Definition: $(G,*)$ is a **Group** $\Leftrightarrow$

1. $G \neq \emptyset$: set is non empty
2. $\forall a, b \in G; a * b \in G$: binary operation is close
3. $\forall a, b, c \in G; a * (b * c) = (a * b) * c$: operation is associative
4. $\exists e \in G, \forall a \in G; a * e = e * a = a$: there is an identity element
5. $\forall a \in G, \exists \bar{a} \in G; a * \bar{a} = \bar{a} * a = e$: each element has its inverse element

Example: Which of the following are groups?

1. $(\mathbb{R}, +)$: set of real numbers with addition
2. $(\mathbb{R}, \cdot)$: set of real numbers with multiplication
3. $(\{1,0\}, +)$: Boolean algebra with OR
4. $(\{1,0\}, \cdot)$: Boolean algebra with AND
5. $(\mathbb{R}^{2\times 2}, +)$: set of 2×2 matrices with real entries with addition
6. $(\mathbb{R}^{2\times 2}, \cdot)$: set of 2×2 matrices with real entries with multiplication
7. $GL_n(\mathbb{R})$: set of invertible $n \times n$ matrices under multiplication
8. set of invertible functions under composition.
9. set of 4^{th} roots of 1 under multiplication
10. D_3: symmetry group of equilateral triangle under composition
11. $E = \{(x,y): y^2 = x^3 + Ax + B, 4A^3 + 27B^2 \neq 0\} \cup \{\mathcal{O}\}$: elliptic curve group, $\mathcal{O}$: point at infinity

Theorem: $(G,*)$ is a **group** $\Rightarrow$

1. e is unique
2. $\bar{a}$ is unique
3. $\forall a \in G; \bar{\bar{a}} = a$
4. $\forall a, b \in G; \overline{a * b} = \bar{b} * \bar{a}$
5. $a * b = a * c \Longrightarrow b = c$: right cancellation
6. $b * a = c * a \Longrightarrow b = c$: left cancellation

Definition: $(G,*)$ is an **Abelian Group** $\Leftrightarrow$

1. $(G,*)$ is a group
2. $\forall a, b \in G; a * b = b * a$: commutative

Example: which of the above examples are Abelian Groups?

Definition: $(F, +, \cdot)$ is a **Field** $\Leftrightarrow$

1. $(F, +)$ is an Abelian Group
2. $(F\backslash\{0\}, \cdot)$ is an Abelian Group: 0 is the identity for $+$
3. $\forall a, b \in F; a \cdot b \in F$: closed under $\cdot$
4. $\forall a, b, c \in F; a \cdot (b + c) = (a \cdot b) + (a \cdot c)$: distributive

Note : we also use the following

1. $-a$ is the inverse of a for $+$
2. a^{-1} is the inverse of a for $\cdot$
3. 1 is the multiplicative identity for $\cdot$

Example: See which of the following are fields

1. $(\mathbb{R}, +, \cdot)$
2. $(\mathbb{R}, \cdot, +)$
3. $(\mathbb{C}, +, \cdot)$: set of complex numbers with addition and multiplication
4. $(\mathbb{Q}, +, \cdot)$: set of rational numbers with addition and multiplication
5. $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : \ a, b \in \mathbb{Q}\}$
6. $(\{1,0\}, +, \cdot)$
7. $(\{T, F\}, \wedge, \vee)$: set of logic with AND and OR
8. $(GL_n(\mathbb{R}), +, \cdot)$
9. $(\xi, \cup, \cap)$: universal set with union and intersection
10. $\mathbb{F}_p$: finite field with prime p elements under $mod\ p$ addition and multiplication

Theorem: If p is prime and a is an integer then $a^p \equiv a \ (mod\ p)$
Theorem: $(F, +, \cdot)$ is a **Field** $\Rightarrow \forall a \in F; 0 \cdot a = 0$

