RIEMANN INTEGRAL

Example: Use n equal partitions of $[0,1]$ to estimate the "area" under the curve $f(x) = x^2$ using
1. left corner of the intervals
2. right corner of the intervals
3. midpoint of the interval
4. line joining the left and right corners of the interval

Definitions:
P is a **partition** of $[a,b]$ iff it is a ordered set of the form $P = (a = x_0, x_1, \cdots, x_n = b)$
P^* is a **refinement** of P iff $P^* \supseteq P$
P is a **common refinement** of P_1, P_2 iff $P = P_1 \cup P_2$
$\mathcal{P}[a,b]$ is the set of all partitions of $[a,b]$

Definition: Upper and Lower Reimann Sums $f: [a,b] \to \mathbb{R}$ is a bounded function
$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$ where $M_i = \sup\{f(x)\,|x_{i-1} \le x \le x_i\}$
$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$ where $m_i = \inf\{f(x)\,|x_{i-1} \le x \le x_i\}$

Definition: **Upper and Lower Reimann Integrals**
$\overline{\int_a^b f(x)dx} = \inf\{U(P,f)|P \in \mathcal{P}[a,b]\}$
$\underline{\int_a^b f(x)dx} = \sup\{L(P,f)|P \in \mathcal{P}[a,b]\}$

Definition:
f is **Reimann Integrable** on $[a,b]$ or $f \in \Re[a,b]$ iff $\underline{\int_a^b f(x)dx} = \overline{\int_a^b f(x)dx}$
Reimann Integral of f is the common value denoted by $\int_a^b f(x)dx$

Theorem: P^* is a refinement of P
1. $L(P,f) \le L(P^*,f)$
2. $U(P^*,f) \le U(P,f)$

Theorem: $\underline{\int_a^b f(x)dx} \le \overline{\int_a^b f(x)dx}$

Theorem: $f \in \mathcal{R}[a,b]$ iff $\forall \varepsilon > 0\ \exists P \in \mathcal{P}[a,b];\ U(P,f) - L(P,f) < \varepsilon$

Theorem: If $f \in \mathcal{R}[a,b]$ and $P \in \mathcal{P}[a,b]$ such that $t_i \in [x_{i-1}, x_i]$ then
$\left|\sum_{i=1}^{n} f(t_i)\Delta x_i - \int_a^b f(x)dx\right| < U(P,f) - L(P,f)$

Theorem: $f \in \mathcal{C}[a,b] \Rightarrow f \in \mathcal{R}[a,b]$

Theorems: $f, g \in \mathcal{R}[a,b]$
1. $f + g \in \mathcal{R}[a,b]$ and $\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
2. $fg \in \mathcal{R}[a,b]$
3. $|f| \in \mathcal{R}[a,b]$ and $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$
4. $f \le g \Rightarrow \int_a^b f(x)dx \le \int_a^b g(x)dx$
5. $f \le M \Rightarrow \int_a^b f(x)dx \le M(b-a)$
6. $c \in [a,b] \Rightarrow f \in \mathcal{R}[a,c], f \in \mathcal{R}[c,b]$ and $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Definition: $f(x)$ is **Uniformly continuous** on $I \subset \mathbb{R}$
$\forall \varepsilon > 0, \exists \delta > 0, \forall x_1, x_2 \in I;\ |x_1 - x_2| < \delta \Longrightarrow |f(x_1) - f(x_2)| < \varepsilon$

Definition: $f(x)$ is **Lipschitz continuous** on $I \subset \mathbb{R}$
$\exists L > 0, \forall x_1, x_2 \in I;\ |f(x_1) - f(x_2)| \le L|x_1 - x_2|$

Theorem: Lipschitz continuous ⇒ Uniformly continuous ⇒ Continuous

Example: Show that $\frac{1}{x}$ is not uniformly continuous on $(0,1]$ but x^2 is.

Theorem: Fundamental Theorem of Calculus
If $f \in \mathcal{R}[a,b]$ and there is a differentiable function F such that $F' = f$ then
$\int_a^b f(x)dx = F(b) - F(a)$

Theorem: Second Fundamental Theorem of Calculus
If $f \in \mathcal{R}[a,b]$ and $x \in [a,b]$ and $F(x) = \int_a^x f(x)dx$ then

1. F is continuous on $[a,b]$.
2. If f is continuous at a point $x_0 \in [a,b]$ then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Theorem: Integration by Parts
F, G differentiable on $[a,b]$, $F' = f \in \mathcal{R}[a,b]$and $G' = f \in \mathcal{R}[a,b]$ *then*
$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx$

Theorem: Change of Variable
g has continuous derivative g' *on* $[c,d]$. f is continous on $g([c,d])$and let $F(x) = \int_{g(c)}^x f(t)dt, x \in g([c,d])$. Then for each $x \in [c,d]$, $\int_c^x f(g(t))g'(t)dt$ exixts and has value $F(g(x))$.

