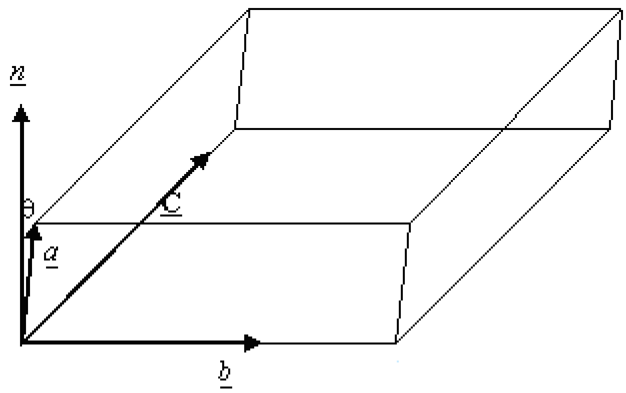
$$\Rightarrow \underline{a} \cdot (\underline{b}x\underline{c}) = \begin{vmatrix} 1 & -3 & 1 \\ -4 & 12 & -4 \\ 2 & 2 & 2 \end{vmatrix} = 0 \Rightarrow \underline{b} \cdot (\underline{a}x\underline{c}) = \begin{vmatrix} -4 & 12 & -4 \\ 1 & -3 & 1 \\ 2 & 2 & 2 \end{vmatrix} = 0$$

Q(16). Prove that if a, b, and c form three edges of a parallelepiped all meeting at a common point, then the volume of this solid figure is given by |a . (b x c)|. Deduce that the vanishing of the triple scalar product implies that the vectors a, b, and c are co-planar (that is, all lie in a common plane).

Solution:

The scalar quantity $\underline{a}.(\underline{b} \times \underline{c})$ is known as the scalar triple product of $\underline{a}$ and $\underline{b}x\underline{c}$. It is often denoted by $[\underline{a}, \underline{b}, \underline{c}]$.

The magnitude of this quantity is the volume of the parallelopied formed by the vectors $\underline{a}, \underline{b},$ and $\underline{c}$, i.e. $|\underline{a}.(\underline{b} \times \underline{c})| = |\underline{a}||\underline{b}x\underline{c}|\cos\theta, \theta$ being the angle between $\underline{a}$, and $\underline{b}x\underline{c}$



Let $\underline{a}, \underline{b}, \underline{c}$ be three given vectors. We can permute the three given vectors in six different ways. Also each manner of writing down the three vectors gives rise to two scalar triple products depending upon the positions of dot and cross. Thus, we have the following twelve scalar triple products

$(\underline{a}\times\underline{b})\cdot\underline{c}$,$(\underline{b}\times\underline{c})\cdot\underline{a}$, $(\underline{c}\times\underline{a})\cdot\underline{b}$ $(\underline{a}\cdot\underline{b})\times\underline{c}$,$(\underline{b}\cdot\underline{c})\times\underline{a}$, $(\underline{c}\cdot\underline{a})\times\underline{b}$

$(\underline{a}\times\underline{c})\cdot\underline{b}$,$(\underline{b}\times\underline{a})\cdot\underline{c}$, $(\underline{c}\times\underline{b})\cdot\underline{a}$, $(\underline{a}\cdot\underline{c})\times\underline{b}$, $(\underline{b}\cdot\underline{a})\times\underline{c}$,$(\underline{c}\cdot\underline{b})\times\underline{a}$

We shall now prove the following two important results

(i). A cyclic permutation of three vectors does not change the value of the scalar triple product and an anti-cyclic permutation changes the value in sign but not in magnitude.

(ii). The positions of dot and cross can be interchanged without any change in the value of the scalar triple product.

Firstly suppose that $\underline{a}, \underline{b}, \underline{c}$ is a right-handed system so have $V=(\underline{a}\times\underline{b})\cdot\underline{c}$.

The vector triads $\underline{b}, \underline{c}, \underline{a}$ and $\underline{b}, \underline{c}, \underline{a}$ are also right-handed and the parallelopiped with OA, OB, OC as adjacent edges is the same as that with OB, OC, OA or with OC, OA, OB as adjacent edges. Thus,

$$V=[\underline{b}, \underline{c}, \underline{a}] \text{ and } V=[\underline{c}, \underline{a}, \underline{b}]$$

1.If $(\underline{a}x\underline{b}).\underline{c}=0$ since $(\underline{a}\times\underline{b})$ is perpendicular to both $\underline{a}$, and $\underline{b}$, then vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ are coplanar.

