UNIVERSITY OF MORATUWA

MSC/POSTGRADUATE DIPLOMA IN OPERATIONAL RESEARCH

POR 504 NUMERICAL METHODS

THREE HOURS **AUGUST 2007**

Answer **FIVE** questions and **NO MORE**.

Question 1:

Briefly describe the meaning of each of the following terms, in relation to numerical algorithms:

(a) *Round off error*

(b) *Truncation error*

(c) *Propagated error*

Write down the *10* th degree *Taylor polynomial* that approximates the function,

$f(x) = \log(1+x)$, near $x = x_0 = 0$ and the corresponding *Remainder Term*.

Using the above *10* th degree Taylor polynomial, calculate $\log\left(\frac{12}{11}\right)$ approximately.

Clearly indicate the *type* and *magnitude* of the various types of errors that are introduced while performing the above calculations.

Estimate the total error in your final answer and show that the *actual error* is much less than the *estimated error*. Explain why it is so.

Question 2:

Briefly describe various methods that are available, under following categories, to solve a system of linear equations:

(a) *Matrix Methods*

(b) *Elimination Methods*

(c) *Iterative Methods*

Solve the system of linear equations

$$\begin{aligned} 5x_1 + x_2 - 2x_3 - x_4 &= 10 \\ 7x_1 + x_2 - 3x_3 - 2x_4 &= 10 \\ x_1 - 9x_2 + 2x_3 + 5x_4 &= 10 \\ 4x_1 + 3x_2 + x_3 + 9x_4 &= 10 \end{aligned}$$

independently by methods belonging to **any two** of the above three categories.

Question 3:

With usual notations, prove that if $y = ax + b$ is the regression line that fits the data set $D = \{(x_i, y_i) \mid x_i, y_i \in \Re, i = 1,2,3,\cdots,n\}$, then the coefficients a and b satisfy the equations:

$$a + (\bar{x})\, b = \bar{y}$$

$$(\bar{x})\, a + \left(\overline{x^2}\right) b = \left(\overline{xy}\right).$$

Write down the corresponding result for the regression curve of degree **2** that fits the same data set.

The following table shows the failure rate y of a certain machine component at various temperatures x (in degrees).

x	**51.73**	**59.73**	**62.04**	**66.04**	**72.12**	**79.80**	**86.85**
y	**0.138**	**0.286**	**0.385**	**0.459**	**0.550**	**0.690**	**0.782**

Obtain the regression line that fits these data and hence estimate

(a) The failure rate at 100 degrees.

(b) The temperature beyond which the component would not work at all.

What would be the answers for **(a) and (b)** above, if we use a quadratic model instead of a linear one for the relationship between x and y.

Question 4:

Given a sequence $\{y_k \mid k = 0,1,2,\cdots n\}$, define the forward differences: Δy_k, $\Delta^2 y_k$, $\ldots \Delta^{n-1} y_k$.
With usual notations prove that

(a) $\Delta^p y_0 = \sum_{i=0}^{p} (-1)^i \binom{p}{i} y_{p-i}$

(b) $y_k = \sum_{i=0}^{k} \binom{k}{i} \Delta^i y_0$

Deduce that if $\{x_k = x_0 + kh \mid k = 0,1,2,\cdots n\}$, is another sequence with constant spacing h, then the unique n^{th} degree polynomial $p(x)$ that takes the value y_k at x_k $\forall\ k = 0, 1, \cdots, n$, is given by

$$p(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2h^2}(x - x_0)(x - x_1) + \cdots + \frac{\Delta^n y_0}{n!\,h^n}(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

The following table gives the probability $y(x)$ that a normally distributed random variable X with mean 0 and variance 1 lies between 0 and x.

x	0.0	0.5	1.0	1.5	2.0	2.5	3.0
y	0.0	0.1915	0.3413	0.4332	0.4772	0.4938	0.4987

Obtain a polynomial of least degree that fits this data and hence estimate the probability that X lies between 0 and $\frac{4}{3}$.

Question 5:

A function f is five times continuously differentiable, and $\left|f^{(5)}(x)\right| < M$ on $[a,b]$. f takes the values y_{-2}, y_{-1}, y_0, y_1 and y_2 respectively at the points $a-2h, a-h, a, a+h$ and $a+2h$.

Prove that $f'(a) \approx \dfrac{y_{-2} - 8y_{-1} + 8y_1 - y_2}{12h}$

Obtain similar expressions for $f'(a-2h)$ and $f'(a+2h)$.