Theorem: Mean Value Theorem for Integrals
$f \in \mathcal{R}[a,b]$with $m \le f \le M$. Then $\exists\, c \in [m,M]$ such that $\int_a^b f(x)dx = c(b-a)$.
If also $f \in \mathcal{C}[a,b]$ then $\exists\, x_0 \in (a,b)$ such that $\int_a^b f(x)dx = f(x_0)(b-a)$.

Definition: Improper Integrals of the first kind
Suppose $\int_a^b f(x)dx$ exixts for each $b \ge a$.
If $\lim_{b\to\infty} \int_a^b f(x)dx$ exists and equal to $I \in \mathbb{R}$ we say that $\int_a^\infty f(x)dx$ converges and has value I
Otherwise we say that $\int_a^\infty f(x)dx$ diverges

Definition: Improper Integrals
$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx, f: [a,\infty) \to \mathbb{R}$
$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx, f: (-\infty, b] \to \mathbb{R}$
$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx,\ f: (-\infty,\infty) \to \mathbb{R}, c \in \mathbb{R}$
$\int_{a^+}^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx, f: (a,b] \to \mathbb{R}$
$\int_a^{b^-} f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx\,, f: [a,b) \to \mathbb{R}$
$\int_a^b f(x)dx = \int_a^{c^-} f(x)dx + \int_{c^+}^b f(x)dx, f: [a,c) \cup (c,b] \to \mathbb{R}, c \in (a,b)$

Example: Find $\int_{-1}^1 \frac{1}{x^2}dx$ if it exists

Example:
Prove that if f is bounded above and increasing, then $\lim_{x\to\infty} f(x)$ is existing and finite
Prove that $\int_a^\infty |f(x)|dx$ converges $\Longrightarrow \int_a^\infty f(x)dx$ converges
Prove that if $|f(x)| \le Me^{ax}$, then the **Laplace Transform** of $f(x), \overline{f}(s) = \int_0^\infty e^{-sx} f(x)dx$ exists for all $s > a$.

Theorem: Comparison Test
Assume that the proper integral $\int_a^b f(x)dx$ exists for each $b \ge a$ and suppose that $0 \le f(x) \le g(x)$ for all $x \ge a$, then $\int_a^\infty g(x)dx$ converges $\Longrightarrow \int_a^\infty f(x)dx$ converges

Theorem: Limit Comparison Test
Assume both proper integrals $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ exist for each $b \ge a$, where $f(x) \ge 0$ and $g(x) > 0$
If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \mathrm{c}$, then

1. $c \neq 0, \infty \Rightarrow \int_a^\infty f(x)dx$ converges $\Leftrightarrow \int_a^\infty g(x)dx$ converges
2. $c = 0$ and $\int_a^\infty g(x)dx$ converges $\Rightarrow \int_a^\infty f(x)dx$ converges
3. $c = \infty$ and $\int_a^\infty g(x)dx$ diverges $\Rightarrow \int_a^\infty f(x)dx$ diverges

Note: There are similar comparison tests for other improper integrals

Example: **Gamma Function** is defined by $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$. Show that

1. $\Gamma(x)$ exists for all $x > 0$
2. $\Gamma(x) = (x-1)\Gamma(x-1)$
3. $\Gamma(n) = (n-1)!$ for integer $n \geq 1$
4. we can use 2. to define $\Gamma(x)$ for $x < 0$
5. $\Gamma(x)$ does not exist for $x = 0, -1, -2, -3, \ldots.$
6. Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and $\int_0^\infty e^{-t^2}dt = \sqrt{\pi}$ using $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$
7. Use the formula for the the n dimesional ball $V_n(r) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} r^n$ to find volumes of 2,3,4,5 dimesional balls
8. Use the fact that $\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x$ asmptotically as $t \to \infty$ to find 10! approximately
9. What is $-\Gamma'(1)$?. It is called the Euler Constant γ and no one knows if it is rational or irrational!
10. Prove that the **Beta function** $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$ exists for all $x, y > 0$. It can be shown that $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

