SYSTEM OF LINEAR EQUATIONS

MODEL QUESTIONS WITH SOLUTIONS

Q(1). Show that there is only one value of k for which the system of equations

$$2x + y - z = 0$$
$$(k-2)x + ky + 2z = 0$$
$$6x + 3y + (k-1)z = 0$$

has non trivial solution. Solve the system of equations for this value of k.

Solution:

Let $\begin{pmatrix} 2 & 1 & -1 \\ k-2 & k & 2 \\ 6 & 3 & k-1 \end{pmatrix}$

$R_3 \to -3R_1 + R_3$ $\begin{pmatrix} 2 & 1 & -1 \\ k-2 & k & 2 \\ 0 & 0 & k+2 \end{pmatrix}$

When $k+2 \neq 0$ r(A)=3

Therefore, System has no solution.
If k =-2

$\begin{pmatrix} 2 & 1 & -1 \\ -4 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ $R_2 \to 2R_1 + R_2$

$\begin{pmatrix} 2 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\therefore r(A) = 1$

$\therefore 2x + y - z = 0$

Let $y = \alpha$, $z = \beta$ (z is Free variables)

$x = \dfrac{\beta - \alpha}{3}$ (x is leading variable)

Q (2) Find the rank of the coefficient matrix and the augmented matrix of the system of the system of equations x + 2y + 3z = 1

2x + y − z = 16

x + 5y + az = b

where a and b are two parameters. Hence find the value of a and b for the system of equations has (1) No solution

(ii) Exactly one solution

(iii) Infinitely many solutions.

Solve the system of equations for which a=8 and b= -3

Solution:

$$(A / b)=\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 1 & -1 & 16 \\ 1 & 5 & a & b \end{array}\right)$$

.

$$\begin{array}{l} R_3 \to -R_1 + R_{.3} \\ R_2 \to -2R_1 + R_2 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -7 & 14 \\ 0 & 3 & a-3 & b-1 \end{array}\right)$$

$$R_3 \to -R_2 + R_3 \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -7 & 14 \\ 0 & 0 & a-10 & b+13 \end{array}\right)$$

When $a \neq 10$ $r(A/b = 3)$, and $r(A) = 3)$

$\therefore$ System is consistent and system has unique solution.

When $a = 10$, $b - 13$, then

$r(A) = 2$, $r(A/b) = 3$

$\therefore$ System is In consistent and system no solution.

When $a = 10$, $b = -13$, then

$r(A) = 2$, $r(A/b) = 2$ $\therefore$ System is consistent and system has many solutions.

If $a = 8$, $b = -3$

Reduce matrix becomes

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -7 & 14 \\ 0 & 0 & 2 & 10 \end{array}\right)$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -7 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{matrix} 1 \\ 14 \\ 10 \end{matrix}$$

$$-2z = 10 \Rightarrow z = -5$$

$$-3y - 7z = +14 \Rightarrow y = 3$$

$$\underline{\underline{x + 2y + 3z = 1 \Rightarrow x = 10}}$$

Q 3. State whether each of the following systems is either "determined", \`underdetermined' or \`overdetermined'. Then reduce the augmented matrix to its echelon form using elementary row transformations (i.e. implement the method of Gaussian elimination). Hence determine which systems are \`consistent' (in which case either state the \`unique solution' if there is one, or classify the set of infinite solutions) or \`not consistent' (there is no solution).

