

MATRICES

MODEL QUESTIONS WITH SOLUTIONS

1. Given $A = \begin{pmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{pmatrix}$

- (a) Compute that $A+B$ and $A-C$
 (b) Verify that $A+(B+C) = (A+B)+C$
 (c) Compute AB , BA and AC^T

Solution: $A + B = \begin{pmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & 1 & 7 \end{pmatrix}$ and $A - C = \begin{pmatrix} -3 & 1 & -5 \\ 5 & -3 & 0 \\ 0 & 1 & -2 \end{pmatrix}$

$$(A + B) + C = A + (B + C) = \begin{pmatrix} 8 & 2 & 1 \\ 9 & 5 & 9 \\ 4 & -3 & 7 \end{pmatrix}$$

$$AB = \begin{pmatrix} 5 & 3 & 3 \\ 19 & -5 & 16 \\ 1 & -3 & 0 \end{pmatrix} \quad BA = \begin{pmatrix} 0 & 4 & -9 \\ 19 & 3 & -3 \\ 5 & 1 & -3 \end{pmatrix}$$

$$AC^T = \begin{pmatrix} 0 & 0 & 12 \\ 24 & 4 & 11 \\ 5 & -1 & 6 \end{pmatrix}$$

Q(2). Evaluate :

$$\begin{pmatrix} 0 & 1 & 3 \\ 4 & 5 & 6 \\ 7 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 4 & 5 \\ 6 & 3 & 2 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 12 & 12 & 5 \\ 46 & 49 & 36 \\ 29 & 43 & 43 \end{pmatrix}$$

Q(3). If $A = \begin{pmatrix} 1 & 2 \\ 4 & -3 \end{pmatrix}$, show that $A^2 = \begin{pmatrix} 9 & -4 \\ -8 & 17 \end{pmatrix}$ and $A^3 = \begin{pmatrix} -7 & 30 \\ 60 & -67 \end{pmatrix}$

Q(4). If $A = \begin{pmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ 1 & -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 2 & 2 \\ -1 & 0 & -1 \\ -3 & -3 & -2 \end{pmatrix}$

Show that $\{kA + (1-k)B\}^2 = I$, k being a scalar.

Solution: $\{kA + (1-k)B\}^2 = k^2A^2 + (1-k)^2B^2 + k(1-k)AB + k(1-k)BA$

Now compute each matrix as follows;

$$k^2A^2 + (1-k)^2B^2 + k(1-k)AB + k(1-k)BA =$$

$$A^2 = \begin{pmatrix} -3 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & -4 & -3 \end{pmatrix}, \quad B^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad AB = \begin{pmatrix} 5 & 4 & 4 \\ -2 & -1 & -2 \\ 4 & 2 & 3 \end{pmatrix}, \quad BA = \begin{pmatrix} 1 & -4 & -4 \\ 0 & 3 & 2 \\ -2 & 2 & 3 \end{pmatrix}$$

Consider element a_{11} ; of $k^2A^2 + (1-k)^2B^2 + k(1-k)AB + k(1-k)BA$

$$-3k^2 + (1-k)^2 + 5k(1-k) + k(1-k) = 1 - 8k^2 + 4k \equiv 1$$

Therefore $k = 1/2$

When this value $(A^2 + B^2 + AB + BA) \frac{1}{4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Q(5). Find x, y such that $\begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 8 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$$\begin{aligned} 2x - y &= 8 \\ -3x + 4y &= 1 \end{aligned} \quad x = \frac{31}{5}, \quad y = \frac{22}{5}$$

Q(6). Express the matrix $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ -3 & 1 & 4 \end{pmatrix}$ as the sum of symmetric and skew

symmetric matrix.

$$S = \begin{pmatrix} 1 & 2 & 1/2 \\ 2 & 3 & 0 \\ 1/2 & 0 & 4 \end{pmatrix} \text{ and } T = \begin{pmatrix} 0 & 7/2 \\ 0 & 0 & -1 \\ -7/2 & 1 & 0 \end{pmatrix}$$

Exercises 3.5

Q(1). If $A = \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$, show that $A^3 = A^{-1}$.