MULTIVARIATE CALCULUS

Definition: **Function of two variables** $f: A \subseteq \mathbb{R}^2 \to \mathbb{R}$

Example: Draw the graphs of the following functions/surfaces

1. $f(x,y) = x^2 + y^2$
2. $f(x,y) = \sqrt{x^2 + y^2}$
3. $\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 1$

Definition: **Limit**

$\lim_{(x,y)\to(a,b)} f(x,y) = L \Leftrightarrow$
$\forall \varepsilon > 0 \exists \delta > 0, 0 < d\big((x,y),(a,b)\big) < \delta \Rightarrow |f(x,y) - L| < \varepsilon$

Note: **Matric**

$0 < d\big((x,y),(a,b)\big) < \delta$ is a region around and excluding (a,b). Some options for the matric d are

1. $\sqrt{(x-a)^2 + (y-b)^2}$
2. $|x-a| + |y-b|$
3. $\max\{|x-a|, |y-b|\}$

We will use the first matric. One can show that they are equivalent, what is needed is a region around (a,b).
Example: Use the definition to show that $\lim_{(x,y)\to(1,2)} x^2y = 6$

Example: Investigate the existence of the limit, $\lim_{(x,y)\to(0,0)} f(x,y)$ for the following functions

1. $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases}$
2. $f(x,y) = \begin{cases} \frac{x^2y^2}{x^2y^2+(x-y)^2} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases}$
3. $f(x,y) = \begin{cases} x\sin\frac{1}{y} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases}$

Theorem:

If $\lim_{(x,y)\to(a,b)} f(x,y) = L$, $\lim_{x\to a} f(x,y)$ and $\lim_{y\to b} f(x,y)$ exists then
$\lim_{x\to a}\lim_{y\to b} f(x,y) = \lim_{y\to b}\lim_{x\to a} f(x,y) = L$.

Example:

Use the above theorem to prove that $\lim_{(x,y)\to(0,0)} f(x,y)$ is not existing for $f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases}$.

Prove by definition that if $\lim_{(x,y)\to(0,0)} f(x,y)$ along $y = x$ and $y = 2x$ are different, then the limit is not existing.

Definition: **Continuity** of f $(f \in \mathcal{C})$ at (a,b)

$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Definition: **Partial derivatives**

$$f_x(a,b) = f_1(a,b) = \frac{\partial f}{\partial x}(a,b) = \lim_{x\to a} \frac{f(x,b)-f(a,b)}{x-a} = \lim_{\Delta x\to 0} \frac{f(a+\Delta x,b)-f(a,b)}{\Delta x}$$
$$f_y(a,b) = f_2(a,b) = \frac{\partial f}{\partial y}(a,b) = \lim_{y\to b} \frac{f(a,y)-f(a,b)}{y-b} = \lim_{\Delta y\to 0} \frac{f(a,b+\Delta y)-f(a,b)}{\Delta y}$$

Definition: $f \in \mathcal{C}^1 \Leftrightarrow f_x \in \mathcal{C}$ and $f_y \in \mathcal{C}$

Theorem: **Mean Value**

1. $f \in \mathcal{C}^1$ in A
2. The disk $(x-a)^2 + (y-b)^2 \leq \delta^2$ is inside A
3. $\Delta x^2 + \Delta y^2 \leq \delta^2$

   Then

1. $f(a+\Delta x, b+\Delta y) - f(a,b) = \Delta x f_x(a+\theta\Delta x, b) + \Delta y f_y(a+\Delta x, b+\alpha\Delta y)$
2. $0 < \theta, \alpha < 1$

Definition: **Differentiability** of f $(f \in \mathcal{D})$ at (a,b)

1. f_x and f_y exists at (a,b)
2. $f(a+\Delta x, b+\Delta y) - f(a,b) = \Delta x f_x(a,b) + \Delta y f_y(a,b) + \Delta x\phi(\Delta x,\Delta y) + \Delta y\psi(\Delta x,\Delta y)$ for some $\Delta x^2 + \Delta y^2 \leq \delta^2$
3. $\lim_{(\Delta x,\Delta y)\to(0,0)} \phi(\Delta x,\Delta y) = \lim\limits_{(\Delta x,\Delta y)\to(0,0)} \psi(\Delta x,\Delta y) = 0$

