1. Show that the matrices $\begin{pmatrix}1 & 2\\3 & 1\end{pmatrix}, \begin{pmatrix}3 & -1\\2 & 2\end{pmatrix}, \begin{pmatrix}1 & -5\\-4 & 0\end{pmatrix}$ are linearly dependent. Find a basis for $\mathbb{R}^{2\times 2}$.

Solution

Here $F = \mathbb{R}$.

We have $2\begin{pmatrix}1 & 2\\3 & 1\end{pmatrix} - 1\begin{pmatrix}3 & -1\\2 & 2\end{pmatrix} + 1\begin{pmatrix}1 & -5\\-4 & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} = \underline{0}$ and $(-2,-1,1) \neq (0,0,0)$

Therefore $\left\{\begin{pmatrix}1 & 2\\3 & 1\end{pmatrix}, \begin{pmatrix}3 & -1\\2 & 2\end{pmatrix}, \begin{pmatrix}1 & -5\\-4 & 0\end{pmatrix}\right\}$ is Linearly Dependent.

We show that $B = \left\{\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\right\}$ is a Basis for $\mathbb{R}^{2\times 2}$

$a\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + b\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} + c\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} + d\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \underline{0}$ for $a, b, c, d \in \mathbb{R}$

$\Rightarrow \begin{pmatrix}a & b\\c & d\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$

$\Rightarrow (a, b, c, d) = (0,0,0,0)$

$\Rightarrow B$ is Linerly Independent

Also for $\begin{pmatrix}a & b\\c & d\end{pmatrix} \in \mathbb{R}^{2\times 2}$, we have

$\begin{pmatrix}a & b\\c & d\end{pmatrix} = a\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + b\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} + c\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} + d\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$

$\mathbb{R}^{2\times 2} = \text{Span } B$ (why?)

Therefore B is a Basis for $\mathbb{R}^{2\times 2}$

2. Show that every set with more than n vectors is linearly dependent in a n-dimensional vector space.

Solution

Let V be the $n-$ dimensional Vector Space over F over with a basis $B = \{u_1, u_2 \cdots, u_n\}$
Let $S = \{v_1, v_2 \cdots, v_m\}$ be the set with $m > n$ elements.
Each v_i can be written in the form $v_i = \sum_{j=1}^{n} b_{ij} u_j$ where $b_{ij} \in F$.

$\sum_{i=1}^{m} a_i v_i = \underline{0}$ for $a_i \in F$.
$\Rightarrow \sum_{i=1}^{m} a_i \sum_{j=1}^{n} b_{ij} u_j = \underline{0}$
$\Rightarrow \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_i b_{ij}\right) u_j = \sum_{j=1}^{n} c_j u_j = \underline{0}$
$\Rightarrow (c_1, c_2, \cdots, c_n) = (0,0,\cdots,0)$ since B is Linearly Independent

Now we have n equations $c_j = \sum_{i=1}^{m} a_i b_{ij} : j = 1,2,\cdots,n$ in m unknowns $a_i : i = 1,2,\cdots,m$ with $m > n$
Therfore we have a $\text{non} - \text{trivial}$ solution $(a_1, a_2, \cdots, a_m) \neq (0,0,\cdots,0)$ (why?)
So S in Lindearly Dependent.