TOPOLOGY

Definition:
$\mathbb{N}_n = \{1,2,\cdots,n\}$
$\mathbb{N}$: set of natural numbers
$\mathbb{Z}$: set of integers
$\mathbb{Q}$: set of rational numbers
$\mathbb{R}$: set of real numbers
$\mathbb{Q}^c$: set of irrational numbers
$\mathbb{C}$: set of complex numbers

Theorem:
There is a rational number and an irrational number between any two real numbers

Definition: **open ball** $B_r(a)$ of **radius** r and **center** a
$B_r(a) = \{x||x-a| < r\}$

Definition: boundary points ∂A of A
$p \in \partial A \Leftrightarrow \forall r\ \exists q,s \in B_r(p) \text{s.t.}\, q \in A \text{ and } s \in A^c$

Definition: interior points $A^{\circ} = \text{int}A$ of A
$p \in A^{\circ} \Leftrightarrow \exists r \text{ s.t.}\, B_r(p) \subset A$

Definition: closure points $\overline{A}$ of A
$p \in \overline{A} \Leftrightarrow \forall r\ \exists q \in B_r(p) \text{s.t.}\, q \in A$

Definition: limit points A' of A
$p \in A' \Leftrightarrow \forall r\ \exists q \in B_r(p) \text{s.t.}\, q \in A \text{ and } q \neq p$

Theorem:
1. $\partial\mathbb{Q} = \mathbb{R}$
2. $\mathbb{Q}^{\circ} = \emptyset$
3. $\overline{\mathbb{Q}} = \mathbb{R}$
4. $\mathbb{Q}' = \mathbb{R}$

Theorem:
1. $A^{\circ} \subseteq A \subseteq A' \subseteq \overline{A}$
2. $\partial A \subseteq \overline{A}$
3. $\overline{A} = A^{\circ} \cup \partial A$
4. $\overline{A} = A \cup A'$

Definition: dense subset B of A
$\overline{B} = A$

Theorem:
Irrational numbers are dense in real numbers

Definition: open set A
$A^{\circ} = A$

Definition: closed set A
$\overline{A} = A$

Theorem:
1) union of open sets is open
2) intersection of closed sets is closed

Example:
1. $\bigcap_{k=1}^{\infty}\left(2-\frac{1}{k},3+\frac{1}{k}\right)=[2,3]$
2. $\bigcup_{k=1}^{\infty}\left[2+\frac{1}{k},3-\frac{1}{k}\right]=(2,3)$

SET THEORY

Definition: set A and B and equivalent $A{\sim}B$
$\exists$ a bijective function $f: A \to B$

Theorem:
$\sim$ is an equivalence relation

Definition: finite set A
$A \sim \mathbb{N}_n$

Definition: countably infinite set A
$A \sim \mathbb{N}$

Definition: countable set A
A is finite or countably infinite

Theorem: The following are equivalent
1) A is countable
2) $\exists$ a one to one function $f: A \to \mathbb{N}$
3) $\exists$ a onto function $f: \mathbb{N} \to A$

Definition: uncountable set A
A is not countable

Theorem:
1) a subset of a countable set is countable
2) a superset of an uncountable set is uncountable

Theorem:
1. $\mathbb{Z}$ is countable
2. $\mathbb{Q}$ is countable (Cantor)
3. $\mathbb{R}$ is uncountable (Cantor)

Definition: Cardinality $\mathrm{card}A=|A|=n(A)=\#A$ of the set A
1. $\mathrm{card}\mathbb{N}_n=n$
2. $\mathrm{card}\mathbb{N}=\mathcal{N}_0$
3. $\mathrm{card}A=\mathrm{card}B$ iff $A{\sim}B$
4. $\mathrm{card}A>\mathrm{card}B$ iff $\exists$ a onto function $f: A \to B$

Theorem:
1. $\mathrm{card}\mathcal{P}(A)=2^{\mathrm{card}\,A}$
2. $\mathrm{card}\mathbb{R}=2^{\mathcal{N}_0}=\mathcal{N}_1=\mathfrak{c}$

Hypothesis: continuum hypothesis
There is no set S with $\mathcal{N}_0 < cardS < \mathcal{N}_1$

Definition: upper bound U of set A
$\forall x \in A, x \leq U$

Definition: A is **bounded above**
$\exists$ an upper bound U of A

Definition: lower bound L of set A
$\forall x \in A, L \leq x$

Definition: A is **bounded below**
$\exists$ a lower bound L of A

Definition: bounded set A
A bound above and below.