2. For the nonzero vectors $\underline{a}, \underline{b}$, and $\underline{c}$, they are coplanar (ie lie on the same plane) if and only if $[\underline{a}, \underline{b}, \underline{c}]=0$.

Q(17). If $\underline{A}=2\underline{i}+3\underline{j}-4\underline{k}$, $\underline{B}=3\underline{i}+5\underline{j}+2\underline{k}$ and $\underline{C}=\underline{i}-2\underline{j}+3\underline{k}$ determine $\underline{A}.\underline{B}$, $\underline{A}x\underline{B}$ and $\underline{A}.(\underline{B}x\underline{C})$.

Solution:

$$\underline{A}\cdot\underline{B}=(2\underline{i}+3\underline{j}-4\underline{k})\cdot(3\underline{i}+5\underline{j}+2\underline{k})=13$$

$$\Rightarrow \underline{A}x\underline{B}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 2 & 3 & 4\\ 3 & 5 & 2\end{vmatrix}=-14\underline{i}+8\underline{j}+4\underline{k}$$

$$\Rightarrow \underline{A}.(\underline{B}x\underline{C}) = \begin{vmatrix} 2 & 3 & 4 \\ 3 & 5 & 2 \\ 1 & -2 & 3 \end{vmatrix} = -27$$

Q(18). If $\underline{A} = \underline{i} + 3\underline{j} + 5\underline{k}$, $\underline{B} = 3\underline{i} + \underline{j} + 2\underline{k}$ and $\underline{C} = \underline{i} - \underline{j} + \underline{k}$ find $\underline{A}x(\underline{B}x\underline{C})$ and $(\underline{A}x\underline{B})x\underline{C}$.

Solution:

Use the expansion $\underline{A}x(\underline{B}x\underline{C}) = (\underline{A}\cdot\underline{C})\underline{B} - (\underline{A}\cdot\underline{B})\underline{C}$

$(\underline{A}\cdot\underline{C}) = 3$, $(\underline{A}\cdot\underline{B}) = 16$, $\Rightarrow \underline{A}x(\underline{B}x\underline{C}) = 3(3\underline{i} + \underline{j} + 2\underline{k}) - 16(\underline{i} - \underline{j} + \underline{k}) = (-7\underline{i} + 19\underline{j} - 10\underline{k})$

$(\underline{A}x\underline{B})x\underline{C} = (\underline{A}\cdot\underline{C})\underline{B} - (\underline{B}\cdot\underline{C})\underline{A}$

$\Rightarrow (\underline{A}x\underline{B})x\underline{C} = 3(3\underline{i} + \underline{j} + 2\underline{k}) - 4(\underline{i} + 3\underline{j} + 5\underline{k}) = (5\underline{i} - 9\underline{j} - 14\underline{k})$

Q(19). If $\underline{F} = x^2\underline{i} + x^4\underline{j} + 2x\underline{k}$ then $\frac{d\underline{F}}{dx} = 2x\underline{i} + 4x^3\underline{j} + 2\underline{k}$ and $\frac{d^2\underline{F}}{dx^2} = 2\underline{i} + 12x^2$

$$\left|\frac{d\underline{F}}{dx}\right| = \sqrt{4x^2 + 16x^6 + 4}$$

Q(20). Find the unit normal vector to the surface $\phi = xz^2 + 3xy - 2yz^2 + 1 = 0$ at the point (1,-2,-1)

Solution:

$$\nabla\phi = (z^2 + 3y)\underline{i} + (3x - 2z^2)\underline{j} + (2xz - 4yz)\underline{k}$$
$$(\nabla\phi)_{(1,-2,-1)} = -5\underline{i} + \underline{j} - 10\underline{k}$$
$$\underline{n} = \frac{(\nabla\phi)}{|(\nabla\phi)|} = \frac{-5\underline{i} + \underline{j} - 10\underline{k}}{\sqrt{62}}$$

Q(21). Determine the directional derivative of $\phi = xe^y + yz^2 + xyz$ at the point (2,0,3) in the direction $\underline{A} = 3\underline{i} - 2\underline{j} + \underline{k}$.