(a) $$\begin{aligned} x - y + 2z &= -2 \\ 3x - 2y + 4z &= -5 \\ 2y - 3z &= 2 \end{aligned}$$

(b) $$\begin{aligned} x - 2y + 2z &= -3 \\ 2x + y - 3z &= 8 \\ -x + 3y + 2z &= -5 \end{aligned}$$

(c) $$\begin{aligned} 2x + 2y + z &= 2 \\ -x + y - 2z &= -5 \\ x - 3y - z &= 4 \end{aligned}$$

(d) $$\begin{aligned} 3x + y + z &= 8 \\ 3x - y + 2z &= 3 \\ x + y + z &= 6 \\ -2x + 2y - 3z &= -7 \end{aligned}$$

(e) $$\begin{aligned} -3w + x - 2y + 13z &= -3 \\ 2w - 3x + y - 8z &= 2 \\ w + 4x + 3y - 9z &= 1 \end{aligned}$$

Solution: (a)

$$\begin{pmatrix}1 & -1 & 2\\ 3 & -2 & 4\\ 0 & 2 & -3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{matrix}2\\ -5\\ 2\end{matrix}$$

System is determined since three are 3 variables and 3 equations
Suppose augment matrix

$$(A/b)\left(\begin{array}{ccc|c}1 & -1 & 2 & -2\\ 3 & -2 & 4 & -5\\ 0 & 2 & -3 & 2\end{array}\right)$$

$$R_3 \to -2R_1 + R_2 \left(\begin{array}{ccc|c}1 & -1 & -2 & -2\\ 0 & 1 & -2 & 1\\ 0 & 2 & -3 & 2\end{array}\right)$$

$$R_3 \to -2R_1 + R_3 \left(\begin{array}{ccc|c}1 & -1 & -2 & -2\\ 0 & 1 & -2 & 1\\ 0 & 0 & 1 & 0\end{array}\right)$$ This is the Echelon from of matrix and

r (A)=3, r(A / b)= 3

$\therefore$ System is consistent and has unique solution.

$$\begin{pmatrix}1 & -1 & 3\\ 0 & 1 & -2\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}-2\\ 1\\ 0\end{pmatrix}$$

$z = 0$

$y - 2z = 1, \quad y = 1$

$x - y + 2z = -2, \quad x = -1$

(b) System is determined

$$(A/b) = \left(\begin{array}{ccc|c}1 & -2 & 2 & -3\\ 2 & 1 & -3 & -8\\ -1 & 3 & -2 & -5\end{array}\right)$$

$$\begin{matrix}R_3 \to -2R_1 + R_2\\ R_3 \to R_1 + R_3\end{matrix} \left(\begin{array}{ccc|c}1 & -2 & 2 & -3\\ 0 & 5 & -7 & 14\\ 0 & 1 & 4 & -8\end{array}\right)$$

$$R_2 \longleftrightarrow R_3 \left(\begin{array}{ccc|c}1 & -2 & 2 & -3\\ 0 & 1 & 4 & -8\\ 0 & 21 & -7 & 14\end{array}\right)$$ This is the Echelon from, r = (A)=3, r(A / b)=3

$\therefore$ System is consistent and system has unique solution

$$\begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & -27 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ -8 \\ 54 \end{pmatrix}$$

$-27z = 54 \Rightarrow z = -2$

$y + 4z = -8 \Rightarrow y = 0$

$x - 2y + 2z = -3 \Rightarrow x = 1$

(c)

$$\begin{pmatrix} 2 & 2 & 1 \\ -1 & 1 & -2 \\ 1 & -3 & -1 \end{pmatrix}\begin{pmatrix} \text{x} \\ \text{y} \\ \text{z} \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix}$$

Now system is determined.
Now define augmented matrix (A / b)

$$(A/b) = \left(\begin{array}{ccc|c} 2 & 2 & 2 & 2 \\ -1 & 1 & -2 & -5 \\ 1 & -3 & -1 & 4 \end{array}\right) R_1 \longleftrightarrow -2R_2$$

$$\left(\begin{array}{ccc|c} 1 & -3 & -1 & 4 \\ -1 & 1 & -2 & -5 \\ 2 & 2 & 1 & 2 \end{array}\right) \begin{array}{l} R_2 \to R_1 + R_2 \\ R_3 \to -2R_1 + R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -3 & -1 & 4 \\ 0 & -2 & -3 & -1 \\ 0 & 8 & 3 & -6 \end{array}\right) R_3 \to 4R_2 + R_3$$

$$\left(\begin{array}{ccc|c} 1 & -3 & -1 & 4 \\ 0 & -2 & -3 & -1 \\ 0 & 0 & -9 & -10 \end{array}\right)$$ This is the Echelon from,r=(A)=3,r(A / b)=3