Definition: $(V,*,\circ)$ is a vector space over $(F,+,\cdot)$

1. $(V,*)$ is an Abelian Group
2. $(F,+,\cdot)$ is a Field
3. $\forall a \in F, \forall x \in V; a \circ x \in V$: closed under scalar multiplication
4. $\forall a,b \in F, \forall x \in V; (a+b) \circ x = (a \circ x) * (b \circ x)$: distributive w.r.t. scalar addition
5. $\forall a \in F, \forall x,y \in V; a \circ (x * y) = (a \circ x) * (a \circ y)$: distributive w.r.t. vector addition
6. $\forall a,b \in F, \forall x \in V; (a \cdot b) \circ x = a \circ (b \circ x)$: associative w.r.t. scalar product
7. $\forall x \in V; 1 \circ x = x$

Note: the above four operations are functions such that

$+: F \times F \to F$: scalar addition
$\cdot: F \times F \to F$: scalar product
$*: V \times V \to V$: vector addtiopn
$\circ: F \times V \to V$: scalar multiplication

Example: which of the above examples are Vector Spaces?

1. $(\mathbb{R}^n,+,\cdot)$ over $(\mathbb{R},+,\cdot)$: set of vectors with n real entries
2. $(\mathbb{R},+,\cdot)$ over $(\mathbb{R},+,\cdot)$
3. $(\mathbb{C},+,\cdot)$ over $(\mathbb{R},+,\cdot)$
4. $(\mathbb{R},+,\cdot)$ over $(\mathbb{C},+,\cdot)$
5. $(\mathbb{C},+,\cdot)$ over $(\mathbb{C},+,\cdot)$
6. $(\mathbb{R}[x],+,\cdot)$ over $(\mathbb{R},+,\cdot)$: set of polynomial in x with real coefficients
7. $(V,+,\cdot)$ over $(\mathbb{R},+,\cdot)$: set of solution of a linear differential equation
8. $(C[0,1],+,\cdot)$ over $(\mathbb{R},+,\cdot)$: continuous functions on $[0,1]$

Theorem:
$(V,*,\circ)$ over $(F,+,\cdot)$ is a vector space $\Rightarrow$

1. $\forall x \in V; 0 \circ x = e$
2. $\forall a \in F; a \circ e = e$
3. $\forall a \in V, \forall a \in F; (-a) \circ x = \overline{(a \circ x)}$
4. $a \circ x = e \Rightarrow a = 0 \; or \; x = e$

Definition:
$(S,*,\circ)$ is a sub vector space of $(V,*,\circ)$ over $(F,+,\cdot) \Leftrightarrow$

1. $(V,*,\circ)$ is a vector space over $(F,+,\cdot)$
2. $S \subseteq V$
3. $(S,*,\circ)$ is a vector space over $(F,+,\cdot)$

Theorem:

1. $(V,*,\circ)$ is a vector space over $(F,+,\cdot)$
2. $S \subseteq V$: subset
3. $\forall x,y \in V; x * y \in V$: closed under addition
4. $\forall a \in F, \forall x,y \in V; a \circ x \in V$: closed under scalar multiplication

$\Rightarrow$

1. $(S,*,\circ)$ is a vector space over $(F,+,\cdot)$
2. $(S,*,\circ)$ is a sub vector space of $(V,*,\circ)$ over $(F,+,\cdot)$

Note: When there is no threat of confusion we may use the following

1. $x * y = x + y$
2. $a \cdot b = ab$
3. $a \circ x = ax$
4. $\bar{x} = -x$
5. $e = \underline{0}$

Example: Write the rules for vector space with this new notation
V is a vector space over F (or V is a F-vector space)
$B \subseteq V,\; u_i \in B,\; a_i \in F$

Definition: $\sum_i a_i u_i$ is a linear combination of B

Definition: $span\ B = \{\sum_i a_i u_i : a_i \in F, u_i \in B\}$: set of all possible linear combinations of the elements of B.