Definition: supremum of A
$\sup A = \max\{U|U \text{ is an upper bound } of\ A\}$

Definition: infemum of A
$\inf A = \inf\{L|L \text{ is a lower bound of } A\}$

Definition:
1. $A + x = \{a + x|a \in A\}$
2. $A + B = \{a + b|a \in A \text{ and } b \in B\}$

Theorem:
1. $\sup(A + B) = \sup \mathrm{A} + \sup\,(B)$
2. $\inf(A + B) = \inf(A) + \inf\,(B)$

Axiom: completeness axiom
1. every non empty subset of real numbers which is bounded above has a sepremum.
2. every non empty subset of real numbers which is bounded below has a infemum.

Definition: open cover $\{I_k\}_k$ of open sets I_k of A
$A \subseteq \bigcup_k I_k$

Definition: length of an interval $I = (a, b)$
$l(I) = b - a$

Definition: (Lebesgue) measure of A
$m(A) = \inf\{\sum_k l(I_k)|\ \{I_k\}_k \text{is an open cover of } A\ \}$

Theorem:
1. $m(A \cup B) \leq m(A) + m(B)$
2. $m(A + x) = m(A)$

Definition: measurable set $A \subseteq X$
$\forall B \subseteq X, m(B) = m(B \cup A) + m(B \cap A^c)$

Theorem:

1. an interval $I = (a, b)$ is measurable and $m(I) = l(I)$
2. $\emptyset$ is measurable and $m(\emptyset) = 0$
3. A is measurable $\Rightarrow m(A \cup B) = m(A) + m(B)$
4. $m(\mathbb{Q}) = 0$

Definition: Cantor set

Let $C_0 = [0,1]$.

$C_1 = C_0 - \left(\frac{1}{3}, \frac{2}{3}\right) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$ set obtained by removing the middle $\frac{1}{3}$ of C_0

$C_2 = C_1 - \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right) = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$ set obtained by removing the middle $\frac{1}{3}$ of closed sets of C_1

and so on

Cantor set $\mathcal{C} = \bigcap_{n=1}^{\infty} C_n$

Theorem: Cantor set $\mathcal{C}$

1. only contains numbers with digits 0 or 2 in base 3.
2. $\frac{1}{4} \in \mathcal{C}$
3. is Uncountable
4. has measure zero

Definition: diameter of a set

$\text{diam}(A) = \sup\{|x - y| : x, y \in A\}$

Definition: $H_\alpha(A) = \inf\{\sum_{n=1}^{\infty} (\text{diam } A)^\alpha : A \subseteq \bigcup_{n=1}^{\infty} A_n\}$

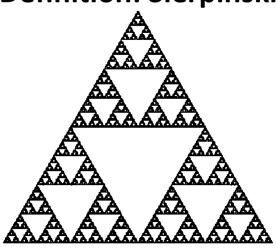
Definition: Hausdorff dimension

$\dim_H(A) = \inf\{\alpha > 0 : H_\alpha(A) = 0\} = \sup\{\alpha \geq 0 : H_\alpha(A) > 0\}$

Theorem:

1. $\dim_H([0,1]) = 1$
2. A is countable $\Rightarrow \dim_H(A) = 0$
3. $\dim_H(\mathcal{C}) = \frac{\log 2}{\log 3} = \log_3 2 \approx 0.6309 \ldots$

Definition: Sierpinski triangle $\mathcal{S}$



Definition: Mandelbrot set $\mathcal{M}$

$\mathcal{M} = \{c | f^n(0) \nrightarrow 0 \; as \; n \to \infty\}$

where $f(z) = z^2 + c$ and $f^n(z) = f(f^{n-1}(z))$ with $f^0(z) = f(z)$

Theorem:

1. $\dim_H(\mathcal{S}) = \frac{\log 3}{\log 2} = \log_2 3 \approx 1.584 \ldots$
2. $\dim_H(\partial\mathcal{M}) = 2$

GRAPH THEORY

Definitions:
graph $G = (V, E)$
V is the set of **vertices**
E is the set of **edges**

Definitions: If $u, v \in V$ are joined by $e \in E$ in a graph $G = (V, E)$ then
we say edges u, v are **adjacent**
if there is no direction of the edge e we write $e = \{u, v\}$ and say G an **undirected** graph
if there is a direction of e from u to v we write $e = (u, v)$ and say G an **directed** graph
say G a **weighted graph** iff $\exists$ a function $W: E \to \mathbb{R}$ and the weight of G is $\sum_{e \in E} W(e)$

Definition: incidence(adjacency) matrix $M(G)$ of an undirected, unweighted graph G
$M(G) = (m_{ij})$ where m_{ij} =no of edges joining v_i to v_j

Definition: degree of a vertex
$\deg(v)$ =no of edges connected to v

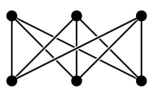
Theorem:
1) number of odd degree vertices are even
2) number of even degree vertices are odd

Definition:
1) **path** is a sequence of vertices and edges connected to them.
2) **cycle** is a closed path.
3) graph is **connected** iff there is a path between any two vertices.
4) graph is **simple** iff it contains no loops or multiple edges. .

