Theorem 1. *Complex Inversion Formula for Laplace Transform*
Let $|f(t)| \leq Me^{at}$ *so the Laplace Transform* $F(s)$ *be valid for* $\text{Re } s > a$ *and let* $\lim_{|s|\to\infty} |F(s)| = 0$*. Then the inverse Laplace transform* $f(t)$ *is given by*
$f(t) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} F(s)e^{st}ds =$ *Sum of Residues of* $F(s)e^{st}$ *for* $\text{Re } s \leq a$.

Proof.
Let $|f(t)| \leq Me^{at}$. Then the Laplace transform is
$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{st}dt$
So $|F(s)| \leq \int_0^\infty Me^{at}e^{\text{Re } st}dt = \frac{1}{\text{Re } s-a}$ whenever $\text{Re } s > a$.

Take any real number $b > a$ and define $g(t) = f(t)e^{-bt}$ for $t \geq 0$ and 0 otherwise.

Fourier Transform of $g(t)$ is
$\mathcal{F}\{g(t)\} = G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = \int_0^\infty f(t)e^{-(b+i\omega)t}dt = F(b+i\omega)$

And the inverse Fourier Transform of $G(\omega)$ is
$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(b+i\omega)e^{i\omega t}d\omega$

In particular for $t \geq 0$,
$g(t) = f(t)e^{-bt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(b+i\omega)e^{i\omega t}d\omega$ or
$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(b+i\omega)e^{(b+i\omega)t}d\omega$

Now changing the variable $s = b + i\omega$, we arrive at
$f(t) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} F(s)e^{st}ds$

Now let L : vertical line from $b - iR$ to $b + iR$ and Γ : left half circle with center $b + i0$ and radius R. Let the closed loop $C = L \cup \Gamma$.

On Γ with $s = b + Re^{i\theta}$ for $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ we have
$\left|\int_\Gamma F(s)e^{st}ds\right| \leq \int_{\pi/2}^{3\pi/2} |F(b + Re^{i\theta})|e^{(b+R\cos\theta)t}Rd\theta \leq \epsilon Re^{bt}\int_{\pi/2}^{3\pi/2} e^{Rt\cos\theta}d\theta$

Note that $\cos\theta \leq -\frac{2}{\pi}\theta + 1$ on $\frac{\pi}{2} \leq \theta \leq \pi$ and $\cos\theta \leq \frac{2}{\pi}\theta - 3$ on $\pi \leq \theta \leq \frac{3\pi}{2}$

Therefore, $\int_{\pi/2}^{3\pi/2} e^{Rt\cos\theta}d\theta \leq \int_{\pi/2}^{\pi} e^{Rt(-\frac{2}{\pi}\theta+1)}d\theta + \int_{\pi}^{\frac{3\pi}{2}} e^{Rt(\frac{2}{\pi}\theta-3)}d\theta$
$= e^{Rt}\frac{\pi}{2Rt}(e^{-Rt} - e^{-2Rt}) + e^{-3Rt}\frac{\pi}{2Rt}(e^{3Rt} - e^{2Rt}) = \frac{\pi}{Rt}(1 - e^{-Rt})$

Now, $\left|\int_\Gamma F(s)e^{st}ds\right| \leq \epsilon Re^{bt}\frac{\pi}{Rt}(1 - e^{-Rt}) = \epsilon e^{bt}\frac{\pi}{t}(1 - e^{-Rt}) \to 0$ as $R \to \infty$

Also $\int_L F(s)e^{st}ds \to \int_{b-i\infty}^{b+i\infty} F(s)e^{st}ds = 2\pi i f(t)$ as $R \to \infty$.

And $\int_C F(s)e^{st}ds \to 2\pi i$ Sum of Residues of $F(s)e^{st}$ to the left of $\text{Re}s < b$.

So we have,
$0 + 2\pi i f(t) = 2\pi i$ Sum of Residues of $F(s)e^{st}$ for $\text{Re } s < b$.

Finally since $F(s)$ has no singularities for $a < \text{Re } s \leq b$ we have
$f(t) =$Sum of Residues of $F(s)e^{st}$ for $\text{Re } s \leq a$

□