Theorem:

1. $span\ B$ is a subspace of V
2. $span\ B$ is the smallest subspace of V containing B

Definition: B is linearly independent $\Leftrightarrow$
$\forall \underline{a} = (a_1, \cdots, a_n) \in F^n; L(\underline{a}) = \sum_i a_i u_i = \underline{0} \Rightarrow \underline{a} = \underline{0}$

Definition: B is **linearly dependent** $\Leftrightarrow$ B is not linearly independent

Theorem:

1. $|B| \geq 2$
2. B is linearly dependent

$\Rightarrow$ At least one element of B can be written as a linear combination of the other elements

Definition: $B \subseteq V$ is a basis for V $\Leftrightarrow$

1. $\text{span } B = V$
2. B is linearly independent

Example: Find a basis for the following vector spaces

1. $\mathbb{R}^n$
2. set of solution of a linear differential equation
3. $\mathcal{P}_n(\mathbb{R})$: set of polynomials with degree n or less
4. $\mathbb{R}[x] = P(\mathbb{R})$: set polynomials with real coefficients
5. set of Taylor series expandable functions on $[0,1]$
6. set of Fourier series expandable functions on $[0,1]$
7. $C[0,1]$: continuous functions on $[0,1]$

Definition:

1. B is a basis for V
2. $|B|$ is finite

$\Leftrightarrow$ V is finite dimensional

Note: From here on we will work with finite dimensional vector spaces

Theorem:

1. V is finite dimensional
2. B is a basis for V

$\Rightarrow$ each vector in V can be uniquely expressed as a linear combination of the vectors in B

Theorem:

1. V is finite dimensional
2. B is a basis for V

$\Rightarrow$ every set with more than $|B|$ vectors is linearly dependent

Theorem: V is finite dimensional $\Rightarrow$ any two bases for V have the same size

Definition: Dimension

1. V is finite dimensional
2. B is a basis for V

$\Leftrightarrow dim\, V = |B|$

Definition: $dim\{\underline{0}\} = 0$

Definition: standard basis for $\mathbb{R}^n$
$B = \{e_i\}_n$
$e_i = (O_{1\times(i-1)}, 1, O_{1\times(n-i)})$

V is a vector space over F
$F = \mathbb{C}, u, v, w \subseteq V, a \in F$

Definition: inner product

1. $\langle \cdot,\cdot \rangle : V \times V \to F = \mathbb{C}$ is a function
2. $\langle u, v \rangle = \overline{\langle v, u \rangle}$
3. $\langle u + v, w \rangle = \langle u, v \rangle + \langle v, w \rangle$
4. $\langle au, v \rangle = a\langle u, v \rangle$
5. $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0 \Leftrightarrow u = \underline{0}$

$\Leftrightarrow \langle \cdot,\cdot \rangle$ is an inner product

Note: we have $a = \bar{a}$ when $F = \mathbb{R}$

Definition: V is an **inner product vector space** $\Leftrightarrow$ V has an inner product defined

Theorem:

1. $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$
2. $\langle u, av \rangle = a\langle u, v \rangle$
3. $\langle \underline{0}, u \rangle = \langle u, \underline{0} \rangle = 0$

Example: Show that the following are inner products

1. $u, v \in \mathbb{R}^n, F = \mathbb{R}; \langle u, v \rangle = u \cdot v$
2. $u, v \in \mathbb{C}^n, F = \mathbb{R}; \langle u, v \rangle = u \cdot \bar{v}$
3. $f, g \in C([a, b]), F = \mathbb{R}; \langle f, g \rangle = \int_a^b f(x)g(x)dx$

Definition: Norm induced by the inner product $\|u\| = \sqrt{\langle u, u \rangle}$

Theorem:

1. $\|u + v\| \leq \|u\| + \|v\|$: triangular inequality
2. $|\langle u, v \rangle| \leq \|u\|\|v\|$: Cauchy-Schwarz inequality