$\therefore$ System is consistent and system has unique solution

$$\begin{pmatrix} 12 & -3 & -1 \\ 0 & -2 & -3 \\ 0 & 0 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ -10 \end{pmatrix}$$

$$9z = 10, \qquad z = 10/9$$

$$-2y - 3z = -1, \; y \qquad y = -7/6$$

$$x - 3y - z = 4 \qquad x = \frac{29}{18}$$

$$\begin{pmatrix} 3 & 1 & 1 \\ 3 & -1 & 2 \\ 1 & 1 & 1 \\ -2 & 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \\ 6 \\ -7 \end{pmatrix}$$

System is over determined.

$$(A/b) = \left(\begin{array}{ccc|c} 3 & 1 & 1 & 8 \\ 3 & -1 & 2 & 3 \\ 1 & 1 & 1 & 6 \\ -2 & 2 & -3 & -7 \end{array}\right) \quad R_1 \longleftrightarrow -2R_3$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 3 & 1 & 1 & 8` \\ 3 & -1 & 2 & 3 \\ -2 & 2 & -3 & -7 \end{array}\right) \begin{array}{l} R_2 \to 3R_1 + R_2 \\ R_3 \to -3R_1 + R_3 \\ R_4 \to 2R_1 + R_4 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & -2 & -10 \\ 0 & -4 & -1 & -15 \\ 0 & 4 & -1 & 5 \end{array}\right) \begin{array}{l} R_3 \to -2R_2 + R4 \\ R_4 \to 2R_2 + R_4 \end{array}$$

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & -2 & -10 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & -5 & -15 \end{pmatrix} \begin{array}{l} R_3 \to -2R_2 + R4 \\ R_4 \to R_3 + R_4 \end{array}$$

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & -2 & -10 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ This is the Echelon from, r=(A)=3, r(A / b)=3

$\therefore$ System is consistent and system has unique solution

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -10 \\ 5 \end{pmatrix}$$

$3z = 5 \Rightarrow z = 5/3$

$-2y - 2z = -10$

$y + z = 5 \Rightarrow y = 10/3$

$x + y + z = 6 \Rightarrow x = 1$

(e)

$$\begin{pmatrix} -3 & 1 & -2 & 13 \\ 2 & -3 & 1 & -8 \\ 1 & 4 & 3 & -9 \end{pmatrix}\begin{pmatrix} u \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$$

Solution is under determined sine three are 4 variables and have 3 equations.

$$\text{define}\,(A/b) = \left(\begin{array}{cccc|c} -3 & 1 & -2 & 13 & -3 \\ 2 & -3 & 1 & -8 & 2 \\ 1 & 4 & 3 & -9 & 1 \end{array}\right)$$

$$R_1 \longleftrightarrow R_3 \left(\begin{array}{cccc|} 1 & 4 & 3 & -9 \\ 2 & -3 & 1 & -8 \\ -3 & 1 & -2 & -13 \end{array}\right)$$

$$\begin{pmatrix} 1 & 4 & 3 & -9 & 1 \\ 0 & -11 & -5 & 10 & 0 \\ 0 & 13 & 7 & 14 & -10 \end{pmatrix} R_2 \to R_3 + R_2 \begin{pmatrix} 1 & 4 & 3 & -9 & 1 \\ 0 & 2 & 2 & 24 & -10 \\ 0 & 13 & 7 & 14 & -10 \end{pmatrix} R_2 \to 7R_2$$

