Q1. Show that the eigenvalues of a Hermitian matrix are real.

Solution

$Ax = \lambda x$
$\Rightarrow (Ax)^H = (\lambda x)^H$
$\Rightarrow (\overline{Ax})^T = (\overline{\lambda x})^T$
$\Rightarrow \bar{x}^T \bar{A}^T = \bar{\lambda}\bar{x}^T$
$\Rightarrow \bar{x}^T A = \bar{\lambda}\bar{x}^T$ since A is Hermetian: $A = A^H = \bar{A}^T$
$\Rightarrow \bar{x}^T Ax = \bar{\lambda}\bar{x}^T x$
$\Rightarrow \bar{x}^T \lambda x = \bar{\lambda}\bar{x}^T x$
$\Rightarrow \lambda \|x\|^2 = \bar{\lambda}\|x\|^2$
$\Rightarrow \lambda = \bar{\lambda} \in \mathbb{R}$ since $x \neq \underline{0}$

Q2. Let $x =$ (sum of the digits of your index number)Mod 5. Select matrix number x.
Find the characteristic polynomial, minimal polynomial, eigenvalues, spectral radius and eigenvectors.

$$\begin{pmatrix} -11 & -10 & 5 \\ 5 & 4 & -5 \\ -20 & -20 & 4 \end{pmatrix}, \begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Solution:

Let $A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$,

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix}$$
$$= (1-\lambda)(-(5+\lambda)(4-\lambda) + 18) + 3(3(4-\lambda) - 18) + 3(-18 + 6(5+\lambda))$$
$$= (1-\lambda)(\lambda^2 + \lambda - 2) + 9(-2-\lambda) + 18(2+\lambda)$$
$$= -\lambda^3 + 3\lambda - 2 + 9\lambda + 18$$
$$= -\lambda^3 + 12\lambda + 16$$

So the Characteristic Polynomial is

$$p(\lambda) = \lambda^3 - 12\lambda - 16$$

Checking the factors of 16: $\pm 1, \pm 2, \pm 4, \pm 8, \pm 16$ hoping for rational solutions:

$$p(-2) = -8 + 24 - 16 = 0$$

Therefore
$p(\lambda) = (\lambda+2)(\lambda^2 - 2\lambda - 8) = (\lambda+2)(\lambda+2)(\lambda-4) = (\lambda+2)^2(\lambda-4)$
$p(\lambda) = 0 \Leftrightarrow \lambda = -2, 4$ are the Eigenvalues

Spectral Radius $\rho(A) = \max\{|-2|, |4|\} = 4$

$$(A+2I)(A-4I) = \left(\begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\right)\left(\begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}\right)$$
$$= \begin{pmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{pmatrix}\begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix} = 3\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 2 & -2 & 2 \end{pmatrix} 3\begin{pmatrix} -1 & -1 & 1 \\ 1 & -3 & 1 \\ 2 & -2 & 0 \end{pmatrix} = 9\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O$$

Therefore the Minimal Polynomial is $m(\lambda) = (\lambda+2)(\lambda-4) = \lambda^2 - 2\lambda - 8$

$$(A+2I)x = \begin{pmatrix} 1+2 & -3 & 3 \\ 3 & -5+2 & 3 \\ 6 & -6 & 4+2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{matrix} -R_1+R_2 \to R_2 \\ \Rightarrow \\ -2R_1+R_3 \to R_3 \end{matrix} \begin{pmatrix} 3 & -3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} R_1/3 \to R_1 \\ \Rightarrow \\ \end{matrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow a - b + c = 0 \Rightarrow b = a + c$

Therefore

$$x = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ a+c \\ c \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \in span\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$$

So the Eigenvactors corresponding to $\lambda = -2$ are $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$$(A-4I)x = \begin{pmatrix} 1-4 & -3 & 3 \\ 3 & -5-4 & 3 \\ 6 & -6 & 4-4 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{matrix} R_1+R_2 \to R_2 \\ \Rightarrow \\ 2R_1+R_3 \to R_3 \end{matrix} \begin{pmatrix} -3 & -3 & 3 \\ 0 & -12 & 6 \\ 0 & -12 & 6 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} -R_2+R_3 \to R_3 \\ \Rightarrow \\ -4R_1+R_2 \to R_1 \end{matrix} \begin{pmatrix} -3 & -3 & 3 \\ 0 & -12 & 6 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{matrix} -2R_1+R_2 \to R_1 \\ \Rightarrow \\ \end{matrix} \begin{pmatrix} 6 & -6 & 0 \\ 0 & -12 & 6 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} R_1/(6) \to R_1 \\ \Rightarrow \\ R_2/(-6) \to R_2 \end{matrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow a - b = 0, 2b - c = 0 \Rightarrow a = b, c = 2b$

Therefore

$$x = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ b \\ 2b \end{pmatrix} = b\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \in span\left\{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}\right\}$$

So the Eigenvector corresponding to $\lambda = 4$ is $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$