Solution:

Directional derivative at a given direction is defined by $\nabla\phi\cdot\underline{n}$

$$\nabla\phi = (e^y + yz)\underline{i} + (xe^y + z^2 + xz)\underline{j} + (2yz + xz)\underline{k}$$

$$(\nabla\phi)_{(2,0,3)} = \underline{i} + 17\underline{j} \Rightarrow \quad (\nabla\phi)\cdot\underline{n} = (\underline{i} + 17\underline{j})\cdot\frac{(3\underline{i} - 2\underline{j} + \underline{k})}{\sqrt{14}} = \underline{\underline{\frac{-31}{\sqrt{14}}}}$$

Q(22). Determine the values of P such that the three vectors $\underline{A}$, $\underline{B}$, and $\underline{C}$ are coplanar, when $\underline{A} = 2\underline{i} + \underline{j} + 4\underline{k}$, $\underline{B} = 3\underline{i} + 2\underline{j} + P\underline{k}$ and $\underline{C} = \underline{i} + 4\underline{j} + 2\underline{k}$.

Solution:

When vectors $\underline{A}$, $\underline{B}$, and $\underline{C}$ are coplanar, then $(\underline{A}\times\underline{B})\cdot\underline{C}=0$

$$(\underline{A}\times\underline{B})\cdot\underline{C}=\begin{vmatrix}2 & 1 & 4\\ 3 & 2 & p\\ 1 & 4 & 2\end{vmatrix}=42-7p=0 \Rightarrow \underline{\underline{p=6}}$$

Q(23). Find Curl ($\underline{F}$) and div($\underline{F}$) for the vector function $\underline{F}$= grad($x^2+y^2+z^2-3xyz$)

Solution:

$$\underline{F}=\nabla\phi=(2x-3yz)\underline{i}+(2y-3xz)\underline{j}+(2z-3xy)\underline{k}$$

$$\text{curl}\underline{F}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ (2x-2yz) & (2y-3xz) & (2z-3xy)\end{vmatrix}=0$$

$$\text{div}\underline{F}=\nabla^2\phi=\frac{\partial}{\partial x}(2x-3yz)+\frac{\partial}{\partial y}(2y-3xz)+\frac{\partial}{\partial z}(2z-3xy)$$

$$=2+2+2=6$$

Q(24). If $\Phi=\frac{x}{r^3}$, show that div(grad(Φ)=0 (or Div(grad) is called $\nabla^2\Phi$ =0)

Solution:

$$\frac{\partial\phi}{\partial x}=\frac{1}{r^3}-\frac{3x^2}{r^5},\quad \frac{\partial\phi}{\partial y}=-\frac{3xy}{r^5},\ \text{and}\quad \frac{\partial\phi}{\partial z}=-\frac{3xz}{r^5}$$

$$\frac{\partial^2\phi}{\partial x^2}=-\frac{3x^2}{r^5}-\frac{6x}{r^5}+\frac{15x^3}{r^7},\quad \frac{\partial^2\phi}{\partial y^2}=-\frac{3x}{r^5}+\frac{15xy^2}{r^7},\quad \frac{\partial^2\phi}{\partial z^2}=-\frac{3x^2}{r^5}+\frac{15xz^2}{r^7},$$

$$\text{Therefore,}\quad \nabla^2\phi=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0$$

Q(25). Show that Div ($\underline{r}$)=3 , Curl($\underline{r}$)=0 , $\text{div}(r^n r)=(n+3)r^n$ $\nabla^2(\frac{1}{r})=0$, curl($r^n\underline{r}$))= 0

Solution:

If $\underline{r}=\underline{x}i+y\underline{i}+z\underline{k}$ then, $\text{div}(\underline{r})=\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z}=3.$ and $\text{curl}\underline{r}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x & y & z\end{vmatrix}=0$