Theorem: $f \in \mathcal{C}^1 \Rightarrow f \in \mathcal{D} \Rightarrow f \in \mathcal{C}$

Example: Let $f(x,y) = g\left(\sqrt{x^2+y^2}\right), g(x) = x\sin\frac{1}{x}, g(0) = 0$. Show that $f \in \mathcal{D}$ but $f \notin \mathcal{C}^1$

Definition: Higher order derivatives

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$
$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\partial x}$$
$$f_{yx} = \left(f_y\right)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\partial y}$$
$$f_{yy} = \left(f_y\right)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$ and so on

Note:

1. We write $f \in \mathcal{C}^2$ to mean $f_{xx}, f_{xy}, f_{yx}, f_{yy} \in \mathcal{C}$
2. In a similar manner we write $f \in \mathcal{C}^n$ to mean that all the n th order partial derivatives are continuous. There are 2^n of them.
3. There are $\binom{n}{m} = {}^n C_m = \frac{n!}{m!(n-m)!}$, n th order partial derivatives that contains x, m times.

Example: Let

$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases}.$$

Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

Theorem: $f \in \mathcal{C}^2 \Rightarrow f_{xy} = f_{yx}$

Theorem: Chain rule

1. $f = f(x,y), y = y(t), x = x(t)$ all in C^1 . Then
   $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$
2. $f = f(x,y), y = y(u,v), x = x(u,v)$ all in C^1 . Then
   $\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$ and $\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$

Note: The above may be written as

$$\frac{\partial f}{\partial t} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \frac{\partial f}{\partial(x,y)}\frac{\partial(x,y)}{\partial t} \text{ and } \frac{\partial f}{\partial(u,v)} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{\partial f}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)}$$

The determinant , $\det\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ is called the **Jacobian** or J

With $\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\underline{u} = \begin{pmatrix} u \\ v \end{pmatrix}$, the above may also be written as
$(f \circ \underline{x})'(t) = (f' \circ \underline{x})(t)\underline{x}'(t)$ and $(f \circ \underline{x})'(\underline{u}) = (f' \circ \underline{x})(\underline{u})\underline{x}'(\underline{u})$

We also see that $\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} = \frac{\partial f}{\partial(x,y)} = f'(\underline{x})$ is acting as the true first derivative of $f = f(x,y)$. Therefore it is also called $\nabla f = \text{grad} f$ or the **Gradient** of f.

Example: If $u = u(x,y) \in C^2$ then prove that the **Laplace operator** $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ becomes
$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$ when $x = r\cos\theta, y = r\sin\theta$.

Example: Assume all functions are C^1
Show that if $x = x(u,v), y = y(u,v), u = u(r,s), v = v(r,s)$ then $\frac{\partial(x,y)}{\partial(r,s)} = \frac{\partial(x,y)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(r,s)}$.
Show that if $u = f(x,y), v = g(x,y)$ then a functional relation of the form $h(u,v) = 0$ exists iff det $\frac{\partial(u,v)}{\partial(x,y)} \equiv 0$.

Definition: **Directional Derivative** of f in the direction of the unit vector $\underline{u} = (u,v)$ at (a,b).
$D_{\underline{u}} f(a,b) = \lim_{\Delta t \to 0} \frac{f(a+u\Delta t, b+v\Delta t) - f(a,b)}{\Delta t}$

Theorem: $f \in C^1, \nabla f(a,b) \neq \underline{0}$

1. $D_{\underline{u}} f(a,b) = \frac{\partial f}{\partial x}(a,b)u + \frac{\partial f}{\partial y}(a,b)v = \nabla f(a,b) \cdot \underline{u}$
2. $\max_{\underline{u}} D_{\underline{u}} f(a,b) = D_{\widehat{\nabla f(a,b)}} f(a,b) = \|\nabla f(a,b)\|$
3. $\min_{\underline{u}} D_{\underline{u}} f(a,b) = D_{-\widehat{\nabla f(a,b)}} f(a,b) = -\|\nabla f(a,b)\|$

Theorem: **Normal vector** to a surface at (a,b)
$\underline{n}(a,b) = (f_x(a,b), f_x(a,b), -1) = (\nabla f(a,b), -1)$

Proof: Let $\underline{r} = \underline{r}(t) = (x(t), y(t), z(t)) \in C^1$ be a curve on the surface of $z = f(x,y) \in C^1$
and $\underline{r}(t_0) = (x(t_0), y(t_0), z(t_0)) = (a, b, f(a,b))$.
Note that $\underline{r}'(t_0) = (x'(t_0), y'(t_0), z'(t_0))$ is the tangent vector to the curve at (a,b).
Now $\underline{n}(a,b) \cdot \underline{r}'(t_0) = f_x(a,b)x'(t_0) + f_y(a,b)y'(t_0) - z'(t_0) = \frac{df}{dt}(t_0) - z'(t_0) = 0$
Ie $\underline{n}(a,b) = (f_x(a,b), f_x(a,b), -1) = (\nabla f(a,b), -1)$ is a vector perpendicular to the surface $z = f(x,y)$ at (a,b).