If $A^3 = A^{-1}$ is satisfied then $AA^3 = AA^{-1}$ or $A^4 = I$

Try

$$A^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ therefore, } A^3 = A^{-1}$$

Q(2). Find the Inverse of (a) $\begin{pmatrix} 3 & 2 & 2 \\ 2 & 5 & 3 \\ 1 & 2 & 1 \end{pmatrix}$,

Use direct method $A^{-1} = \underline{\underline{\begin{pmatrix} 1/3 & -2/3 & 4/3 \\ -1/3 & -1/3 & 5/3 \\ 1/3 & 4/3 & -11/3 \end{pmatrix}}}$

$$(b) A = \begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$$

Use elementary row operations

$$\text{Define } (A|I) = \left(\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & 0 \\ 4 & 2 & 3 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array} \right)$$

$$\begin{array}{l} R_2 \rightarrow -4R_1 + R_2 \\ R_3 \rightarrow -R_1 + R_3 \end{array} \left(\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & 0 \\ 0 & -10 & -25 & -4 & 1 & 0 \\ 0 & -1 & -6 & -1 & 0 & 1 \end{array} \right)$$

$$R_3 \leftrightarrow -R_3 \left(\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & 0 \\ 0 & -10 & -25 & -4 & 1 & 0 \\ 0 & 1 & 6 & 1 & 0 & -1 \end{array} \right)$$

$$R_3 \leftrightarrow R_2 \left(\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & 0 \\ 0 & 1 & 6 & 1 & 0 & -1 \\ 0 & -10 & -25 & -4 & 1 & 0 \end{array} \right)$$

$$\begin{array}{l} R_1 \rightarrow 3R_2 + R_1 \\ R_3 \rightarrow 10R_1 + R_3 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & -11 & -2 & 0 & 3 \\ 0 & 1 & 6 & 1 & 0 & -1 \\ 0 & 0 & 1 & 6/35 & 1/35 & -10/35 \end{array} \right)$$

$$\begin{array}{l} R_1 \rightarrow 11R_3 + R_1 \\ R_2 \rightarrow -6R_3 + R_2 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -4/35 & 11/35 & -5/35 \\ 0 & 1 & 0 & -1/35 & -6/35 & 25/35 \\ 0 & 0 & 1 & 6/35 & 1/35 & -10/35 \end{array} \right)$$

$$\text{Therefore, } A^{-1} = \underline{\underline{\begin{pmatrix} -4/35 & 11/35 & -5/35 \\ -1/35 & -6/35 & 25/35 \\ 6/35 & 1/35 & -10/35 \end{pmatrix}}}$$

Find the inverse of the matrix $\begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$ by elementary row

transformations.

$$\text{Let } A = \begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

$$\text{Define } (A|I) = \left(\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array} \right)$$

$$R_2 \rightarrow -2R_1 + R_2 \quad \left(\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 2 & -2 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array} \right)$$

$$R_1 \leftrightarrow -R_2 \quad \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & -2 & 1 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array} \right)$$

$$R_2 \rightarrow -2R_1 + R_2 \quad \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & -2 & 1 & 0 \\ 0 & -2 & -5 & 5 & -2 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array} \right)$$

$$R_3 \leftrightarrow -R_2 \quad \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & -2 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & -2 & -5 & 5 & -2 & 0 \end{array} \right)$$

$$R_3 \rightarrow 2R_2 + R_3 \quad \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & -2 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{array} \right)$$

$$R_1 \rightarrow -R_2 + R_1 \quad \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & -1 \\ 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{array} \right)$$

$$\begin{array}{l} R_2 \rightarrow -3R_3 + R_2 \\ R_1 \rightarrow R_2 + R_1 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & 1 \\ 0 & 1 & 0 & -15 & 6 & -5 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{array} \right)$$

$$\text{Therefore, } A^{-1} = \underline{\underline{\begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}}}$$

Q(5). If $A = \begin{pmatrix} 1 & -a & 1 \\ b & 0 & 2b \\ 0 & a & 0 \end{pmatrix}$ then, show that

(i) $A^3 = abA + A^2 - abI$, is satisfied

Use method of induction .Multiply the above equation by A

$$A^4 = abA^2 + A^3 - abA \text{ and eliminate } A^3$$

$$\text{It gives, } A^4 - abA^2 - A^2 + abI = 0$$

Therefore, the given statement is true hen $n = 4$.

Assume that it is also true upto $n = p$ (even) number

$$A^p - abA^{p-2} - A^2 + abI = 0$$

$$\text{Multiply this relation with } A^2, \text{ we get, } A^{p+2} - abA^p - A^4 + abA^2 = 0$$

Now eliminate A^4

$$A^{p+2} - abA^p - A^2 + abI = 0$$

Therefore it I true for $n = p+2$.

Hence by Methodical induction the given expression is true for every integer values of n

(ii) Show also that $A^{2n} = \left\{ \frac{(ab)^n - 1}{ab - 1} \right\} A^2 - \left\{ \frac{ab \{(ab)^{n-1} - 1\}}{ab - 1} \right\} I$, where n is a positive integer.

$$\text{When } n = 2, \quad A^4 = \left\{ \frac{(ab)^2 - 1}{ab - 1} \right\} A^2 - \left\{ \frac{ab \{(ab) - 1\}}{ab - 1} \right\} I$$

$A^4 = (ab + 1)A^2 - abI$ which satisfied the above expression.