Use the results, $\text{div}(\phi\underline{A})=\phi\text{div}(\underline{A})+\nabla\phi\cdot\underline{A}$ and $\nabla f(r)=\frac{df}{dr}\frac{\underline{r}}{r}$

$$\text{div}(r^n r)=r^n\text{div}\underline{r}+\nabla(r^n)\cdot\underline{r}$$

$$=3r^n+nr^{n-2}\underline{r}\cdot\underline{r}$$

$$=3r^n+nr^n=\underline{\underline{(3+n)r^n}}$$

If $\Phi = \frac{1}{r}$,

$$\frac{\partial\phi}{\partial x} = -\frac{x}{r^3},\quad \frac{\partial\phi}{\partial y} = -\frac{y}{r^3},\ \text{and}\ \frac{\partial\phi}{\partial z} = -\frac{z}{r^3}$$

$$\frac{\partial^2\phi}{\partial x^2} = \frac{3x^2}{r^5} - \frac{1}{r^3},\quad \frac{\partial^2\phi}{\partial y^2} = \frac{3y^2}{r^5} - \frac{1}{r^3},\quad \frac{\partial^2\phi}{\partial z^2} = \frac{3z^2}{r^5} - \frac{1}{r^3}$$

Therefore, $\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$

Again use the results, $\text{curl}(\phi\underline{A}) = \phi\,\text{curl}(\underline{A}) + \nabla\phi\times\underline{A}$ and $\nabla f(r) = \frac{df}{dr}\frac{\underline{r}}{r}$

$$\begin{aligned}\text{curl}(r^n r) &= r^n\text{curl}\,\underline{r} + \nabla(r^n)\times\underline{r}\\ &= 0 + nr^{n-2}\,\underline{r}\times\underline{r}\\ &= 0\end{aligned}$$

Q(26). Show that Curl($\underline{r}\times\underline{a}$)=-2 $\underline{a}$, $\text{div}(\underline{r}\times\underline{a}) = 0$, $\text{grad}(\underline{r}\circ\underline{a}) = \underline{a}$ where $\underline{a}$ is a constant vector.

Solution:

Let $\underline{r} = x\underline{i} + y\underline{i} + z\underline{k}$ and $\underline{a} = a\underline{i} + b\underline{i} + c\underline{k}$ then,

$$\Rightarrow\ \underline{r}x\underline{a} = \begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ x & y & z\\ a & b & c\end{vmatrix} = (yc - zb)\underline{i} + (az - cx)\underline{j} + (bx - ay)\underline{k}$$

Therefore, $\text{curl}(\underline{r}\times\underline{a}) = \begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ yc - zb & az - cx & bx - ay\end{vmatrix} = -2\underline{a}$

$$\Rightarrow\ \text{div}(\underline{r}x\underline{a}) = \frac{\partial}{\partial x}(yc - zb) + \frac{\partial}{\partial y}(az - cx) + \frac{\partial}{\partial z}(bx - ay) = 0$$

$$\Rightarrow\ \underline{r}\cdot\underline{a} = ax + by + cz$$
$$\nabla(\underline{r}\cdot\underline{a}) = \nabla(ax + by + cz) = \underline{a}$$

Q(27). Show that $\text{div}((\underline{r}\times\underline{a})\times\underline{b}) = 0$, $\text{Curl}((\underline{r}\times\underline{a})\times\underline{b}) = 2\underline{b}\times\underline{a}$, where $\underline{a}$ and $\underline{b}$ are constant vectors.

Solution:

Let $\underline{r} = x\underline{i} + y\underline{i} + z\underline{k}$ and $\underline{a} = a\underline{i} + b\underline{i} + c\underline{k}$ then, and $\underline{b} = d\underline{i} + e\underline{i} + f\,\underline{k}$

$$\Rightarrow\ \underline{r}x\underline{a} = \begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ x & y & z\\ a & b & c\end{vmatrix} = (yc - zb)\underline{i} + (az - cx)\underline{j} + (bx - ay)\underline{k}$$

$$\Rightarrow \quad (\underline{r}x\underline{a})\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ yc - zb & az - cx & bx - ay \\ d & e & f \end{vmatrix}$$

$$= [f(az - cx) - e(bx - ay)]\underline{i} + [d(bx - ay) - f(yc - zb)]\underline{j} + [e((yc - zb) - d(az - cx)]\underline{k}$$