Definition: u and v are orthogonal $\Leftrightarrow \langle u, v \rangle = 0$

Definition: u and v are orthonormal $\Leftrightarrow$

1. u and v are orthogonal
2. $\|u\| = \|v\| = 1$

Definition: B is orthogonal $\Leftrightarrow \forall i \neq j; u_i$ and u_j are orthogonal

Definition: B is orthonormal$\Leftrightarrow \forall i \neq j; u_i$ and u_j are orthonormal

Theorem: $\langle \sum_{i=1}^{n} a_i\, u_i, v \rangle = \sum_{i=1}^{n} a_i \,\langle u_i, v \rangle$

Theorem:

1. B is orthogonal
2. $\underline{0} \notin B$

$\Rightarrow B$ is linearly independent

Theorem: If $\{u_i\}$ is an orthonormal basis for a vector space V then any $v \in V$ can be written as
$v = \sum_{i=1}^{n} \langle u_i, v \rangle u_i$

Definition: Orthogonal complement of a subspace W in an inner product space V
$W^{\perp} = \{u \in V : < u, w >= 0 \ \forall w \in W\}$

Theorem: $W^{\perp}$ is a subspace of V

Theorem: Let V be an inner product space and W be a subspace spanned by the orthonormal set $\{w_1, w_2, \cdots, w_n\}$. Then we can write $u = Pu + Qu \in V$ where $Pu = \sum_{i=1}^{n} < u, w_i > w_i \in W$ and $Qu \in W^{\perp}$

Example: Let $V = \mathbb{R}^3$ and $W = span\{(1,0,0),(0,1,0)\}$. Find Pu and Qu for $u = (a, b, c) \in V$.

Theorem: Pu is the best approximation in W to u with respect to the norm defined by the inner product.

Example:
1. Let $V = C[-1,1]$ and the inner product for $f, g \in V$ be defined by $\int_{-1}^{1} f(x)g(x)dx$. Find the best approximation to e^x in $W = span\{1, x, x^2\}$.
2. Let $V = C[-\pi,\pi]$ and the inner product for $f, g \in V$ be defined by by $\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx$. Find the best approximation to f in $W = span\{e^{ijx}\}_{j=1}^{n}$.

Theorem: Gram-Schmidt Process
Let $U = \{u_1, u_2, \cdots, u_n\}$ be linearly independent in an inner product space. Then if
$v_j = u_j - \sum_{i=1}^{j-1} < u_i, w_i > w_i$ for $j = 1,2,\cdots,n$
$w_j = \frac{v_j}{\|v_j\|}$ for $j = 1,2,\cdots,n$
Then $W = \{w_1, w_2, \cdots, w_n\}$ is orthonormal and $spanU = spanW$

V, W are vector spaces with the common field F
$u, v \in V, r \in F$
Definition: Linear Transformation
T is a linear transformation $\Leftrightarrow$
1. $T: V \to W$ is a function
2. $T(u + v) = T(u) + T(v)$
3. $T(ru) = rT(u)$

Example: Show that the following are linear transformation
1. $T: \mathbb{R}^2 \to \mathbb{R}^3; T: (x, y) \mapsto (x + y, 2x + y, 3x + y)$
2. $T: C(\mathbb{R}) \to \mathbb{R}; T: u \mapsto \int_0^1 u(t)dt$
3. $T: \mathbb{P}_n(\mathbb{R}) \mapsto \mathbb{P}_n(\mathbb{R}); T: p(x) \mapsto xp'(x) + p(x)$

Theorem:
1. $T(au + bv) = aT(u) + bT(v)$
2. $T(\underline{0}) = \underline{0}$
3. $T(-u) = -T(u)$

Definition:
1. $\ker T = \{u | T(u) = \underline{0} \text{ } and \text{ } u \in V\}$: Kernal
2. $\operatorname{ran} T = T(V) = \{T(u) | u \in V\}$: Range
3. $\operatorname{dom} T = V$: Domain

