1. Let $u \in V$ be an inner product space and W be a subspace spanned by the orthonormal set $\{w_1, w_2, \cdots, w_n\}$. Let the projection of $u \in V$ onto W be defined by $Pu = \sum_{i=1}^{n} \langle u, w_i \rangle w_i$. Show that Pu is the best approximation in W to u with respect to the norm defined by the inner product.

Solution

Let $F = \mathbb{C}$. We will show that $\|u - w\| \geq \|u - Pu\|$ for any $w \in W$.

$\|u - w\|^2$
$= \|u - Pu + Pu - w\|$
$= \|s + t\|$ where $s = u - Pu \in V$ *and* $t = Pu - w \in W$(why?)
$= \langle s + t, s + t \rangle$
$= \langle s, s \rangle + \langle s, t \rangle + \langle t, s \rangle + \langle t, t \rangle$
$= \|s\|^2 + \langle s, t \rangle + \overline{\langle s, t \rangle} + \|t\|^2$ since $\langle t, s \rangle = \overline{\langle s, t \rangle}$
$= \|s\|^2 + 2\mathrm{Re}\langle s, t \rangle + \|t\|^2$

Now let w_j be any element in the orthonormal set, we have
$\langle Pu, w_j \rangle = \langle \sum_{i=1}^{n} \langle u, w_i \rangle w_i, w_j \rangle = \sum_{i=1}^{n} \langle u, w_i \rangle \langle w_i, w_j \rangle = \langle u, w_j \rangle \langle w_j, w_j \rangle = \langle u, w_j \rangle 1 = \langle u, w_j \rangle$
and $\langle s, w_j \rangle = \langle u - Pu, w_j \rangle = \langle u, w_j \rangle - \langle Pu, w_j \rangle = \langle u, w_j \rangle - \langle u, w_j \rangle = 0$

Now let $t = \sum_{i=1}^{n} a_i w_i$ be any element of W
Then $\langle s, t \rangle = \langle s, \sum_{j=1}^{n} a_j w_j \rangle = \sum_{j=1}^{n} \bar{a}_j \langle s, w_j \rangle = \sum_{j=1}^{n} \bar{a}_j \, 0 = 0.$
This actually shows that $s = u - Pu \in W^{\perp} \subseteq V$(why?)

Finally $\|u - w\|^2 = \|s\|^2 + 2\mathrm{Re}0 + \|t\|^2 = \|s\|^2 + \|t\|^2 \geq \|s\|^2 = \|u - Pu\|^2$
Therefore $\|u - w\| \geq \|u - Pu\|$ for any $w \in W$ *and* $u \in V$ (why?)

2. Let $V = \mathcal{C}[-1,1]$ and the inner product for $f, g \in V$ be defined by $\int_{-1}^{1} f(x)g(x)dx$. Find the best approximation to e^x in $W = \text{span}\{1, x, x^2\}$.

Solution

Let $W = \text{span}\{1, x, x^2\} = \text{span}\{u_1, u_2, u_3\}$.
First we find an Orthonormal Basis for W using Gram − Schimdt Process(why is it working?)

Let $v_1 = u_1 = 1$, so $\|v_1\|^2 = \int_{-1}^{1} 1^2\, dx = [x]_{-1}^{1} = 2$ and $\|v_1\| = \sqrt{2}$
So $w_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}$ is the first element in the Orhonormal set

Also $v_2 = u_2 - \langle u_2, w_1\rangle w_1$ and $\langle u_2, w_1\rangle = \int_{-1}^{1} x \frac{1}{\sqrt{2}} dx = \frac{1}{2\sqrt{2}}[x^2]_{-1}^{1} = 0$
So $v_2 = x - 0 = x$ and $\|v_2\|^2 = \int_{-1}^{1} x^2\, dx = \frac{1}{3}[x^3]_{-1}^{1} = \frac{2}{3}$ and $\|v_2\| = \frac{\sqrt{2}}{\sqrt{3}}$
Then $w_2 = \frac{v_2}{\|v_2\|} = \frac{\sqrt{3}}{\sqrt{2}}x$ is the second element in the Orhonormal set