The population growth of a certain kind of biological organism was investigated in a laboratory and the following results were obtained:

T	0.000	1.000	2.000	3.000	4.000	5.000	6.000	7.000
P	1.234	1.364	1.507	1.666	1.841	2.035	2.248	2.485

Here, T denotes the time (day) and P denotes the population (units/volume). Obtain the rate of growth $\frac{dP}{dT}$ of the population P at times $T = 0:7$. Indicate any special relationship that you observe between $\frac{dP}{dT}$ and P. Hence, obtain an exact expression for P as a function of T.

Question 6:

Derive the Trapezoidal Rule for the integration of a function f over an interval $[a,b]$.

Show that the truncation error in this method is $-\frac{(b-a)h^2}{12} f''(\xi)$, where h is the length of a smaller sub interval and ξ is some point in (a,b).

(a) The following table indicates the readings of a speedometer of a moving train at various times.

Time (min)	0	1	2	3	4	5	6
Speed (kms/min)	0.00	0.42	0.76	1.03	1.58	1.78	1.82

Use Trapezoidal Rule to get an estimate of the distance traveled by the train during these six minutes. What is the accuracy of your answer. Justify.

(b) Use Simpson's Rule to evaluate the integral $\int_0^{\pi} e^{e^x} dx$ correct to three decimal places.

Clearly show how you determine the sub interval length h to ensure the required accuracy.

(You may assume that the truncation error in Simpson's method is $-\frac{(b-a)h^4}{180} f^{(4)}(\xi)$)

Question 7:

Second order Runge-Kutta Method to solve the initial value problem, $\dfrac{dy}{dx} = f(x,y)$,

$y(x_0) = y_0$ is given by the scheme: $y(x_0 + h) \approx y_1 = \frac{1}{2}(k_1 + k_2)$, where $k_1 = hf(x_0, y_0)$,

$k_2 = hf(x_0 + h, y_0 + k_1)$. By expanding the one-variable function y and the two variable function f as Taylor series in h, keeping x_0 and y_0 fixed, show that the truncation error in the scheme is of the form

$$y(x_0 + h) - y_1 = \frac{h^3}{12}\left(ff_x + 2ff_{xy} + f^2 f_{yy} - 2f_x f_y - 2ff_y^2\right) + O\left(h^4\right)$$

(Here f and its partial derivatives denote the values evaluated at (x_0, y_0).

Use ***RK2*** method <u>iteratively</u> to solve the IVP: $\dfrac{dy}{dx} = \cos(x + y)$, $y(0) = 0$ to find $y(1)$ correct to three decimal places. Clearly indicate how you choose your step length to ensure the required accuracy.

Solve the same IVP using Taylor series method, without subdividing the interval $[0,1]$.

Clearly indicate how do you choose the degree of the Taylor polynomial to ensure the required accuracy.

Question 8

Suppose $u = u(x,t)$ is a solution to the PDE $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2}$ for $x \in [0, L]$ and $t > 0$.

Let, $D = \{(x_i, y_j) \mid x_0 < x_1 < \cdots < x_m,\ y_0 < y_1 < \cdots < y_m\}$ be a set of grid points with a horizontal spacing h and vertical spacing k. Let $U_{i,j}$ denote the value of U at (x_i, y_j), $\forall i, j$. Show that $U_{i,j}$ satisfies the difference equation

$U_{i,j} = \alpha U_{i-1,j-1} + (1 - 2\alpha)U_{i,j-1} + \alpha U_{i+1,j-1}$, where $\alpha = \dfrac{ka}{h^2}$. Show also that the local truncation error due to this discretization is $\left(\dfrac{ak(6ak - h^2)}{12}\right)u_{xxxx} + o(h^4)$, where subscripts denote partial differentiation at (x_i, y_j).

A thin uniform metallic rod of length π and heat constant $a = 0.1$, has an initial temperature distribution given by $u(x,0) = 50(\sin(x) + \sin(3x))$, $x \in [0, \pi]$. The temperatures at the two end points of the rod are kept fixed for all $t \geq 0$. Write down the initial boundary value problem(IBVP) satisfied by the temperature distribution $u(x,t)$ on this rod. Solve this IBVP numerically to find the temperature of the mid point of the rod at times $t = 1,2,3,4,5,6$. (Choose grid spacing suitably to optimize accuracy).

What is the maximum temperature reached at the mid point?.