1. Show that $T: \mathbb{R}^3 \to \mathbb{R}^3$ given by $T: (x, y, z) \mapsto (x - y + 2z, 2x + y, -x - 2y + 2z)$ is a Linear Transformation. Find rank $T = \dim(\operatorname{ran} T)$ and null $T = \dim(\ker T)$ and verify the Dimension Theorem in Q2.

Solution

Let $F = \mathbb{R}$. We will verify the Dimension Theorem.

Let $(x, y, z) \in \ker T \subseteq \mathbb{R}^3$
$\Leftrightarrow \mathrm{T}(x, y, z) = \underline{0}$
$\Leftrightarrow (x - y + 2z, 2x + y, -x - 2y + 2z) = (0,0,0)$

$$\Leftrightarrow \begin{array}{c} x - y + 2z = 0 \cdots (1) \\ 2x + y = 0 \cdots (2) \\ -x - 2y + 2z \cdots (3) \end{array} \Leftrightarrow \begin{array}{c} x - y + 2z = 0 \cdots (1) \\ 2x + y = 0 \cdots (2) \Leftrightarrow x = -\frac{1}{2}y \\ -3y + 4z = 0 \cdots (1) + (3) = (4) \Leftrightarrow z = \frac{3}{4}y \end{array}$$

$\Leftrightarrow (x, y, z) = \left(-\frac{1}{2}y, y, \frac{3}{4}y\right) = 4y(-2,4,3), y \in \mathbb{R} = F$

So $\ker T = \operatorname{span}\{(-2,4,3)\}$. Also $\{(-2,4,3)\}$is Linearly Independent(why?).
Therefore $\{(-2,4,3)\}$ is a Basis for $\ker T$ and $\mathrm{nullT} = \dim(\ker T) = 1$

Let $T(x, y, z) \in \operatorname{ran} T$ where $x, y, z \in \mathbb{R}$. We have

$T(x, y, z) = (x - y + 2z, 2x + y, -x - 2y + 2z) = (x, 2x, -x) + (-y, y, -2y) + (2z, 0,2z)$
$= x(1,2,-1) + y(-1,1,-2) + 2z(1,0,1)$
So $\operatorname{ran} T = \operatorname{span}\{(1,2,-1), (-1,1,-2), (1,0,1)\}$ (why?)

To see whether $\{(1,2,-1), (-1,1,-2), (1,0,1)\}$ is linearly independent
we will form a matrix by using these vectors as rows and do Row Operations as follows

$$\begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & -2 \\ 1 & 0 & 1 \end{pmatrix} \underset{-R_1 + R_3 \to R_3}{\overset{R_1 + R_2 \to R_2}{\sim}} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -3 \\ 0 & -2 & 2 \end{pmatrix} \underset{R_2 + R_3/2 \to R_3}{\overset{R_2/3 \to R_2}{\sim}} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \overset{-2R_2 + R_1 \to R_1}{\sim} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

So $\operatorname{ran} T = \operatorname{span}\{(1,2,-1), (-1,1,-2), (1,0,1)\} = \operatorname{span}\{(1,0,1), (0,1,-1)\}$ (why?)
The set $\{(1,0,1), (0,1,-1)\}$ is aslo linearly independent (why?)
Therefore $\{(1,0,1), (0,1,-1)\}$ is a Basis for $\operatorname{ran} T$ and $\operatorname{rank} T = \dim(\operatorname{ran} T) = 2$

We also have the Domain of T, $\operatorname{dom} T = \mathbb{R}^3$ *and* $\dim(\operatorname{dom} T) = \dim(\mathbb{R}^3) = 3$
Finally $\dim(\operatorname{dom} T) = 3 = 2 + 1 = \operatorname{rank} T + \operatorname{null} T$, is a verification of the Dimension Theorem.

2. Prove the Dimension Theorem: $\dim(\operatorname{dom} T) = \operatorname{rank} T + \operatorname{null} T$

Solution

Let $T: V \to W$ be the Linear Transfromation over the Field F and $\{u_1, u_2, \cdots, u_m\}$ be a basis for kerT
Therefore $\text{null}T = \dim(\ker T) = m$.

Since $\ker T \subseteq V$, we can extend this basis of $\ker T$ to form a Basis
$\{u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_n\}$ of V (Theorem, why?).
Therefore $\dim(\text{dom}T) = \dim V = n + m$.

We will show that $\{T(v_1), T(v_2), \cdots, T(v_n)\}$ is a basis for $\text{ran}T$.
Let $w \in \text{ran}T, \exists v = a_1u_1 + a_2u_2 + \cdots + a_mu_m + b_1v_1 + b_2v_2 + \cdots + b_nv_n \in V$ such that

$w = T(v) = T(a_1u_1 + a_2u_2 + \cdots + a_mu_m + b_1v_1 + b_2v_2 + \cdots + b_nv_n)$
$= a_1T(u_1) + a_2T(u_2) + \cdots + a_mT(u_m) + b_1T(v_1) + b_2T(v_2) + \cdots + b_nT(v_n)$
$= a_1\underline{0} + a_2\underline{0} + \cdots + a_m\underline{0} + b_1T(v_1) + b_2T(v_2) + \cdots + b_nT(v_n)$ since $u_i \in \ker T$
$= b_1T(v_1) + b_2T(v_2) + \cdots + b_nT(v_n)$

So $\text{ran}T = \text{span}\{T(v_1), T(v_2), \cdots, T(v_n)\}$

Also $b_1T(v_1) + b_2T(v_2) + \cdots + b_nT(v_n) = \underline{0}$ for some $b_i \in F$
$\Rightarrow T(b_1v_1 + b_2v_2 + \cdots + b_nv_n) = \underline{0}$
$\Rightarrow b_1v_1 + b_2v_2 + \cdots + b_nv_n \in \ker T$
$\Rightarrow b_1v_1 + b_2v_2 + \cdots + b_nv_n = a_1u_1 + a_2u_2 + \cdots + a_mu_m$ for some $a_i \in F$
since $\{u_1, u_2, \cdots, u_m\}$ is a Basis for $\ker T$
$\Rightarrow b_1v_1 + b_2v_2 + \cdots + b_nv_n - a_1u_1 - a_2u_2 - \cdots - a_mu_m = \underline{0}$
$\Rightarrow (b_1, b_2, \cdots, b_n, -a_1, -a_2, \cdots, -a_m) = (0,0, \cdots, 0,0,0, \cdots, 0)$ since $\{u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_n\}$ is a Basis for V
$\Rightarrow (b_1, b_2, \cdots, b_n) = (0,0, \cdots, 0)$

So $\{T(v_1), T(v_2), \cdots, T(v_n)\}$ is Linearly Independent and is a Basis for $\text{ran}T$
Therefore $\text{rankT} = \dim(\text{ranT}) = n$.

Finally $\dim(\text{dom}T) = n + m = \text{rank}T + \text{null}T$