1. Show that $(\mathbb{R}^{+},\cdot,\circ)$ over $(\mathbb{R},+,\cdot)$ is a vector space, where for $a \in \mathbb{R}$ and $x \in \mathbb{R}^{+}$ the operation $\circ$ is defined as $a \circ x = x^{a}$. What does the Theorem $(-a) \circ x = \overline{a \circ x}$ look like in this vector space?.

Solution

1. $(\mathbb{R}^{+},\cdot)$ is an Abelian Group:
   1.1 $1 \in \mathbb{R}^{+} \Rightarrow \mathbb{R}^{+} \neq \emptyset$
   1.2 $\forall a, b \in \mathbb{R}^{+}; a \cdot b \in \mathbb{R}^{+}$
   1.3 $\forall a, b \in \mathbb{R}^{+}; a \cdot (b \cdot c) = (a \cdot b) \cdot c$
   1.4 $\exists 1 \in \mathbb{R}^{+}, \forall a \in \mathbb{R}^{+}; a \cdot 1 = 1 \cdot a = a$
   1.5 $\forall a \in \mathbb{R}^{+}, \exists a^{-1} \in \mathbb{R}^{+}; a \cdot a^{-1} = a^{-1} \cdot a = 1$
   1.6 $\forall a, b \in \mathbb{R}^{+}; a \cdot b = b \cdot a$

2. $(\mathbb{R},+,\cdot)$ is a Field(why?)

3. $\forall a \in \mathbb{R}, \forall x \in \mathbb{R}^{+}; a \circ x = x^{a} \in \mathbb{R}^{+}$

4. $\forall a \in \mathbb{R}, \forall x, y \in \mathbb{R}^{+}; a \circ (x \cdot y) = (x \cdot y)^{a} = x^{a} \cdot y^{a} = (a \circ x) \cdot (a \circ y)$

5. $\forall a, b \in \mathbb{R}, \forall x \in \mathbb{R}^{+}; (a+b) \circ x = x^{a+b} = x^{a} \cdot x^{b} = (a \circ x) \cdot (b \circ x)$

6. $\forall a, b \in \mathbb{R}, \forall x \in \mathbb{R}^{+}; (a \cdot b) \circ x = x^{a \cdot b} = x^{b \cdot a} = \left(x^{b}\right)^{a} = a \circ (b \circ x)$

7. $\forall x \in \mathbb{R}^{+}; 1 \circ x = x^{1} = x$

   $\therefore (\mathbb{R}^{+},\cdot,\circ)$ over$(\mathbb{R},+,\cdot)$ is a Vector Space

Here $(-a) \circ x = x^{-a}$
and $\overline{a \circ x} = (a \circ x)^{-1} = (x^{a})^{-1}$
Therefore accoding to the theorem $x^{-a} = (x^{a})^{-1}$

2. Prove that the intersection of sub vector spaces is also a sub vector space.

Solution

Let S, T be sub vector spaces of V over F, then

1. $\underline{0} \in S \text{ } and \text{ } \underline{0} \in T \Rightarrow \underline{0} \in S \cap T \Rightarrow S \cap T \neq \emptyset$

2. $S, T \subseteq V \Rightarrow S \cap T \subseteq V$
   Therefore $S \cap T$ is a non empty subset of V. Also,

3. $x, y \in S \cap T \Rightarrow x, y \in S$ and $x, y \in T \Rightarrow x + y \in S$ and $x + y \in T \Rightarrow x + y \in S \cap T$

4. $a \in F$ and $x \in S \cap T \Rightarrow a \in F$ and $x \in S$ and $x \in T \Rightarrow ax \in S$ and $ax \in T \Rightarrow ax \in S \cap T$
   Therefore $S \cap T$ is closed under vector addition and scalara multiplication

   Therefore $S \cap T$ is a sub vector space of V (by Theorem, why?)