$$\begin{pmatrix} 1 & 4 & 3 & -9 & 1 \\ 0 & 14 & 14 & 168 & -70 \\ 0 & 13 & 7 & 14 & -10 \end{pmatrix} R_2 \to -R_3 + R_2$$

$$\begin{pmatrix} 1 & 4 & 3 & -9 & 1 \\ 0 & 1 & 7 & 154 & -60 \\ 0 & 13 & 7 & 14 & -10 \end{pmatrix} R_3 \to -13R_2 + R_3$$

$$\begin{pmatrix} 1 & 4 & 3 & -9 & 1 \\ 0 & 1 & 7 & 154 & -60 \\ 0 & 13 & 7 & 14 & -10 \end{pmatrix} R_3 \to -13R_2 + R_3$$

$$\begin{pmatrix} 1 & 4 & 3 & -9 & 1 \\ 0 & 1 & 7 & 154 & -60 \\ 0 & 0 & 84 & -1988 & 770 \end{pmatrix} \quad \text{This is the Echelon form}$$

$r(A) = 3, \quad r(A/b) = 3 \therefore$ System is consistent and has many solutions.

Q(4) Solve the following systems of equations using Gaussian elimination. For what values of b are these systems consistent?

(a) $\begin{array}{l} x - 2y + 2z = -3 \\ 2x + y - 3z = 8 \\ 9x - 3y - 3z = b \end{array}$ (b) $\begin{array}{l} 2x + 3y = 7 \\ x - y = 1 \\ bx + 2y = 8 \end{array}$

Solution: (a)

$$\begin{pmatrix} 1 & -2 & 2 \\ 2 & 1 & -2-3 \\ 9 & -3 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \\ b \end{pmatrix}$$

Define augmented matrix (A / b)

$$\left(\begin{array}{ccc|c} 1 & -2 & 2 & -3 \\ 2 & +1 & -3 & 8 \\ 9 & -3 & -3 & b \end{array}\right) \begin{array}{l} R_2 \to -2R_1 + R_2 \\ R_3 \to -9R_1 + R_3 \end{array}$$

$$\begin{pmatrix} 1 & -2 & 2 & -3 \\ 2 & 5 & -7 & 14 \\ 0 & 15 & -21 & 27+b \end{pmatrix} R_3 \to -3R_2 + R_3$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 2 & -3 \\ 2 & 5 & -7 & +4 \\ 0 & 15 & -21 & b-15 \end{array}\right)$$ This is the Echelon form of matrix

when $b-15 \neq 0 \quad r(A/b)=3,\ r(A)=2$

when $b=15,\ r(A)=2,\ r(A/b)=2$

System is consistent and has many solutions.
5y-7z=14
X-2y+2e=-3

Let z = t (t is a paramater)

$$y=\frac{14+7t}{5}$$

$$x=\frac{13+4t}{5}$$

(b)

$$\begin{pmatrix} 2 & 2 \\ 1 & -1 \\ b & +2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 7 \\ 1 \\ 8 \end{pmatrix}$$

Define $r(A/b)=\left(\begin{array}{cc|c} 2 & 2 & 7 \\ 1 & -1 & 1 \\ b & 2 & 8 \end{array}\right)$

$$R_2 \rightarrow -2R_1+R_2 \quad \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 4 & 5 \\ 0 & b+2 & 8-b \end{array}\right)$$

$$R_2 \rightarrow -\frac{1}{4}R_2 \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 5/4 \\ 0 & b+2 & \frac{11}{2}-13\frac{b}{4} \end{array}\right)$$

This is the Echelon form of matrix and r(A)=2 r(A/b)=3 of

$$\frac{11}{2}-\frac{13b}{4} \neq 0 \Rightarrow 13b-22 \neq 0 \Rightarrow b \neq \frac{22}{13}$$

When 13b-22=0, then r(A)=r(a/b)=2 system is consistent and has unique solution.