1 Vector Calculus

Definition 1. *Curve*
A function $\boldsymbol{r} : [a, b] \subset \mathbb{R} \to \mathbb{R}^3$ *given by* $\boldsymbol{r}(t) =< x(t), y(t), z(t) >$ *is a curve. The direction of the curve is said to be from* a *to* b.

Definition 2. *Length of a curve*
Let $\boldsymbol{r}$ *be a curve in* $\mathbb{R}^3$. *Let* $P = \{t_0, t_1, \ldots, t_n\}$ *with* $t_0 = a, t_n = b$ *with* $t_k > t_{k-1}$ *be a partition of* $[a, b]$ *i.e.* $P \in \mathcal{P}[a, b]$. *Define* $s(\boldsymbol{r}, P) = \sum_{k=1}^{n} \|\boldsymbol{r}(t_k) - \boldsymbol{r}(t_{k-1})\|$ *and the length of the curve by* $s(\boldsymbol{r}) = \sup\{s(\boldsymbol{r}, P) \,|P \in \mathcal{P}[a, b]\}$

Definition 3. $\boldsymbol{r}$ *is Rectifiable iff* $s(\boldsymbol{r}) \in \mathbb{R}$

Theorem 1. *If* $\boldsymbol{r} \in \mathcal{C}^1$ *then* $s(\boldsymbol{r}) = \int_a^b \|\boldsymbol{r}'(t)\| dt$ *and* $\frac{ds}{dt} = \|\boldsymbol{r}'(t)\|$.

Note 1. *MVT does not exist for a curve* $\boldsymbol{r} = \boldsymbol{r}(t)$. *However in* $\mathbb{R}^2$ *there exists* $c \in (a, b)$ *such that* $\mathbf{r}'(t)$ *is parallel to* $\mathbf{r}(b) - \mathbf{r}(a)$. *Let* $\mathbf{r}'(t) \neq \mathbf{0}$ *for all curves we discuss below.*

Definition 4. *Unit Tangent Vector* $\boldsymbol{T} = \frac{d\boldsymbol{r}}{ds}$, *Unit Normal Vector* $\boldsymbol{N} = \frac{1}{\kappa}\frac{d\boldsymbol{T}}{ds}$
Unit Binormal Vector $\boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N}$, *Curvature* $\kappa = \|\frac{d\boldsymbol{T}}{ds}\|$, *Torsion* $-\tau\boldsymbol{N} = \frac{d\boldsymbol{B}}{ds}$

Theorem 2. *Frenet-Serret Formulas*

$$\begin{pmatrix} \frac{d\boldsymbol{T}}{ds} \\ \frac{d\boldsymbol{N}}{ds} \\ \frac{d\boldsymbol{B}}{ds} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix}$$

Definition 5. *Velocity* $\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \dot{\boldsymbol{r}}$, *Acceleration* $\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \dot{\boldsymbol{v}}$

Theorem 3. $\kappa = \frac{\|\boldsymbol{v}\times\boldsymbol{a}\|}{\|\boldsymbol{v}\|^3}$, $\tau = \frac{\boldsymbol{v}\times\boldsymbol{a}\cdot\dot{\boldsymbol{a}}}{\|\boldsymbol{v}\times\boldsymbol{a}\|^2}$

Example 1.
1. Let $\boldsymbol{r}(t) = \langle a\cos t, a\sin t, ct\rangle$. *Find* $\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}, \kappa, \tau$
2. Show that the curvarute of a circle with radius R *is* $\frac{1}{R}$.
3. Find velocity and acceleration on a parametric curve and deduce the same for circular motion.

Definition 6. *Path* C *is a function* $\boldsymbol{r} : [a, b] \subset \mathbb{R} \to \mathbb{R}^n$ *which is smooth(i.e.* $\mathcal{C}^\infty$*) and one to one on* (a, b). *Also the Direction of the path is from* a *to* b

Definition 7. *Loop is a Path such that* $\boldsymbol{r}(a) = \boldsymbol{r}(b)$. *Usually the Direction of a Loop is taken anticlockwise.*

Definition 8. *Vector Field* $\boldsymbol{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$ *is a* $\mathcal{C}^1$ *function.*

Definition 9. *Line Integral of the Vector Field* $\boldsymbol{F}$ *over the Path* C *given by* $\boldsymbol{r}(t)$ *is defined as* $\int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)dt$. *We simply write* $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ *and* $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$ *when the Path is a Loop.*

Definition 10. *Vector Field* $\boldsymbol{F}$ *is Path Independent iff its Line Integral* $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ *is independent of the path* C *between any two given points of its domain.*

Theorem 4. *$\boldsymbol{F}$ is Path Independent* $\Leftrightarrow \oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = 0$ *for every Loop* C.

Definition 11. *Gradient(Dell) Operator where*
$\text{grad} = \nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}$
For a function $\phi = \phi(x, y, z)$ *we define,*
$\text{grad}\phi = \nabla\phi = \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right)\phi = \boldsymbol{i}\frac{\partial\phi}{\partial x} + \boldsymbol{j}\frac{\partial\phi}{\partial y} + \boldsymbol{k}\frac{\partial\phi}{\partial z}$

Definition 12. *Vector Field $\boldsymbol{F}$ is Conservative iff there exists a* $\mathcal{C}^2$ *function* $\phi : D \subset \mathbb{R}^n \to \mathbb{R}$ *such that* $\boldsymbol{F} = \nabla\phi$

Theorem 5. *Path Independent* $\Leftrightarrow$ *Conservative.*
The $\Leftarrow$ *direction* $\int_a^b \nabla\phi(\boldsymbol{r}(t))\cdot\boldsymbol{r}'(t)dt = \phi(\boldsymbol{r}(b)) - \phi(\boldsymbol{r}(a))$ *is called the First Fundamental Theorem of Line Integrals.*
The $\Rightarrow$ *direction* $\nabla\int_a^s \boldsymbol{F}(\boldsymbol{r}(t))\cdot\boldsymbol{r}'(t)dt = \boldsymbol{F}(\boldsymbol{r}(s))$ *is called the Second Fundamental Theorem of Line Integrals.*

Example 2.
1.Consider the vector field $\boldsymbol{F} = < 3x^2y^2, 2x^3y >$. *Find the work done along the two paths given by* $y = x$ *and* $y = x^2$ *from* $(0, 0)$ *to* $(1, 1)$. *Find a scalar potential* ϕ *given by* $\boldsymbol{F} = \nabla\phi$ *and explain the reason for the same work done along the two paths.*
2. Consider the Newton's gravitational force given by $\boldsymbol{F} = \frac{GMm}{r^3}\boldsymbol{r}$. *Directly integrate the gravitational field* $\boldsymbol{E} = -\frac{GM}{r^3}\boldsymbol{r}$ *to find a scalar gravitational potential* V.

Definition 13. *Divergence and Curl of a Vector Field* $\boldsymbol{F} = \boldsymbol{i}F_1 + \boldsymbol{j}F_2 + \boldsymbol{k}F_3$
$\text{div}\boldsymbol{F} = \nabla\times\boldsymbol{F} = \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right)\cdot(\boldsymbol{i}F_1 + \boldsymbol{j}F_2 + \boldsymbol{k}F_3) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$
And
$$\text{curl}\boldsymbol{F} = \nabla\times\boldsymbol{F} = \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right)\times(\boldsymbol{i}F_1 + \boldsymbol{j}F_2 + \boldsymbol{k}F_3) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Definition 14. *Vector Field $\boldsymbol{F}$ is Irrotational iff* $\text{curl}\boldsymbol{F} = \boldsymbol{0}$.

Example 3. *Show that* $\text{curl}(\text{grad}\phi) = \nabla\times\nabla\phi = \mathbf{0}$ *for a scalar field* $\phi \in \mathcal{C}^2$.

Theorem 6. *Conservative* $\Rightarrow$ *Irrotational.*

Definition 15. *A Path Connected domain is a domain D where there is a path between every two points of the domain.*

Definition 16. *D is Simply Connected iff it is Path Connected and*
Paths with the same end points can be continuously deformed to each other.
i.e. every Loop can be continuously deformed to a point.
i.e. the interior of every Loop is also belongs to the set.

Example 4. *Consider the vector field* $\boldsymbol{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$.
1. Attempt to find a scalar potential.
2. Find the work done from $(1, 0)$ *to* $(-1, 0)$ *along several paths.*
3. Find $\text{curl}\boldsymbol{F}$.

Definition 17. *Reimann Integral on a rectangle*
Let the rectangle $A = [0, a] \times [0, b]$ *be partitioned into* n *rectangles* $P = \{A_1, A_2, \cdots, A_n\} \in \mathcal{P}(A)$*. The* k *th rectangle is having area* $\Delta A_k = \Delta x_k \Delta y_k$*. Let the function* $f(x, y)$ *be defined on* A*. Let* $m_k = \inf\{f(x, y)|(x, y) \in A_k\}, M_k = \sup\{f(x, y)|(x, y) \in A_k\}$ *and* $L(P, f) = \sum_{k=1}^{n} m_k \Delta A_k, U(P, f) = \sum_{k=1}^{n} M_k \Delta A_k$*. Define* $L(f) = \sup\{L(P, f)|P \in \mathcal{P}(A)\}, U(f) = \inf\{U(P, f)|P \in \mathcal{P}(A)\}$
Iff $L(f) = U(f)$ *we say that* f *is Riemann Integrable on* A *or* $f \in \mathcal{R}(A)$ *and the common value denoted by* $\int_A f(x, y)dA$ *as the Riemann Integral.*
Note that we get the area of $A = ab$ *when* $f(x, y) = 1$*.*

Theorem 7. *Fubini's*
Let $f \in \mathcal{C}$ *then* $\iint_{[0,a]\times[0,b]} f(x, y)dA = \int_0^a \left(\int_0^b f(x, y)dy\right) dx = \int_0^b \left(\int_0^a f(x, y)dx\right) dy$*.*
Note that we write $dA = dxdy$ *motivated by the above theorem.*

Definition 18. *Reimann Integral of on an arbitrary region.*
Let f *be defined* $A \subset [0, a] \times [0, b]$
Define $g(x, y) = f(x, y)$ *if* $(x, y) \in A$ *and* 0 *otherwise.*
Now we define: $\int_A f(x, y)dA = \int_{[0,a]\times[0,b]} g(x, y)dA$*.*
Note that we can define the are of A *a when* $f(x, y) = 1$*.*

Theorem 8. *Extended Fubini's theorem*
Let f *be* $\mathcal{C}$ *and* $\mathcal{R}$ *in a region* A *bounded by two* $\mathcal{C}$ *curves* $h(x)$ *and* $g(x)$ *for* $x \in [0, a]$*, then* $\iint_A f(x, y)dA = \int_0^a \left(\int_{g(x)}^{h(x)} f(x, y)dy\right) dx$

Let f *be* $\mathcal{C}$ *and* $\mathcal{R}$ *in a region* A *bounded by two* $\mathcal{C}$ *curves* $p(y)$ *and* $q(y)$ *for* $y \in [0, b]$*, then* $\iint_A f(x, y)dA = \int_0^b \left(\int_{p(y)}^{q(y)} f(x, y)dx\right) dy$

Example 5.
1. Evaluate the integral $\int_0^3 \int_{x^2}^9 xy^2 dydx$ *as it is and after changing the order of integration.*
2. Find the integral $\int_0^\infty e^{-x^2} dx$ *by considering a double integral.*

Theorem 9. *Change of variable*
Let $x = x(u, v), y = y(u, v)$ *and the Jacobian* $J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \neq 0$ *so that we can locally invert* (x, y) *to get* (u, v)*. Let* $u, v, \boldsymbol{k}$ *makes a right handed system. Then* $J = ||\boldsymbol{r}_u \times \boldsymbol{r}_v||$ *and* $dA = Jdudv$

Example 6. *Change variables by* $u = y + x$ *and* $v = y - x$ *to evaluate the integral* $\int\int_A xydxdy$ *where* A *is the region bounded by* $y = x, y = x + 1, y = -x, y = -x + 1$*.*

Theorem 10. *Green's*
Let C *be a Loop and also the boundary of a Simply Connected region* $A \subset \mathbb{R}^2$ *and let* $\boldsymbol{F} = \langle F_1, F_2\rangle$ *be a Vector Field. Then*

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_A \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

.

We can make the area dA a vector quantity by multiplying it with the unit vector in the direction of an advancement of a right hand screw when the rotation is the anticlockwise direction of the curve C. So $d\boldsymbol{A} = \boldsymbol{k}dA$. Now the above theorem can be written as

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = \iint_A \mathrm{curl}\boldsymbol{F}\cdot d\boldsymbol{A}$$

Theorem 11. *Divergence theorem on the plane*
Let $\boldsymbol{n}$ be the unit normal vector out from the loop C where s is the arc length and C is the boundary of a simply Connected region $A \subset \mathbb{R}^2$. $\boldsymbol{F} = \langle F_1, F_2\rangle$ be a Vector Field. Then

$$\oint_C \boldsymbol{F}\cdot\boldsymbol{n}ds = \iint_A \left(\frac{\partial F_2}{\partial x} + \frac{\partial F_1}{\partial y}\right) dA = \iint_A \mathrm{div}\boldsymbol{F}dA$$

Example 7. *Let $\boldsymbol{F} =< 3x^2y^2 + 2x, 2x^3y + 1 >$ and A be the region bounded by the curves $y = x$ and $y = x^2$. verify*
1. Green's theorem
2. Divergence theorem on the plane.

Definition 19. *Surface*
A function $\boldsymbol{r} : \mathbb{R}^2 \to \mathbb{R}^3$ given by $\boldsymbol{r}(u, v) =< x(u, v), y(u, v), z(u, v) >$ is a surface. If a normal vector $\boldsymbol{n}$ exists, then surface is said to be Orientable. In such a case $\boldsymbol{n} = \frac{\boldsymbol{r}_u\times\boldsymbol{r}_v}{||\boldsymbol{r}_u\times\boldsymbol{r}_v||}$. There are non-orientable surfaces, examples are Mobius Strip and Klein Bottle.
If $u, v, \boldsymbol{n}$ makes a right handed system, we have $d\boldsymbol{S} = \boldsymbol{n}dS = (\boldsymbol{r}_u \times \boldsymbol{r}_v)dudv$

Theorem 12. *Stoke's*
Let C be a Loop and also the boundary of a Simply Connected surface $S \subset \mathbb{R}^3$ and let $\boldsymbol{F}$ be a Vector Field. Then

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = \iint_S \mathrm{curl}\boldsymbol{F}\cdot d\boldsymbol{S}$$

Here we select $\boldsymbol{n}$ such that it makes a right handed screw with the direction of the Loop C.

Theorem 13. *Let F be defined on a Simply Connected domain, then*
$\boldsymbol{F}$ is Irrotational $\Rightarrow$ $\boldsymbol{F}$ is Conservative.

Example 8. *Let $S_1 : z = 2x + 3, S_2 : z = x^2 + y^2$ be two surfaces and let C be the curve on which they intersect. Verify the Stoke's Theorem on each surface for the vector field $\boldsymbol{F} =< xy, yz, zx >$.*

Definition 20. *Volume*
A function $\boldsymbol{r} : \mathbb{R}^3 \to \mathbb{R}^3$ given by $\boldsymbol{r}(u, v, w) =< x(u, v, w), y(u, v, w), z(u, v, w) >$ is a volume.
When (u, v, w) makes a right handed system, we have $J = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \boldsymbol{r}_u \times \boldsymbol{r}_v \cdot \boldsymbol{r}_w$ and the element volume is $dV = Jdudvdw$.