Theorem: Equation of the **tangent plane** to the surface $z = f(x,y) \in C^1$ at (a,b).

$$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) = \nabla f(a,b)\begin{pmatrix} x-a \\ y-b \end{pmatrix} = \nabla f(a,b)\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix}\right)$$

Example: Let $f(x,y) = x^4 + y^4 - x^2 - y^2 + 1$. At the point (1,2) find

1. Direction in which the function increases most rapidly
2. Directional derivative in that direction
3. Equation of the tangent plane.

Theorem: **Taylor's expansion** for one variable $f: I \in \mathbb{R} \to \mathbb{R}$
If $f \in C^{n+1}$ and $a, a+h \in I$
then $f(a+h) = \sum_{m=0}^{n} \frac{1}{k!} \frac{d^m f}{dx^m}(a) h^m + \frac{1}{(n+1)!} \frac{d^{n+1} f}{dx^{n+1}}(c) h^{n+1}$
where c is between a and $a+h$.

Note: We can also write the above as
If $f \in C^{n+1}$ and $a + th \in I$ for all $t \in [0,1]$
Then $f(a+h) = \sum_{m=0}^{n} \frac{1}{k!} \left(h \frac{d}{dx}\right)^m f(a) + \frac{1}{(n+1)!} \left(h \frac{d}{dx}\right)^{n+1} f(c)$
for some $c = a + \theta h$ with $\theta \in (0,1)$.
We agree to use the notation $\left(h \frac{d}{dx}\right)^m f(a) \equiv h^m \frac{d^m f}{dx^m}(a)$

Note: The first two terms are the equation of the tangent line.

Proof: Use generalized mean value theorem on
$F(t) = \sum_{m=0}^{n} \frac{1}{m!} f^{(m)}(t)(x-t)^m$ and $G(t) = (x-t)^{n+1}$

Example: When $n = 1$
$f(a+h) = f(a) + \frac{1}{1!} f'(a) h + \frac{1}{2!} f''(c) h^2$

Example: Write the Taylor's expansion for $f(x) = e^x$ at $a = 0$.

Example: Derive the second derivative test to find the extrema of $f(x)$. What to do when $f''(a) = 0$?

Theorem: Taylor's for two variables $f: A \subset \mathbb{R}^2 \to \mathbb{R}$
$f \in C^{n+1}$ and $(a+th, b+tk) \in A$ for all $t \in [0,1]$
Then $f(a+h, b+k) = \sum_{m=0}^{n} \frac{1}{k!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial x}\right)^m f(a,b) + \frac{1}{(m+1)!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial x}\right)^{m+1} f(\boldsymbol{c})$
for some $\boldsymbol{c} = (a + \theta h, b + \theta k)$ with $\theta \in (0,1)$.