$$\Rightarrow \ \text{div}\,[(\underline{r}x\underline{a})\times\underline{b}]$$

$$= \frac{\partial}{\partial x}[f(az - cx) - e(bx - ay)] + \frac{\partial}{\partial y}[d(bx - ay) - f(yc - zb)] + \frac{\partial}{\partial z}[e((yc - zb) - d(az - cx)]$$

$$\text{curl}[(\underline{r}\times\underline{a})\times\underline{b}] = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fac - fcx - ebx + eay & dbx - day - fyc + fzb & eyc - ezb - daz + dcx \end{vmatrix} = 2\underline{b}\times\underline{a}$$

Q(28). (i) Expand $\text{Curl}\{\frac{(\underline{a}\circ\underline{r})}{r^3}\underline{r}\}$

(ii) $\text{div}\{\frac{(\underline{a}\circ\underline{r})}{r^3}\underline{r}\}$ where $\underline{a}$ is a constant vector.

(iii) Show that $\text{Curl}\left(\frac{\underline{a}\times\underline{r}}{r^3}\right) = \frac{\underline{a}}{r^3} + \frac{3\underline{r}}{r^3}(\underline{a}\circ\underline{r})$

Solution:

(i) Let $\underline{r} = x\underline{i} + y\underline{i} + z\underline{k}$ and $\underline{a} = a\underline{i} + b\underline{i} + c\underline{k}$ then $(\underline{a}\circ\underline{r}) = ax + by + cz$

Therefore, $\frac{(\underline{a}\circ\underline{r})}{r^3}\underline{r} = \frac{ax + by + cz}{r^3}\underline{r}$

$$\text{curl}\frac{(\underline{a}\circ\underline{r})}{r^3}\underline{r} = \text{curl}\left(\frac{ax + by + cz}{r^3}\underline{r}\right) = \frac{ax + by + cz}{r^3}\text{curl}\,\underline{r} + \nabla\left(\frac{ax + by + cz}{r^3}\right)\times\underline{r}$$

$$= 0 + \nabla\left(\frac{ax + by + cz}{r^3}\right)\times\underline{r}$$

$$= \left\{(ax + by + cz)\nabla\left(\frac{1}{r^3}\right) + \frac{1}{r^3}\nabla(ax + by + cz)\right\}\times\underline{r}$$

$$= \left\{(ax + by + cz)\frac{(-\underline{r})}{r^5} + \frac{1}{r^3}\underline{a}\right\}\times\underline{r}$$

$$= \frac{\underline{a}\times\underline{r}}{r^3}$$

(ii)

$$\text{div}\frac{(\underline{a}\circ\underline{r})}{r^3}\underline{r} = \text{div}\left(\frac{ax + by + cz}{r^3}\underline{r}\right) = \frac{ax + by + cz}{r^3}\text{div}\,\underline{r} + \nabla\left(\frac{ax + by + cz}{r^3}\right)\cdot\underline{r}$$

$$= \frac{ax + by + cz}{r^3}3 + \nabla\left(\frac{ax + by + cz}{r^3}\right)\cdot\underline{r}$$

$$= \frac{3(ax+by+cz)}{r^3} + \left\{(ax+by+cz)\nabla\left(\frac{1}{r^3}\right) + \frac{1}{r^3}\nabla(ax+by+cz)\right\}\cdot \underline{r}$$

$$= \frac{3(ax+by+cz)}{r^3} + \left\{(ax+by+cz)\frac{(-3\underline{r})}{r^5} + \frac{1}{r^3}\underline{a}\right\}\cdot \underline{r} \qquad a$$

$$= \frac{\underline{a}\cdot\underline{r}}{r^3}$$

(iii) $\Rightarrow \ \underline{a}x\underline{r} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a & b & c \\ x & y & z \end{vmatrix} = (bz-cy)\underline{i} + (cx-az)\underline{j} + (ay-bx)\underline{k}$

Use the results, $\text{curl}(\phi\underline{A}) = \phi\,\text{curl}(\underline{A}) + \nabla\phi\times\underline{A}$ and $\nabla f(r) = \frac{df}{dr}\frac{\underline{r}}{r}$

$$\text{curl}(\underline{a}\times\underline{r}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -(yc-zb) & -(az-cx) & -(bx-ay) \end{vmatrix} = 2\underline{a}$$

$$\text{Curl}\left(\frac{\underline{a}\times\underline{r}}{r^3}\right) = \frac{\text{curl}(\underline{a}\times\underline{r})}{r^3} + \nabla\left(\frac{1}{r^3}\right)\times(\underline{a}\times\underline{r})$$

$$= \frac{2\underline{a}}{r^3} + \nabla\left(\frac{1}{r^3}\right)\times(\underline{a}\times\underline{r})$$

$$= \frac{2\underline{a}}{r^3} + \left(-\frac{3\underline{r}}{r^5}\right)\times(\underline{a}\times\underline{r})$$

$$= \frac{2\underline{a}}{r^3} + \left(-\frac{3}{r^5}\right)\underline{r}\times(\underline{a}\times\underline{r}) = \frac{2\underline{a}}{r^3} + \left(-\frac{3}{r^5}\right)[r^2\underline{a} - (\underline{r}\cdot\underline{a})\underline{r}$$

$$\text{Curl}\left(\frac{\underline{a}\times\underline{r}}{r^3}\right) = -\frac{\underline{a}}{r^3} + \frac{3\underline{r}}{r^3}(\underline{a}\circ\underline{r})$$

Q(29). If $\underline{a}$ and $\underline{b}$ are constant vectors and α is a scalar quantity satisfy a vector equation $\alpha\underline{x} + \underline{a}\times\underline{x} = \underline{b}$, solve the vector equation for $\underline{x}$ for $\begin{cases}\alpha \neq 0 \\ \alpha = 0\end{cases}$

When $\alpha \neq 0$

Solution:

$\alpha\underline{x} + \underline{a}\times\underline{x} = \underline{b}$ (1)

$\underline{a}\times(1) \Rightarrow \alpha\underline{a}\times\underline{x} + \underline{a}\times(\underline{a}\times\underline{x}) = \underline{a}\times\underline{b}$ (2)

$\underline{a}\cdot(1) \Rightarrow \alpha\underline{a}\cdot\underline{x} + \underline{a}\cdot(\underline{a}\times\underline{x}) = \underline{a}\cdot\underline{b}$ (3)

From (1) $\Rightarrow \alpha\underline{a}\times\underline{x} + (\underline{a}\cdot\underline{x})\underline{a} - (\underline{a}\cdot\underline{a})\underline{x} = \underline{a}\times\underline{b}$ (4)

From (3) $\Rightarrow \alpha\underline{a}\cdot\underline{x} = \underline{a}\cdot\underline{b}$ and $\underline{a}\cdot\underline{x} = \frac{\underline{a}\cdot\underline{b}}{\alpha}$ (5)

|From (4) and (5) $\Rightarrow \alpha(\underline{b} - \alpha\underline{x}) + \frac{(\underline{a}\cdot\underline{b})}{\alpha}\underline{a} - (\underline{a}\cdot\underline{a})\underline{x} = \underline{a}\times\underline{b}$

$$\Rightarrow \quad \underline{x} = \frac{\underline{a}\times\underline{b} - \dfrac{(\underline{a}\cdot\underline{b})}{\alpha}\,\underline{a} - \alpha\underline{b}}{\alpha^2 + a^2}$$

When $\alpha = 0$

$\underline{a}\times\underline{x} = \underline{b}$ (1)

$\underline{a}\times(1) \quad \Rightarrow \quad \underline{a}\times(\underline{a}\times\underline{x}) = \underline{a}\times\underline{b}$ (2)

From (1) $\Rightarrow$ $(\underline{a}\cdot\underline{x})\,\underline{a} - (\underline{a}\cdot\underline{a})\,\underline{x} = \underline{a}\times\underline{b}$ (3)

Let $\Rightarrow$ $(\underline{a}\cdot\underline{x}) = t(\text{parameter})$

|From (3) $\Rightarrow$ $\underline{x} = \dfrac{t\,\underline{a} - \underline{a}\times\underline{b}}{a^2}$

$(5,\ 2,\ 0),\ (2, 1, 3)$ and $(4,\ 1,\ -1)$