Definition 21. *Contractible*
Volume V is Contractible iff very Closed Surface in that can be continuously deformed to a point. i.e. the interior of every Volume is also belongs to the set.

Theorem 14. *Gauss's(Divergence)*
Let S be a Closed Orientable Surface and also the boundary of a Contractible Space $V \subset \mathbb{R}^3$ and let $\boldsymbol{F}$ be a Vector Field. Then

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_V \operatorname{div} \boldsymbol{F} dV$$

Here we select $\boldsymbol{n}$ such that it is out from the volume V.

Note 2. *Similar definitions and theorems leading to the Stoke's theorem can be done here. For example iff $\boldsymbol{F} = \operatorname{curl} \boldsymbol{A}$ ($\boldsymbol{A}$ is Vector Potential, note also that $\operatorname{div}(\operatorname{curl} \boldsymbol{A}) = 0$) then closed surface integral is 0(Solenoidal). On the other hand on a Contractible Space $\operatorname{div} \boldsymbol{F} = 0$(Incompressible) implies the same.*

Example 9. *Let V be the volume bounded by the surfaces $S_1 : z = 2x + 3, S_2 : z = x^2 + y^2$. Verify the Divergence Theorem on each surface for the vector field $\boldsymbol{F} =< xy, yz, zx >$.*

Example 10. *Us the Divergence Theorem to prove the Archimedes Principle: upthrust=weight of the liquid displaced.*
Also prove that the resultant forces on the horizontal plane are zero.

Example 11. *Let V be the volume bounded by the two surfaces S_1 and S_2 and let $S = S_1 \cup S_2$. Two surfaces intersect on the curve C. Apply both Stoke's and Divergence theorems to show that $\int\int_S \operatorname{curl} \boldsymbol{F} \cdot d\boldsymbol{S} = 0$*

Example 12. *Given*
1. two scalar fields ϕ, ψ
2. two vector fields $\boldsymbol{F}, \boldsymbol{G}$
3. three vector algebraic operations: addition,scalar product, cross product, dot product
4. three differential operators: $\operatorname{grad}, \operatorname{div}, \operatorname{curl}$
Find how many scalar fields and vector fields can be made with only one operation. Also apply the operations given in 4 to each of the above fields and try to expand the answer by simply writing $\nabla = \sum_k \boldsymbol{e}_k \frac{\partial}{\partial x_k}$ and $\boldsymbol{F} = \sum_k \boldsymbol{e}_k F_k$. Some answers are given below
$\nabla \cdot (\nabla \times \boldsymbol{F}) = 0$
$\nabla \times (\nabla \phi) = \boldsymbol{0}$
$\nabla \times (\nabla \times \boldsymbol{F}) = \nabla(\nabla \cdot \boldsymbol{F}) - (\nabla \cdot \nabla)\boldsymbol{F}$
$\nabla \times (\boldsymbol{F} \times \boldsymbol{G}) = (\boldsymbol{G} \cdot \nabla)\boldsymbol{F} - (\boldsymbol{F} \cdot \nabla)\boldsymbol{G}$
$\nabla \cdot (\phi \boldsymbol{F}) = \phi \nabla \cdot \boldsymbol{F} + \nabla \phi \cdot \boldsymbol{F}$
$\nabla(\boldsymbol{F} \cdot \boldsymbol{G}) = (\boldsymbol{F} \cdot \nabla)\boldsymbol{G} + (\boldsymbol{G} \cdot \nabla)\boldsymbol{F} + \boldsymbol{F} \times (\nabla \times \boldsymbol{G}) + \boldsymbol{G} \times (\nabla \times \boldsymbol{F})$

Example 13. *Consider Maxwell's equations*
Gauss Laws: $\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}$, $\nabla \cdot \boldsymbol{B} = 0$

Faraday's Law: $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$
Amphere's Law: $\nabla \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{J} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\right)$ *simplified to* $\nabla \times \boldsymbol{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}$ *in vacuum.*
Show that the Electric Field in vacuum $\boldsymbol{E}$ *satisfies the wave equation:* $\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = c^2 \nabla^2 \boldsymbol{E}$ *where c=speed of light in vacuum.*