In the same way $v_3 = u_3 - \langle u_3, w_1\rangle w_1 - \langle u_3, w_2\rangle w_2$
and $\langle u_3, w_1\rangle = \int_{-1}^{1} x^2 \frac{1}{\sqrt{2}} dx = \frac{1}{3\sqrt{2}}[x^3]_{-1}^{1} = \frac{2}{3\sqrt{2}} = \frac{\sqrt{2}}{3}$
and $\langle u_3, w_2\rangle = \int_{-1}^{1} x^2 \frac{\sqrt{3}}{\sqrt{2}} x\, dx = \frac{\sqrt{3}}{4\sqrt{2}}[x^4]_{-1}^{1} = 0$
So $v_3 = x^2 - \frac{\sqrt{2}}{3}\frac{1}{\sqrt{2}} - 0 = x^2 - \frac{1}{3}$ and $\|v_3\|^2 = \int_{-1}^{1}\left(x^2 - \frac{1}{3}\right)^2 dx = \left[\frac{x^5}{5} - \frac{2}{3}\frac{x^3}{3} + \frac{1}{9}x\right]_{-1}^{1} = 2\left[\frac{1}{5} - \frac{2}{9} + \frac{1}{9}\right] = \frac{2(4)}{5(9)}$
Then $\|v_3\| = \frac{2\sqrt{2}}{3\sqrt{5}}$ and $w_3 = \frac{v_3}{\|v_3\|} = \frac{3\sqrt{5}}{2\sqrt{2}}\left(x^2 - \frac{1}{3}\right) = \frac{\sqrt{5}}{2\sqrt{2}}(3x^2 - 1)$ is the third element in the Orhonormal set

Finallt the Orthonornal Set required by the previous Theorem is $\{w_1, w_2, w_3\} = \left\{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}x, \frac{\sqrt{5}}{2\sqrt{2}}(3x^2 - 1)\right\}$
and the best approximation to $e^x \in \mathcal{C}[-1,1]$ in $W = \text{span}\{1, x, x^2\} = \text{span}\{w_1, w_2, w_3\}$ is

$P(e^x) = \sum_{i=1}^{3}\langle e^x, w_i\rangle w_i$ where

$\langle e^x, w_1\rangle = \int_{-1}^{1} e^x \frac{1}{\sqrt{2}} dx = \frac{1}{\sqrt{2}}[e^x]_{-1}^{1} = \frac{e - e^{-1}}{\sqrt{2}}$

$\langle e^x, w_2\rangle = \int_{-1}^{1} e^x \frac{\sqrt{3}}{\sqrt{2}} x\, dx = \frac{\sqrt{3}}{\sqrt{2}}[xe^x - e^x]_{-1}^{1} = \frac{\sqrt{3}}{\sqrt{2}}[e^x(x-1)]_{-1}^{1} = \frac{\sqrt{3}}{\sqrt{2}}[e(0) - e^{-1}(-2)] = \sqrt{6}e^{-1}$

$\langle e^x, w_2\rangle = \int_{-1}^{1} e^x \frac{\sqrt{5}}{2\sqrt{2}}(3x^2 - 1)\, dx$
$= \frac{\sqrt{5}}{2\sqrt{2}}[3(x^2e^x - 2(xe^x - e^x)) - e^x]_{-1}^{1} = \frac{\sqrt{5}}{2\sqrt{2}}[e^x(3x^2 - 6x + 5)]_{-1}^{1} = \frac{\sqrt{5}}{2\sqrt{2}}[e(2) - e^{-1}(14)] = \frac{\sqrt{5}(e - 7e^{-1})}{\sqrt{2}}$

Finally
$P(e^x)$
$= \sum_{i=1}^{3}\langle e^x, w_i\rangle w_i = \frac{e - e^{-1}}{\sqrt{2}}\frac{1}{\sqrt{2}} + \sqrt{6}e^{-1}\frac{\sqrt{3}}{\sqrt{2}}x + \frac{\sqrt{5}(e - 7e^{-1})}{\sqrt{2}}\frac{\sqrt{5}}{2\sqrt{2}}(3x^2 - 1)$
$= \frac{e - e^{-1}}{2} + 3e^{-1}x + \frac{5(e - 7e^{-1})}{4}(3x^2 - 1)$
$= \frac{3(-e + 11e^{-1})}{4} + 3e^{-1}x + \frac{15(e - 7e^{-1})}{4}x^2$
$\approx 0.9962940183 + 1.103638324x + 0.5367215260x^2$

is the best 2nd degree polynomial approximation to e^x with respect to norm defined by the above inner product.