Theorem:
1. $\ker T$ is a subspace of V
2. $\operatorname{ran} T$ is a subspace of W

Definition:
1. $\operatorname{null} T = \dim(\ker T)$: Nullity
2. $\operatorname{rank} T = \dim(\operatorname{ran} T)$: Rank

Example: Find null T and rank T of
$T: \mathbb{R}^3 \to \mathbb{R}^3; T: (x, y, z) \mapsto (x - y + 2z, 2x + y, -x - 2y + 2z)$

Theorem: Dimension theorem $\dim(\operatorname{dom} T) = \operatorname{rank} T + \operatorname{null} T$

Example: Coordinate rotation
(x, y) coordinate system is rotated anticlockwise by an angle θ to get a new coordinate system (X, Y). The associated linear transformation is

$$T: \mathbb{R}^2 \to \mathbb{R}^2;\ T: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$B = \{u_1, u_2, \cdots, u_n\} = \{u\}_i$ $\quad$ $(B) = (u_1, u_2, \cdots, u_n) = (u_i)_n = (u_i)_{1\times n}$

$(B)^T = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = (u_i)_n^T = (u_i)_{1\times n}^T$ $\quad$ $(u_{ij})_{m\times n} = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{m1} & u_{m2} & \cdots & u_{mn} \end{pmatrix}$

$T: V \to W$ is a linear transformation
F is the common field of V and W
$B_V = \{u_i\}_n$ is a basis for V, $B_W = \{w_i\}_m$ is a basis for W

Theorem:
$\exists A \in F^{m\times n};\ (T(u_i))_n = (w_i)_m A$

Definition: Matrix of a linear transformation
$A = T_{(B_V),(B_W)} \in F^{m\times n}$

Theorem: $v = \sum_i b_i u_i \in V \Rightarrow$
1. $v = (u_i)_n (b_i)_n^T$
2. $T(v) = (T(u_i))_n (b_i)_n^T$

Theorem:
1. $B_V' = \{u_i'\}_n$ is a basis for V
2. $B_W' = \{w_i'\}_m$ is a basis for W
3. $(T(u_i))_n = (w_i)_m A$
4. $(T(u_i'))_n = (w_i')_m A'$

$\Rightarrow$
1. $\exists P \in F^{n\times n};\ (u_i')_n = (u_i)_n P$
2. $\exists Q \in F^{m\times m};\ (w_i')_m = (w_i)_m Q$
3. $(T(u_i'))_n = \left(T(u_i)\right)_n P$
4. $A' = Q^{-1}AP$

Theorem:
1. $V = F^n$
2. $W = F^m$
3. $B_V = \{e_i\}_n$ the standard basis for V
4. $B_W = \{e_i\}_m$ the standard basis for W

$\Rightarrow$
1. $(u_i)_n = (e_i)_n = I_n$
2. $(w_i)_m = (e_i)_m = I_m$
3. $(T(u_i))_n = A$
4. $v = (b_i)_n^T$
5. $T(v) = Av$
6. $P = (u_i')_n$
7. $Q = (w_i')_m$
8. $A' = (w_i')_m^{-1} A (u_i')_n$

Example: $T: \mathbb{R}^2 \to \mathbb{R}^3;\ T: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ x-2y \\ 2x+3y \end{pmatrix}$

1. Find the matrix of T with $B_V = \left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$ and $B_W = \left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}\right\}$
2. Find the coefficients a_{ij} such that

$$T\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = a_{11}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + a_{12}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + a_{13}\begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$$

$$T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = a_{21}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + a_{22}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + a_{23}\begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$$

Note: In the previous discussion we understood that once we fix the bases, a linear transformation has a unique Matrix of the Linear Transformation. So from here we will discuss properties of matrices. Understand that all these definitions has an equivalent definition in Linear Transformations.