Example 14.
1. Prove $\iint_S \phi d\boldsymbol{S} = \iiint_V \nabla \phi dV$
2. Prove the Chandrasekhar-Wentzel lemma $\oint_C \boldsymbol{r} \times (d\boldsymbol{r} \times \boldsymbol{n}) = -\iint_S (\boldsymbol{r} \times \boldsymbol{n}) \nabla \cdot \boldsymbol{n} dS$ *using any method different to the one found on Wikipedia.*
Also verify the above result on the upper half sphere with center at the origin and radius R.

Definition 22. *Curvilinear Coordinates*
$x = x(u_1, u_2, u_3), y = y(u_1, u_2, u_3), z = z(u_1, u_2, u_3), J = \left|\frac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)}\right| \neq 0$

Scaling Factors and Unit Vectors: $\frac{\partial \boldsymbol{r}}{\partial u_1} = h_1 \boldsymbol{e}_1, \frac{\partial \boldsymbol{r}}{\partial u_2} = h_2 \boldsymbol{e}_2, \frac{\partial \boldsymbol{r}}{\partial u_3} = h_3 \boldsymbol{e}_3$

Element position vector: $d\boldsymbol{r} = h_1 \boldsymbol{e}_1 du_1 + h_2 \boldsymbol{e}_2 du_2 + h_3 \boldsymbol{e}_3 du_3$

Element arc length: $ds^2 = d\boldsymbol{r} \cdot d\boldsymbol{r} = \begin{pmatrix} du_1 & du_2 & du_3 \end{pmatrix} g \begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix}$

g is the Matric Tensor

Definition 23. *Orthogonal Curvilinear Coordinates*
$\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{e}_3, \boldsymbol{e}_2 \times \boldsymbol{e}_3 = \boldsymbol{e}_1, \boldsymbol{e}_3 \times \boldsymbol{e}_1 = \boldsymbol{e}_2$

Theorem 15. *results in orthogonal curvilinear coordinates*

$$g = \begin{pmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{pmatrix}$$

Element volume: $dV = h_1 h_2 h_3 du_1 du_2 du_3 = \sqrt{\det g} du_1 du_2 du_3$

$$\text{grad}\boldsymbol{F} = \frac{1}{h_1}\frac{\partial(h_1 F_1)}{\partial u_1} + \frac{1}{h_2}\frac{\partial(h_2 F_2)}{\partial u_2} + \frac{1}{h_3}\frac{\partial(h_3 F_3)}{\partial u_3}$$

$$\text{div}\boldsymbol{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(F_1 h_2 h_3)}{\partial u_1} + \frac{\partial(h_1 F_2 h_3)}{\partial u_2} + \frac{\partial(h_1 h_2 F_3)}{\partial u_3}\right]$$

$$\text{curl}\boldsymbol{F} = \begin{vmatrix} h_1 \boldsymbol{e}_1 & h_2 \boldsymbol{e}_2 & h_3 \boldsymbol{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}$$

Definition 24.
Cylindrical Polar Coordinates: $x = \rho \cos\theta, y = \rho \sin\theta, z = z$
Spherical Polar Coordinates: $x = r \sin\phi \cos\theta, y = r \sin\phi \sin\theta, z = r \cos\phi$

Example 15. *Find* grad *and* div *in the cylindrical polar coordinate system.*

2 Complex Analysis

Definition 25. *Let* $z = x + iy = re^{i\theta}$ *then*
Real Part: $\mathrm{Re}z = x$, *Imaginary Part:* $\mathrm{Im}z = y$, *Conjugate:*$\overline{z} = x - iy$
Absolute Value: $|z| = r = \sqrt{x^2 + y^2}$
Principal Argument: $\mathrm{Arg}\ z = \theta$ *such that* $-\pi < \theta \leq \pi$
Principal Logarithm: $\mathrm{Log}\ z = \log|z| + i\mathrm{Arg}\ z$
Principal Square Root: $\sqrt{z} = \sqrt{|z|}e^{\mathrm{Arg}z/2}$

Example 16. *Express* $z\overline{w}$ *in terms of* $\underline{z}, \underline{w}$ *and find conditions for* $z = x + iy$ *and* $w = a + ib$ *to be perpendicular and parallel. Here* $\underline{z} =< x, y >$ *and* $\underline{w} =< a, b >$.