Proof: Use Taylor'r expansion for $F(t) = f(a+th, b+tk)$

Example: When $n = 1$

$$
\begin{aligned}
& f(a+h, b+k) \\
&= \sum_{m=0}^{1} \frac{1}{k!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial x}\right)^m f(a,b) + \frac{1}{(1+1)!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial x}\right)^{1+1} f(a+\theta h, b+\theta k) \\
&= f(a,b) + \frac{1}{1!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial x}\right) f(a,b) + \frac{1}{2!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial x}\right)^2 f(\boldsymbol{c}) \\
&= f(a,b) + \frac{1}{1!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial x}\right) f(a,b) + \frac{1}{2!} \left(h^2 \frac{\partial^2}{\partial x^2} + 2hk \frac{\partial^2}{\partial xy} + h^2 \frac{\partial^2}{\partial x^2}\right) f(\boldsymbol{c}) \\
&= f(a,b) + f_x(a,b) h + f_y(a,b) k + \frac{1}{2!} \left(f_{xx}(\boldsymbol{c}) h^2 + 2 f_{xy}(\boldsymbol{c}) hk + f_{yy}(\boldsymbol{c}) k^2\right) \\
&= f(a,b) + \begin{pmatrix} f_x(a,b) & f_y(a,b) \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} + \frac{1}{2!} \begin{pmatrix} h & k \end{pmatrix} \begin{pmatrix} f_{xx}(\boldsymbol{c}) & f_{xy}(\boldsymbol{c}) \\ f_{yx}(\boldsymbol{c}) & f_{yy}(\boldsymbol{c}) \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} \\
&= f(a,b) + \nabla f(a,b) \begin{pmatrix} h \\ k \end{pmatrix} + \frac{1}{2!} \begin{pmatrix} h & k \end{pmatrix} Hf(\boldsymbol{c}) \begin{pmatrix} h \\ k \end{pmatrix} \\
&= f(a,b) + \frac{1}{1!} f'(a,b) \begin{pmatrix} h \\ k \end{pmatrix} + \frac{1}{2!} \begin{pmatrix} h & k \end{pmatrix} f''(\boldsymbol{c}) \begin{pmatrix} h \\ k \end{pmatrix}
\end{aligned}
$$

Note: The first two terms are the equation of the tangent plane.

Definition: $f'' = Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$: **Hessian** of f
$\det Hf = f_{xx} f_{yy} - {f_{xy}}^2$: determinant
$\operatorname{tr} Hf = f_{xx} + f_{yy}$: **trace**

Note: $\det Hf > 0$ and $f_{xx} > 0 (< 0) \Rightarrow f_{yy} > 0 (< 0) \Rightarrow \operatorname{tr} Hf > 0 (< 0)$

Example:
Write the Taylor's expansion for $f(x,y) = e^{xy}$ and $f(x,y) = \sin(\sin x + xe^y)$ at $(a,b) = (0,0)$.
Get the same answer by applying multiple one variable Taylor series expansions at 0.

Definition: (a,b) is a **critical point** of $f \in C^1 \Leftrightarrow \nabla f(a,b) = \mathbf{0}$ or f is not defined

Definition: Let $f \in C^1$.

1. f has a **relative maximum** at $(a,b) \Leftrightarrow f$ is below its tangent plane at (a,b) in a neighbourhood of (a,b)
2. f has a **relative minimum** at $(a,b) \Leftrightarrow f$ is above its tangent plane at (a,b) in a neighbourhood of (a,b)
3. f has a **saddle point** at $(a,b) \Leftrightarrow f$ is both above and below its tangent plane at (a,b) in a neighbourhood of (a,b).

Theorem: $f \in C^1$and (a,b) is a relative maximum/minimum (relative extrema) of $f \Rightarrow \nabla f(a,b) = \mathbf{0}$

Theorem: $f \in C^2$ and $\nabla f(a,b) = \mathbf{0}$ then

1. $\det Hf(a,b) > 0$ and $\operatorname{tr} Hf(a,b) > 0$ then (a,b) is a relative mimimum
2. $\det Hf(a,b) > 0$ and $\operatorname{tr} Hf(a,b) < 0$ then (a,b) is a relative maximum
3. $\det Hf(a,b) < 0$ then (a,b) is a saddle point
4. $\det Hf(a,b) = 0$ inconclusive(why?)

Example: Find the critical points and determine the nature of them (relative maxima/minima/saddle points).
$f(x,y) = x^3 - 12x + y^3 - 27y + 5$
$f(x,y) = x^4 + y^4 - x^2 - y^2 + 1$
$f(x,y) = x^4 + y^4$

Example: Propose a method to determine the nature of critical points when $\det Hf = 0$.

Theorem: Lagrange Multipliers
If $f, g \in C^1$ and $g_x^2 + g_y^2 > 0$ then the maxima/minima of $f(x,y)$ subjected to $g(x,y) = 0$ are included in the set of solutions of $\nabla f(x,y) = \lambda \nabla g(x,y)$ and $g(x,y) = 0$.

Example:
Find the shortest distance from the point $(1,0)$ to the parabola $y^2 = 4x$.
Find the directions of the axes of the ellipse $5x^2 - 6xy + 5y^2 - 4x - 4y - 4 = 0$.
Find the absolute maximum/minimum of $f(x,y) = x^4 + y^4 - x^2 - y^2 + 1$ on the closed disk $(x-1)^2 + y^2 \leq 4$.