Assume that the statement is true upto $n = p$

$$A^{2p} = \left\{ \frac{(ab)^p - 1}{ab - 1} \right\} A^2 - \left\{ \frac{ab \{(ab)^{p-1} - 1\}}{ab - 1} \right\} I \quad (A)$$

Multiply (A) with A^2

$$A^{2(p+1)} = \left\{ \frac{(ab)^p - 1}{ab - 1} \right\} A^4 - \left\{ \frac{ab \{(ab)^{p-1} - 1\}}{ab - 1} \right\} A^2$$

Eliminate A^4 and get,

$$A^{2(p+1)} = \left\{ \frac{(ab)^p - 1}{ab - 1} \right\} ((ab + 1)A^2 - abI) - \left\{ \frac{ab \{(ab)^{p-1} - 1\}}{ab - 1} \right\} A^2$$

$$A^{2(p+1)} = \left\{ \frac{((ab)^p - 1)(ab + 1) - ab \{(ab)^{p-1} - 1\}}{ab - 1} \right\} A^2 - \left\{ \frac{ab \{(ab)^p - 1\}}{ab - 1} \right\} I$$

$$A^{2(p+1)} = \left\{ \frac{(ab)^{p+1} - 1}{ab - 1} \right\} A^2 - \left\{ \frac{ab \{(ab)^p - 1\}}{ab - 1} \right\} I$$

Therefore, the above statement is true for $n = p+1$, hence by method of induction it is true for every positive integers n,

$$\text{i.e, } \underline{\underline{A^{2n} = \left\{ \frac{(ab)^n - 1}{ab - 1} \right\} A^2 - \left\{ \frac{ab \{(ab)^{n-1} - 1\}}{ab - 1} \right\} I}}$$

6.(a) Show that every 2×2 matrix such that $X^T A X = B$, where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ has one of the forms } \begin{pmatrix} a & \frac{1}{2a} \\ a & \frac{1}{-2a} \end{pmatrix} \text{ or } \begin{pmatrix} a & \frac{1}{2a} \\ -a & \frac{1}{2a} \end{pmatrix}$$

Solution:

Let form of matrix X is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Then

$$X^T A X = B, \Rightarrow \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a^2 - c^2 & ab - cd \\ ab - cd & b^2 - d^2 \end{pmatrix}$$

Equating respective elements,

$$a^2 - c^2 = 0$$

$$b^2 - d^2 = 0$$

$$ac - bd = 1$$

If $a = t$ then $c^2 = t^2$, and $c = \pm t$

Suppose $b = k$ $b^2 = d^2 \Rightarrow d = \pm k$ and $tk - (-tk) = 2tk = 1$

$$\Rightarrow t = \frac{1}{2k}$$

Therefore form of matrix is $\begin{pmatrix} t & \frac{1}{2t} \\ t & \frac{1}{-2t} \end{pmatrix}$ or $\begin{pmatrix} t & \frac{1}{2t} \\ -t & \frac{1}{2t} \end{pmatrix}$

(b) If $P = QRQ^{-1}$, show that $P^n = QR^nQ^{-1}$ where n is a positive integer.

$$\text{Try } P^2 = (QRQ^{-1})(QRQ^{-1}) = QR^2Q^{-1}$$

$$\text{Similarly } P^3 = (QR^2Q^{-1})(QRQ^{-1}) = QR^3Q^{-1}$$

In general $P^n = QR^nQ^{-1}$

(c) Let $P = \begin{pmatrix} 5 & -2 \\ 7 & -4 \end{pmatrix}$ and $Q = \begin{pmatrix} 2 & 1 \\ 7 & 1 \end{pmatrix}$, show that $Q^{-1}PQ = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$. Hence

find P^n .

$$\text{When } Q = \begin{pmatrix} 2 & 1 \\ 7 & 1 \end{pmatrix}, \Rightarrow Q^{-1} = -\frac{1}{5} \begin{pmatrix} 1 & -1 \\ -7 & 2 \end{pmatrix}$$

$$Q^{-1}PQ = \frac{1}{5} \begin{pmatrix} 1 & -1 \\ -7 & 2 \end{pmatrix} \begin{pmatrix} 5 & -2 \\ 7 & -4 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 7 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$$

Hence $P = QRQ^{-1}$, and $P^n = QR^nQ^{-1}$

$$P^n = QR^nQ^{-1} = \frac{1}{5} \begin{pmatrix} 2 & 1 \\ 7 & 1 \end{pmatrix} \begin{pmatrix} (-2)^n & 0 \\ 0 & 3^n \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -7 & 2 \end{pmatrix} = \begin{pmatrix} 2(-2)^n - 7 \times 3^n & 0(-2)^{n+1} + 2 \times 3^n \\ 7(-2)^n - 7 \times 3^n & -7(-2)^n + 2 \times 3^n \end{pmatrix}$$