Definition 26. *Complex Limit*
$\lim_{z\to a} f(z) = L \Leftrightarrow \forall\epsilon > 0\exists\delta > 0\forall z, 0 < |z - a| < \delta \Rightarrow |f(z) - L| < \epsilon.$

Definition 27. *Differentiability*
Iff $\lim_{z\to a} \frac{f(z)-f(a)}{z-a} = \lim_{\Delta z\to 0} \frac{f(a+\Delta z)-f(a)}{\Delta z}$ *exists we say that* f *is differentiable* $(f \in \mathcal{D})$ *at* a *and denote its value by* $f'(a)$.

Theorem 16. *Iff* $f = u+iv \in \mathcal{D}$ *then* $u, v \in \mathcal{D}$ *and satisfy the Cauchy-Riemann(CR) Equations* $u_x = v_y, u_y = -v_x$.

Theorem 17. *Let* $f = u + iv \in \mathcal{D}$. *Then* $\nabla^2 u = 0$ *and* $\nabla^2 v = 0$.
Such functions are called Harmonic.

Example 17. *Express the derivative of* $f = u + iv$ *in terms of* div *and* curl *of the vector field* $\boldsymbol{F} =< u, v >$.

Definition 28. *Analytic(Holomorphic) Function* $(f \in \mathcal{A})$
$f \in \mathcal{D}$ *on a neighbourhood of* a.
It follows that for a open region $B \subset \mathbb{C}$, $f \in \mathcal{D}$ *on* $B \Leftrightarrow f \in \mathcal{A}$ *on* B

Example 18. *Find the differentiable points of* $z^2, |z|^2, \overline{z}$ *and determine their analytic points.*
Also express the derivative whenever it is existing.

Definition 29. *Let* C *be a path given by* $z(t) = x(t) + iy(t)$ *for* $t \in [a, b]$ *and* $f(z) = u(x, y) + iv(x, y)$. *Then the complex integral* $\int_C f(z)dz$ *is defined as the line integral* $\int_a^b f(z)\frac{dz}{dt}dt = \int_C(udx - vdy) + i\int_C(vdx + udy)$.

Example 19. *Show that* $\int f'(z)dz = f(z) + c$

Theorem 18. *Let* C *be a loop in a simply connected region.*
Iff $f \in \mathcal{A}$ *then* $\oint_C f(z)dz = 0$.

Theorem 19. *Cauchy Integral Formula*
Let $f \in \mathcal{A}$ *and* C *be a loop in a simply connected region and* a *be a point inside* C. *Then* $f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz$.

Theorem 20. *Let* $f \in \mathcal{A}$ *and* C *be a loop in a simply connected region and* a *be a point inside* C. *Then* $f^{(k)}(a) = \frac{k!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{k+1}}dz$.

Example 20. *Find* $\oint_C \frac{z^2-2}{(z-2)^2(z-3)(z-4)}dz$ *where points* $2, 3$ *are inside and* 4 *is outside of the curve* c.

Theorem 21. $f \in \mathcal{A} \Rightarrow f \in \mathcal{C}^\infty$. *i.e Analytic functions are infinitely differentiable.*

Theorem 22. *Taylor Series*
Let $f \in \mathcal{A}$ *in the region* $|z-a| < R$ *and let* C *be a loop in that region. Then we have* $f(z) = \sum_{k=0}^{\infty} a_k(z-a)^k$ *where* $a_k = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{k+1}}dz = \frac{f^{(k)}(a)}{k!}$.
$\sup R$ *is called the Radius of Convergence and the corresponding region is called the Region of Convergence.*

Example 21.
Find the Taylor series of $f(z) = \frac{1}{1+z^2}$ *at* 0 *and use it to explain the reason for the convergence of the real valued function* $\frac{1}{1+x^2}$ *for* $-1 < x < 1$.