$A = (a_{ij}) \in F^{n\times n}, x \in F^n, \lambda \in F$

Definition: Eigenvalue λ and corresponding **Eigenvector** $x_\lambda \neq 0$
$Ax_\lambda = \lambda x_\lambda$

Theorem: λ is an eigenvalue of $A \Leftrightarrow |A - \lambda I| = 0$

Definition: Characteristic Polynomial $p(\lambda) = |\lambda I - A| = (-1)^n |A - \lambda I|$

Theorem: Cayalay-Hamilton $p(A) = O$

Definition: Minimal Polynomial
The lease degree monic polynomial(coefficient of the highest power is 1) satisfying $m(A) = O$

Theorem:
1. $m(\lambda)$ is unique
2. $m(\lambda)$ divides any $q(A)$with $q(A) = O$
3. $m(\lambda)$ divides $p(\lambda)$
4. $m(\lambda)$ and $p(\lambda)$ have the same roots

Definition: Spectral Radius $\rho(A) = \max\{|\lambda_i|\}$

Definition: Trace $\operatorname{tr}(A) = \sum_{i=1}^n a_{ii}$

Theorem:
1. $\operatorname{tr}(AB) = \operatorname{tr}(BA)$
2. $\sum_{i=1}^n \lambda_i = \operatorname{tr}(A)$
3. $\prod_{i=1}^n \lambda_i = |A|$

Example: Find the characteristic polynomial, minimal polynomial, eigenvalues, spectral radius and eigenvectors of the following matrices.

$$\begin{pmatrix} -11 & -10 & 5 \\ 5 & 4 & -5 \\ -20 & -20 & 4 \end{pmatrix}, \begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Definition: $A \in \mathbb{C}^{n\times n}$
1. $A^T = (a_{ji})$: Transpose
2. $\bar{A} = (\bar{a}_{ij})$: Conjugate
3. $A^H = (\bar{A})^T$: Conjugate Transpose
4. $A = A^T$: Symmetric, $A = A^H$: Hermitian
5. $A^{-1} = A^T$: Orthogonal, $A^{-1} = A^H$: Unitary

Theorem: Eigenvalues of a Hermitian/Real symmetric matrix are real.

Definition: Eigenspace corresponding to λ:
$V_\lambda = \{\lambda | Ax_\lambda = \lambda x_\lambda\} \cup \{\underline{0}\}$

Theorem: V_λ is a subvector space of F^n over F

Theorem:
If the eigenvalues are different their corresponding eigenvectors are linearly independent

Definition:
$g_i =$ geometric multiplicity of $\lambda_i =$ multiplicity of λ_i as a root of $p(\lambda)$
$a_i =$ algebraic multiplicity of $\lambda_i =$ no. of independent eigen vectors corresponding to $\lambda_i = \dim(V_i)$

Theorem: With $p(\lambda) = (\lambda - \lambda_1)^{g_1}(\lambda - \lambda_2)^{g_2}(\lambda - \lambda_3)^{g_3}\cdots$and $m(\lambda) = (\lambda - \lambda_1)^{b_1}(\lambda - \lambda_2)^{b_2}(\lambda - \lambda_3)^{b_3}\cdots$
we have $1 \leq b_i \leq g_i$ and $1 \leq a_i \leq g_i$

Definition: $A \in \mathbb{C}^{n\times n}$ is **diagonalizable** $\Leftrightarrow \exists$ invertible P such that $P^{-1}AP = \Lambda$ is a diagonal matrix

Theorem: The following are equivalent(TFAE) for $A \in \mathbb{C}^{n\times n}$

1. geometric multiplicity=algebraic multiplicity for each eigen value($a_i = g_i$)
2. minimal polynomial has *n* distinct roots($b_i = 1$)
3. it has *n* independent eigenvectors
4. it is diagonalizable(with Λ is the diagonal matrix formed by eigenvalues and columns of P are the corresponding eigenvectors)

Example:

1. Solve the differential equation $\frac{d^2u}{dt^2} + \frac{du}{dt} + u = 0, u(0) = 1, u'(0) = 2$ as a system of differential equations $\dot{y} = Ay, y(0) = \left(u(0), u'(0)\right)^T$
2. Solve the system of differential equations $\ddot{y} = y, y(0) = (1,2,3)^T, y'(0) = (4,5,6)^T$ where $A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$

Definition: Normed vector space V with **norm** $\|\cdot\|$ over the field $F = \mathbb{C}$

1. $\|\cdot\|: V \to F$ is a function
2. $\|v\| \geq 0$ and $\|v\| = 0 \Leftrightarrow v = \underline{0}$ for all $v \in V$
3. $\|av\| = |a|\|v\|$ for all $v \in V$ and $a \in F$
4. $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in V$

Definition/Theorem: vector norms $V = \mathbb{C}^n$ and $F = \mathbb{C}$

1. $\|x\|_p = \sqrt[p]{\sum_i |x_i|^p}$: p norm
2. $\|x\|_1 = \sum_i |x_i|$: 1 norm
3. $\|x\|_2 = \sqrt{\sum_i |x_i|^2}$: 2 norm, norm coming from the inner product $\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{x^T\bar{x}}$
4. $\|x\|_\infty = \max\{|x_i|\}$

Definition: Matrix norm $\|\cdot\|$ of matrix A over $F = \mathbb{C}$

1. $\|\cdot\|$ is a norm
2. $\|AB\| \leq \|A\|\|B\|$ for all matrices A, B

Definition/Theorem: Matrix norms for $A \in \mathbb{C}^{n\times n}$ over $F = \mathbb{C}$

1. $\|A\|_p = \sup_{x \neq \underline{0}} \left\{\frac{\|Ax\|_p}{\|x\|_p}\right\}$: p norm induced by the vector p norm
2. $\|A\|_1 = \max\{\sum_j |a_{ij}|\}$: maximum absolute raw sum norm
3. $\|A\|_\infty = \max\{\sum_i |a_{ij}|\}$: maximum absolute column sum norm
4. $\|A\|_E = \sqrt{\sum_{i,j} |a_{ij}|^2}$: Frobenous norm
5. $\|A\|_2 = \sqrt{\rho(A^H A)}$: specral norm

Theorem: The following are equivalent(TFAE) for $A \in \mathbb{C}^{n\times n}$

1. $\lim_{k\to\infty} A^k = O$
2. $\lim_{k\to\infty} A^k x = \underline{0}$ for all x
3. $\rho(A) < 1$
4. There exists a matrix norm such that $\|A\| < 1$

Theorems:

1. $\|A\|_2^2 \leq \|A\|_1\|A\|_\infty$
2. $\|Ax\|_p \leq \|A\|_p\|x\|_p$: matrix and vector *p* norms are compatible
3. $f(x) = \sum_{n=0}^\infty a_n x^n$ converges for $|x| < R \Rightarrow f(A) = \sum_{n=0}^\infty a_n A^n$ converges for $\rho(A) < R$
4. $A\lambda = \lambda x \Rightarrow f(A)x = f(\lambda)x$
5. A is diagonalizable $\Rightarrow f(A) = P^{-1}f(\Lambda)P$ where $f(\Lambda) = \mathrm{diag}(f(\lambda_i))$

Definition: A complete(all Cauchy Sequences converge) normed space is called a **Banach Space** and a complete inner product space is called a **Hilber Space.**

Algorithm: For solution of the system $Ax = b$ with $A = D + U + L$

1. **Jacobi method**:
   $Ax = (D + U + L)x = b \Rightarrow Dx = -(U + L)x + b \Rightarrow x = -D^{-1}(U + L)x + D^{-1}b \Rightarrow x_{k+1} = Mx_k + N$
2. **Gauss-Seidel method**
   $Ax = (D + U + L)x = b \Rightarrow (D + L)x = -Ux + b \Rightarrow x = -(D + L)^{-1}Ux + (D + L)^{-1}b \Rightarrow x_{k+1} = Mx_k + N$

