FIELD	
NAME	
INDEX NUMBER	
L=Last 3 Digits of the Index Number	
M=Number obtained by taking Mod 2 (remainder after dividing by 2) of each digit of L	
Write this number M on the diagonal of the matrix A, starting from the top left hand corner $A = \begin{pmatrix} \boxed{\cdot} & 1 & 2 \\ 1 & \boxed{\cdot} & 1 \\ 1 & 2 & \boxed{\cdot} \end{pmatrix}$.	

Q1. Find a basis for the vector space spanned by the columns of the matrix A.

Solution: Let $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$.

$V = \text{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\2\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}\right\}$ is the required vector space(it is called the column space of A).

We do column operations:

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \underset{-C_1 + C_2 \to C_2}{\overset{C_1 + C_2 + C_3 \to C_3}{\Rightarrow}} \begin{pmatrix} 1 & 0 & 4 \\ 1 & 1 & 4 \\ 1 & 1 & 4 \end{pmatrix} \underset{-C_2 + C_1 \to C_1}{\overset{-4C_1 + C_3 \to C_3}{\Rightarrow}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

So $B = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$ is a basis for V

Note:

1. You can back-track the columns operations to justify our claims:

   $C_3' = C_1 + C_2 + C_3$

   $C_3'' = -4C_1 + C_3' = -4C_1 + C_1 + C_2 + C_3 = -3C_1 + C_2 + C_3 = \underline{0}$

   or $-3\begin{pmatrix}1\\1\\1\end{pmatrix} + 1\begin{pmatrix}1\\2\\2\end{pmatrix} + 1\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ showing that the columns are linearly dependent.

   or $C_3 = 3C_1 - C_2$

   Also

   $C_2' = -C_1 + C_2$

   $C_1' = -C_2' + C_1 = -(-C_1 + C_2) + C_1 = 2C_1 - C_2$

   Showing that $\text{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\2\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\2\end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$ (why?)

2. Clearly $B = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$ is linearly independent since $a\begin{pmatrix}1\\0\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}a\\b\\b\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow a = b = 0$.

   Alternatively, the inner product $\left\langle\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}.\begin{pmatrix}0\\1\\1\end{pmatrix} = 0$ also tells us that B is linearly independent.

3. $B = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$ is orthogonal. We can make it orthonormal by dividing each vector by its norm: $B' = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$.

   This is as answer to Q2, without using Gram-Schmidt(what will happen if we use Gram-Schmidt on the columns of A?)

4. Any two or all the 3 vectors of $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ not a basis for V (why?).

5. Finding $\det A = 1(2-2) - 1(1-1) + 2(2-2) = 0$ will tell you that the columns are dependent. But it will not tell which columns will be in the basis nor how many of them are there.

6. Doing row operations

   $$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \underset{-R_1 + R_2 \to R_3}{\overset{-R_1 + R_2 \to R_2}{\Rightarrow}} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \underset{-2R_2 + R_1 \to R_1}{\overset{-R_2 + R_3 \to R_3}{\Rightarrow}} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

   will tell you that $\text{rowrank}\, A = \text{columnrank}\, A = 2 = \text{rank}\, A$, or that a basis will have 2 elements. But it will not tell you which columns will be in the basis. In general you can't say that any two columns will be in the basis, take $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$ for example. Row operations will not find a basis for the columns space in general.

Q2. Use Gram-Schmidt process to find an orthonormal basis for the vector space in Q1(keep the square roots, they will remain only in the answer)

Solution: Let $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$.

$\det A = 1(1-2) - 1(1-1) + 2(2-1) = 1 \neq 0$ so the columns are linearly independent.

So $B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a basis for $V = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\}$

Using-Gram Schmidt

Let $v_1 = u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, so $\|v_1\| = \sqrt{3}$, so $w_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is the first element in the Orhonormal set

Also $v_2 = u_2 - \langle u_2, w_1 \rangle w_1$ and $\langle u_2, w_1 \rangle = \langle \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{4}{\sqrt{3}}$

So $v_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \frac{4}{\sqrt{3}}\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \frac{4}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3-4 \\ 3-4 \\ 6-4 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$ and $\|v_2\| = \frac{\sqrt{6}}{3}$

Then $w_2 = \frac{v_2}{\|v_2\|} = \frac{3}{\sqrt{6}}\frac{1}{3}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$ is the second element in the orhonormal set

Also $v_3 = u_3 - \langle u_3, w_1 \rangle w_1 - \langle u_3, w_2 \rangle w_2$ and $\langle u_3, w_1 \rangle = \langle \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{4}{\sqrt{3}}$

and $\langle u_3, w_2 \rangle = \langle \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \rangle = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \frac{-1}{\sqrt{6}}$

So $v_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \frac{4}{\sqrt{3}}\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{-1}{\sqrt{6}}\frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \frac{4}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 12-8-1 \\ 6-8-1 \\ 6-8+2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

and $\|v_3\| = \sqrt{2}$. Then $w_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ is the third element in the orhonormal set

So the required orthonormal basis is $W = \{w_1, w_2, w_3\} = \left\{ \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$

Note: QR factorization

Rewriting u_1, u_2, u_3 as a linear combination of w_1, w_2, w_3 using the Gram-Schmidt formula, we have

$v_1 = u_1 = \|v_1\| w_1$ or $u_1 = \|v_1\| w_1 + 0w_2 + 0w_3$

$v_2 = u_2 - \langle u_2, w_1 \rangle w_1 = \|v_2\| w_2$ or $u_2 = \langle u_2, w_1 \rangle w_1 + \|v_2\| w_2 + 0w_3$

$v_3 = u_3 - \langle u_3, w_1 \rangle w_1 - \langle u_3, w_2 \rangle w_2 = \|v_3\| w_3$ or $u_3 = \langle u_3, w_1 \rangle w_1 + \langle u_3, w_2 \rangle w_2 + \|v_3\| w_3$.

So we can write

$$A = (u_1 \quad u_2 \quad u_3) = (w_1 \quad w_2 \quad w_3)\begin{pmatrix} \|v_1\| & \langle u_2, w_1 \rangle & \langle u_3, w_1 \rangle \\ 0 & \|v_2\| & \langle u_3, w_2 \rangle \\ 0 & 0 & \|v_3\| \end{pmatrix}$$

or $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 2/\sqrt{6} & 0 \end{pmatrix}\begin{pmatrix} \sqrt{3} & 4/\sqrt{3} & 4/\sqrt{3} \\ 0 & \sqrt{6}/3 & -1/\sqrt{6} \\ 0 & 0 & \sqrt{2} \end{pmatrix} = QR$ is the QR factorization of A.

Here $Q = (w_1 \quad w_2 \quad w_3) = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 2/\sqrt{6} & 0 \end{pmatrix}$ is a unitary/orthogonal matrix(why?)

$R = \begin{pmatrix} \sqrt{3} & 4/\sqrt{3} & 4/\sqrt{3} \\ 0 & \sqrt{6}/3 & -1/\sqrt{6} \\ 0 & 0 & \sqrt{2} \end{pmatrix}$ = is an upper triangular matrix. The factorization is unique up to $\pm$.