Definition 30. *Singular Points*
Non-Analytic points a *of* f *are called singular points.*
1. Isolated Singular Point: $\exists \delta > 0 \forall z, 0 < |z-a| < \delta \Rightarrow f \in \mathcal{A}$. *ie* f *is analytic on some punctured disk centered at* a. *There are three types as we will see below.*
2. Non-Isolated Singular Point: Singular points which are not isolated
2.1 Branch Cuts: Ex. $\text{Arg } z, \text{Log } z, \sqrt{z}$ *along the non-positive real axis.*
2.2 Other: Ex. $\tan \frac{1}{z}$ *at* $z = 0$.

Theorem 23. *Laurent Series*
Let $f \in \mathcal{A}$ *in the region* $R_1 < |z-a| < R_2$ *and let* C *be a loop in that region. Then we have* $f(z) = \sum_{k=-\infty}^{\infty} a_k(z-a)^k$ *where* $a_k = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{k+1}}dz$.
$\sup R_2$ *and* $\inf R_1$ *corresponds to the Region of Convergence.*
We also have $a_{-1} = \frac{1}{2\pi i}\oint_C f(z)dz$. *If* a *is an isolated singular point of* f *then we call* a_{-1} *as* $\text{Res}(f, a)$, *Residue of* f *at* a.

Example 22. *Consider the function* $f(z) = \frac{1}{(z-1)(z-2)^2}$.
1. Find the Laurent series expansions of $f(z)$ *at* 0 *for* $|z| < 1, 1 < |z| < 2, 2 < |z|$.
2. Find the Laurent series expansions of $f(z)$ *at* 2 *for* $0 < |z-2| < 1$.

Definition 31. *Further classification of Isolated Singularities.*
If a *is an isolated singularity of* f *we can find the Laurent Series expansion* $f(z) = \sum_{k=-\infty}^{\infty} a_k(z-a)^k$ *valid for* $0 < |z-a| < R_2$
1.1 Removable Singularity: $a_k = 0$ *for all* $k < 0$.
1.2 Pole of Order n: $a_{-n} \neq 0$ *and* $a_k = 0$ *for all* $k < -n$.
1.3 Essential Singularity: $a_k \neq 0$ *for an infinite number of* $k < 0$.

Theorem 24. *Let* a *be an isolated singularity of* f.
Then $|f(z)| \to \infty$ *as* $z \to a$ *iff* a *is a pole.*

Theorem 25. *If* a *is a pole of order* n *of* f, *then*
$\text{Res}(f, a) = \frac{1}{(n-1)!}\lim_{z\to a}\frac{d^{n-1}}{dz^{n-1}}f(z)(z-a)^n$

Theorem 26. *Let* b_j *are isolated singularities of* f *which are the only singularities of* f *inside the loop* C. *Then* $\oint_C f(z)dz = 2\pi i\sum_j \text{Res}(f, b_j)$.

Theorem 27. *Complex Inversion Formula for Laplace Transform*
Let $|f(t)| \leq Me^{at}$ *so the Laplace Transform* $F(s)$ *be valid for* $\operatorname{Re} s > a$ *and let* $\lim_{|s|\to\infty} |F(s)| = 0$. *Then the inverse Laplace transform* $f(t)$ *is given by*
$f(t) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} F(s)e^{st} ds =$ *Sum of Residues of* $F(s)e^{st}$ *for* $\operatorname{Re} s \leq a$.

Example 23. *Find the Inverse Laplace Transform of* $\frac{1}{(s-1)(s-2)^2}$ *and confirm the answer by the usual method.*

Example 24. *Find the following real integrals*
$\int_0^\infty \frac{1}{1+x^4} dx$, $\int_0^\infty \frac{\sin x}{x(1+x^2)} dx$, $\int_0^\infty \frac{\sqrt{x}}{(x+1)^2} dx$