Algorithm: To find the eigenvalue of A

1. **Power method** $y_k = Ax_k, \frac{\|y_{k+1}\|}{\|y_k\|} \to |\lambda_1| > |\lambda_i|$
2. **QR method** $A_k = Q_k R_k, A_{k+1} = R_k Q_k, A_k \to$ matrix with eigen values of A are on the diagonal

Note:
Practically **Householder Transformations** and and **Givens Rotations** are used for finding QR decomposition.
There are requirements for convergence.

Example: Use Jacobi and Gauss-Seidel methods to solve the system $\begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix} x = \begin{pmatrix} 10 \\ 14 \\ 14 \end{pmatrix}$. Comment on the convergence by finding norms $\|M\|$ and by finding $\rho(M)$ using Power method and QR method.

Definition: Real Quadratic Form, $A \in \mathbb{R}^{n \times n}, x \in \mathbb{R}^n$
$f(x) = x^T Ax = \sum_{i,j=1}^{n} A_{ij} x_i x_j$
We can always assume that A is symmetric ie. $A = A^T$

Theorem: Every real symmetric matrix is Orthogonally Diagonalizable

Theorem:

1. $\lambda_{min}\|x\|^2 \leq x^T Ax \leq \lambda_{max}\|x\|^2$
2. Equality occurs when x is the corresponding eigenvector

Definition:

1. $\forall x \neq \underline{0}, x^T Ax > 0$: A is positive definite: $A \succ 0$
2. $\forall x \neq \underline{0}, x^T Ax \geq 0$: A is positive semi definite: $A \succcurlyeq 0$
3. $\forall x \neq \underline{0}, x^T Ax < 0$: A is negative semi definite: $A \prec 0$
4. $\forall x \neq \underline{0}, x^T Ax \leq 0$: A is negative semi definite: $A \preccurlyeq 0$
5. Neither of the above: A is indefinite

Definition: Leading Principal Minors of Order k
$M_k =$ Determinant obtained by taking the first k rows and first k columns of A

Theorem:

1. $A \succ 0 \Leftrightarrow \lambda_{min}(A) > 0$: all eigen values are positive $\Leftrightarrow M_k > 0$
2. $A \succcurlyeq 0 \Leftrightarrow \lambda_{min}(A) \geq 0$: all eigen values are non negative $\Leftrightarrow M_k < 0$
3. $A \prec 0 \Leftrightarrow \lambda_{min}(A) < 0$: all eigen values are nagative $\Leftrightarrow M_k > 0$ for even k and $M_k < 0$ for odd k
4. $A \preccurlyeq 0 \Leftrightarrow \lambda_{min}(A) \geq 0$: all eigen values are non positive $\Leftrightarrow M_k \geq 0$ for even k and $M_k \leq 0$ for odd k

Theorem:

1. $A \succcurlyeq B \Leftrightarrow A - B \succcurlyeq 0$
2. $A \succcurlyeq B$ and $C \succcurlyeq D \Rightarrow A + C \succcurlyeq B + D$
3. $A^2 \succcurlyeq 0$
4. $A \succ 0 \Rightarrow A^{-1} \succ 0$
5. We can have $A \not\succcurlyeq B$ and $A \not\preccurlyeq B$

Definition: Diagonal Form $f(x) = x^T Dx$, D is a diagonal matrix

Example:

1. Investigate the positivity of the function $f(x, y, z) = 2x^2 + 12xy + y^2 - 4xz - 8yz - 3z^2$
2. Identify the surface $f(x, y, z) = 0$ by rotating the coordinate axis

Example: eigenproblem for general linear transformations (extra)
Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by $(x, y) = (x + y, x + y)$. Find the eigenvalues and eigenvectors of T.