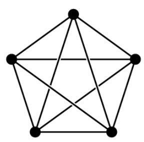
Definition: complete graph K_n
is a simple undirected graph with every pair of vertex is adjacent

Definition: bipartite graph $K_{m,n}$
is a simple undirected graph such that the set of vertices can be decomposed into two disjoint sets of size m, n and every vertex in one set is adjacent to every vertex on the other set.

Example:



$K_{3,3}$



K_5

Definition:
a graph is **planar** if it can be drown on the plane such that no edges crossing.

Theorem: Kuratowski
a graph is planar iff it does not contain K_5 and $K_{3,3}$ as subgraphs

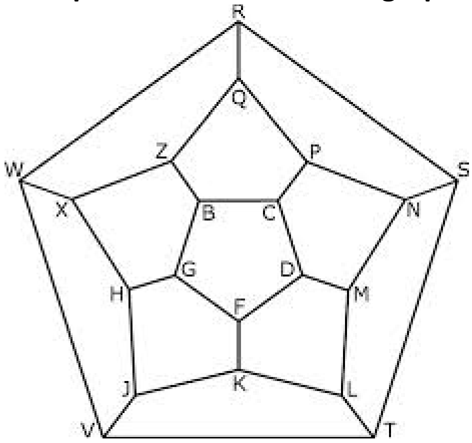
Definition: chromatic number of G
$\chi(G) =$ the smallest number of colors needed to color the vertices of G so that no two adjascent vertices share the same color.

Theorem: five color theorem
$\chi(G) \leq 5$

Conjecture: four color conjecture
$\chi(G) \leq 4$

Definition:
1) **Hamiltonian cycle** is cycle containing each vertex exactly once.
2) a graph is **Hamiltonian** iff it contains a Hamiltonian cycle.

Example: the **dodecahedron graph** is Hamiltonian



Theorem: $G = (V, E)$
1) $\text{card}V \geq 3$
2) every vertex has degree $\geq \text{card}V/2$

then G is Hamiltonian.

Definition: Travelling Salesman Problem on a weighted graph G
is to find the Hamiltonian cycle with the least weight.

Definition:
1) **Eular cycle** is cycle containing each edge exactly once.
2) a graph is **Eularian** iff it contains an Euler cycle.

Theorem: $G = (V, E)$
1) G is connected
2) each vertex has even degree

Iff G is Eularian

Definition: Travelling Postman Problem on a weighted graph G
is to find a cycle with least weight containing every edge.

Definition: Tree
is a connected graph with no cycles.

Definition:
1) **spanning tree** is a tree containing all vertices.
2) **minimal spanning tree** is a spanning tree with minimal weight.

Algorithm: Dijkstra: to find the shortest path

$DIJKSTRA(G, w, s)$

INITIALIZE-SINGLE-SOURCE (G, s)
$S \leftarrow \emptyset$
$Q \leftarrow V[G]$
while $Q \neq \emptyset$
    do $u \leftarrow \mathrm{Extract} - \mathrm{Min}(Q)$
        $S \leftarrow S \cup \{u\}$
        for each $v \in Adj[u]$
            do $RELAX(u, v, w)$

Algirithm:Prim: to find the minimal spanning tree

$MST - PRIM(G, w, r)$

for each $u \in V(G)$
    do $key[u] \leftarrow \infty$
        $\pi[u] \leftarrow NIL$
$key[r] \leftarrow 0$
$Q \leftarrow V(G)$
while $Q \neq \emptyset$
    do $u \leftarrow \mathrm{Extract} - \mathrm{Min}(Q)$
        for each $v \in Adj[u]$
            do if $v \in Q$ and $w(u, v) < key[v]$
                then $\pi[v] \leftarrow u$
                    $key[v] \leftarrow w(u, v)$

Algorithm: Breath-First Search: To find the vertices of G reachable from s

$BFS(G, s)$

for each vertex $u \in V(G) - \{s\}$
    do $color[u] \leftarrow WHITE$
        $d[u] \leftarrow \infty$
        $\pi[u] \leftarrow NIL$
$color[s] \leftarrow GRAY$
$d[s] \leftarrow 0$
$\pi[s] \leftarrow NIL$
$Q \leftarrow \emptyset$
$ENQUEUE(Q, s)$
while $Q \neq \emptyset$
    do $u \leftarrow \mathrm{DEQUEUE}(Q)$
        for each $v \in Adj[u]$
            do if $color[v] = WHITE$
                then $color[v] \leftarrow GRAY$
                    $d[v] \leftarrow d[u] + 1$
                    $\pi[v] \leftarrow u$
                    $ENQUEUE(Q, v)$
        $color[u] \leftarrow BLACK$

References: *Introduction to Algotithms*, Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein.