Blue colored sections will not be tested for the final exam.

Definition 1. *Let* $\boldsymbol{r} : [a,b] \to \mathbb{R}^3$ *given by* $\boldsymbol{r}(t) = \langle x(t), y(t), z(t) \rangle$ *be a curve in* $\mathbb{R}^3$*. Let* $P = \{t_0, t_1, \ldots, t_n\}$ *with* $t_0 = a, t_n = b$ *with* $t_k > t_{k-1}$ *be a partition of* $[a,b]$ *i.e.* $P \in \mathcal{P}[a,b]$*. Define* $\ell(\boldsymbol{r}, P) = \sum_{k=1}^{n} \|\boldsymbol{r}(t_k) - \boldsymbol{r}(t_{k-1})\|$ *and the length of the curve by* $\ell(\boldsymbol{r}) = \sup\{\ell(\boldsymbol{r}, P) \,|P \in \mathcal{P}[a,b]\}$

Definition 2. $\boldsymbol{r}$ *is Rectifiable iff* $\ell(\boldsymbol{r}) \in \mathbb{R}$

Theorem 1. *If* $\boldsymbol{r} \in \mathcal{C}^1$ *then* $\ell(\boldsymbol{r}) = \int_a^b \|\boldsymbol{r}'(t)\| dt$ *and* $\frac{d\ell}{dt} = \|\boldsymbol{r}'(t)\|$

Definition 3. *Unit Tangent Vector* $\boldsymbol{T} = \frac{d\boldsymbol{r}}{d\ell}$*, Unit Normal Vector* $\boldsymbol{N} = \frac{1}{\kappa}\frac{d\boldsymbol{T}}{d\ell}$
Unit Binormal Vector $\boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N}$*, Curvature* $\kappa = \|\frac{d\boldsymbol{T}}{d\ell}\|$*, Torsion* $-\tau\boldsymbol{N} = \frac{d\boldsymbol{B}}{d\ell}$

Theorem 2. *Frenet-Serret Formulas*

$$\begin{pmatrix} \frac{d\boldsymbol{T}}{d\ell} \\ \frac{d\boldsymbol{N}}{d\ell} \\ \frac{d\boldsymbol{B}}{d\ell} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix}$$

Definition 4. *Velocity* $\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \dot{\boldsymbol{r}}$*, Acceleration* $\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \dot{\boldsymbol{v}}$

Theorem 3. $\kappa = \frac{\|\boldsymbol{v} \times \boldsymbol{a}\|}{\|\boldsymbol{v}\|^3}$, $\tau = \frac{\boldsymbol{v} \times \boldsymbol{a} \cdot \dot{\boldsymbol{a}}}{\|\boldsymbol{v} \times \boldsymbol{a}\|^2}$

Definition 5. *Path* C *is a function* $\boldsymbol{r} : [a,b] \subset \mathbb{R} \to \mathbb{R}^n$ *which is smooth(i.e.* $\mathcal{C}^\infty$*) and one to one on* (a,b)*. Also the Direction of the path is from* a *to* b

Definition 6. *Loop is a Path such that* $\boldsymbol{r}(a) = \boldsymbol{r}(b)$*. Usually the Direction of a Loop is taken anticlockwise.*

Definition 7. *Vector Field* $\boldsymbol{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$ *is a* $\mathcal{C}^1$ *function.*

Definition 8. *Line Integral of the Vector Field* $\boldsymbol{F}$ *over the Path* C *given by* $\boldsymbol{r}(t)$ *is defined as* $\int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t) dt$*. We simply write* $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ *and* $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$ *when the Path is a Loop.*

Definition 9. *Vector Field* $\boldsymbol{F}$ *is Path Independent iff its Line Integral* $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ *is independent of the path* C *between every two points of its domain.*

Theorem 4. $\boldsymbol{F}$ *is Path Independent* $\Leftrightarrow \oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0$ *for every Loop* C*.*

Definition 10. *Vector Field* $\boldsymbol{F}$ *is Conservative iff there exists a* $\mathcal{C}^2$ *function* $\phi : D \subset \mathbb{R}^n \to \mathbb{R}$ *such that* $\boldsymbol{F} = \nabla\phi$

Theorem 5. *Path Independent* $\Leftrightarrow$ *Conservative.*
The $\Leftarrow$ *direction* $\int_a^b \nabla\phi(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t) dt = \phi(\boldsymbol{r}(b)) - \phi(\boldsymbol{r}(a))$ *is called the First Fundamental Theorem of Line Integrals.*
The $\Rightarrow$ *direction* $\nabla \int_a^s \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t) dt = \boldsymbol{F}(\boldsymbol{r}(s))$ *is called the Second Fundamental Theorem of Line Integrals.*

Definition 11. *Curl of a Vector Field* $\boldsymbol{F}$ *is* $\text{curl}\boldsymbol{F} = \nabla \times \boldsymbol{F}$ *where*
$\text{grad} = \nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}$ *is the gradient operator.*

Definition 12. *Vector Field* $\boldsymbol{F}$ *is Irrotational iff* $\text{curl}\boldsymbol{F} = \boldsymbol{0}$.

Theorem 6. *Conservative* $\Rightarrow$ *Irrotational.ı.e.* $\text{curl}(\text{grad}\phi) = \nabla \times \nabla\phi = \boldsymbol{0}$

Definition 13. *A Path Connected domain is a domain D where there is a path between every two points of the domain.*

Definition 14. *A Homeomorphism is a continuous and one to one function $f : X \to Y$ which is having a continuous inverse f^{-1}.*
We say that X and Y are Homeomorphic iff there is a Homeomorphism between them.

Definition 15. *D is Simply Connected iff it is Path Connected and*
Paths with the same end points are Homeomorphic.
i.e. every Loop is Homeomorphic to a point.
i.e. the interior of every Loop is also belongs to the set.

Theorem 7. *Green's*
Let C be a Loop and also the boundary of a Simply Connected region $A \subset \mathbb{R}^2$ and let $\boldsymbol{F} = \langle F_1, F_2 \rangle$ be a Vector Field. Then

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_A \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) ds$$

Theorem 8. *Stoke's*
Let C be a Loop and also the boundary of a Simply Connected surface $S \subset \mathbb{R}^3$ and let $\boldsymbol{F}$ be a Vector Field. Then

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_S \text{curl}\boldsymbol{F} \cdot d\boldsymbol{s}$$

Here $d\boldsymbol{s} = \boldsymbol{n}ds$ where $\boldsymbol{n}$ is a unit normal vector to the surface S which makes a right handed system with the direction of the Loop C.
When $\boldsymbol{r}(u, v) \in \mathcal{C}^1$ is the surface S then $d\boldsymbol{s} = (\boldsymbol{r}_u \times \boldsymbol{r}_v)\, dudv$

Theorem 9. *If F is defined on a Simply Connected domain and $\boldsymbol{F}$ is Irrotational $\Rightarrow$ $\boldsymbol{F}$ is Conservative.*

Definition 16. *Divergence of a Vector Field* $\boldsymbol{F}$ *is* $\text{div}\boldsymbol{F} = \nabla \cdot \boldsymbol{F}$

Theorem 10. *Gauss's(Divergence)*
Let S be a Closed Surface and also the boundary of a Contractible Space $V \subset \mathbb{R}^3$ and let $\boldsymbol{F}$ be a Vector Field. Then

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{s} = \iiint_V \text{div}\boldsymbol{F} dV$$

Here $d\boldsymbol{s} = \boldsymbol{n}ds$ where $\boldsymbol{n}$ is a unit normal vector to S out from the region V.
When $\boldsymbol{r}(u, v, w) \in \mathcal{C}^1$ is the space V then $dV = (\boldsymbol{r}_u \times \boldsymbol{r}_v \cdot \boldsymbol{r}_w)\, dudvdw$.
Similar definitions and theorems leading to the Stoke's theorem can be done here. For example iff $\boldsymbol{F} = \text{curl}\boldsymbol{A}$ ($\boldsymbol{A}$ is Vector Potential, note also that $\text{div}(\text{curl}\boldsymbol{A}) = 0$) then closed surface integral is 0(Solenoidal). On the other hand on a Contractible Space $\text{div}\boldsymbol{F} = 0$(Incompressible) implies the same.

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Definition 17. *Let* $z = x + iy = re^{i\theta} = |z|e^{i(\theta+2n\pi)}, n \in \mathbb{Z}$ *then*
$\text{Arg } z = \theta$ *such that* $-\pi < \theta \leq \pi$
$\text{Log } z = \log|z| + i\text{Arg } z$*:Principal Logarithm*
$\sqrt{z} = \sqrt{|z|}e^{Argz/2}$

Definition 18. *Complex Limit*
$\lim_{z\to a} f(z) = L \Leftrightarrow \forall\epsilon > 0 \exists\delta > 0 \forall z, 0 < |z - a| < \delta \Rightarrow |f(z) - L| < \epsilon.$

Definition 19. *Differentiability*
Iff $\lim_{z\to a} \frac{f(z)-f(a)}{z-a} = \lim_{\Delta z\to 0} \frac{f(a+\Delta z)-f(a)}{\Delta z}$ *exists we say that* f *is differentiable* ($f \in \mathcal{D}$) *at* a *and denote its value by* $f'(a)$.

Theorem 11. *Iff* $f = u+iv \in \mathcal{D}$ *then* $u, v \in \mathcal{D}$ *and satisfy the Cauchy-Riemann(CR) Equations* $u_x = v_y, u_y = -v_x$.

Theorem 12. *Let* $f = u + iv \in \mathcal{D}$. *Then* $\nabla^2 u = 0$ *and* $\nabla^2 v = 0$.
Such functions are called Harmonic.

Definition 20. *Analytic(Holomorphic) Function* ($f \in \mathcal{A}$)
$f \in \mathcal{D}$ *on a neighbourhood of* a.
It follows that for $B \subset \mathbb{C}$, $f \in \mathcal{D}$ *on* $B \Leftrightarrow f \in \mathcal{A}$ *on* B

Definition 21. *Singular Points*
Non-Analytic points a *of* f *are called singular points.*
1. Isolated Singular Point: $\exists\delta > 0 \forall z, 0 < |z - a| < \delta \Rightarrow f \in \mathcal{A}$. *ie* f *is analytic on some punctured disk centered at* a. *There are three types as we will see below.*
2. Non-Isolated Singular Point: Singular points which are not isolated
2.1 Branch Cuts: Ex. $\text{Arg } z, \text{Log } z, \sqrt{z}$ *along the non-positive real axis.*
2.2 Other: Ex. $\tan\frac{1}{z}$ *at* $z = 0$.

Definition 22. *Let* C *be a path given by* $z(t) = x(t) + iy(t)$ *for* $t \in [a, b]$ *and* $f(z) = u(x, y) + iv(x, y)$. *Then the complex integral* $\int_C f(z)dz$ *is defined as the line integral* $\int_a^b f(z)\frac{dz}{dt}dt = \int_C(udx - vdy) + i\int_C(vdx + udy)$.

Theorem 13. *Let* C *be a loop in a simply connected region.*
Iff $f \in \mathcal{A}$ *then* $\oint_C f(z)dz = 0$.

Theorem 14. *Cauchy Integral Formula*
Let $f \in \mathcal{A}$ *and* C *be a loop in a simply connected region and* a *be a point inside* C. *Then* $f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz$.

Theorem 15. *Mean Value Property*
Let $f \in \mathcal{A}$ *and* C *and* D *be the circle and disk with center* a *and radius* R. *Then* $f(a) = \frac{1}{2\pi}\int_0^{2\pi} f\left(a + Re^{i\theta}\right)d\theta = \frac{1}{2\pi R}\int_C f d\ell$ *and* $f(a) = \frac{1}{\pi R^2}\iint_D f ds$.
Similar properties hold for Harmonic functions.

Theorem 16. *Let* $f \in \mathcal{A}$ *and* C *be a loop in a simply connected region and* a *be a point inside* C. *Then* $f^{(k)}(a) = \frac{k!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{k+1}}dz$.

Theorem 17. *$f \in \mathcal{A} \Rightarrow f \in \mathcal{C}^\infty$. i.e Analytic functions are infinitely differentiable.*

Theorem 18. *Taylor Series*
Let $f \in \mathcal{A}$ in the region $|z-a| < R$ and let C be a loop in that region. Then we have $f(z) = \sum_{k=0}^{\infty} a_k (z-a)^k$ where $a_k = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{k+1}} dz = \frac{f^{(k)}(a)}{k!}$.
$\sup R$ is called the Radius of Convergence and the corresponding region is called the Region of Convergence.

Theorem 19. *Laurent Series*
Let $f \in \mathcal{A}$ in the region $R_1 < |z-a| < R_2$ and let C be a loop in that region. Then we have $f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k$ where $a_k = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{k+1}} dz$.
$\sup R_2$ and $\inf R_1$ corresponds to the Region of Convergence.
We also have $a_{-1} = \frac{1}{2\pi i} \oint_C f(z) dz$. If a is an isolated singular point of f then we call a_{-1} as $\operatorname{Res}(f, a)$, Residue of f at a.

Definition 23. *Further classification of Isolated Singularities.*
If a is an isolated singularity of f we can find the Laurent Series expansion
$f(z) = \sum_{-\infty}^{\infty} a_k (z-a)^k$ valid for $0 < |z-a| < R_2$
1.1 Removable Singularity: $a_k = 0$ for all $k < 0$.
1.2 Pole of Order n: $a_{-n} \neq 0$ and $a_k = 0$ for all $k < -n$.
1.3 Essential Singularity: $a_k = 0$ for an infinite number of $k < 0$.

Theorem 20. *Let a be an isolated singularity of f.*
Then $|f(z)| \to \infty$ as $z \to a$ iff a is a pole.

Theorem 21. *If a is a pole of order n of f, then*
$\operatorname{Res}(f, a) = \frac{1}{(n-1)!} \lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} f(z)(z-a)^n$

Theorem 22. *Let b_j are isolated singularities of f which are the only singularities of f inside the loop C. Then $\oint_C f(z) dz = \sum_j \operatorname{Res}(f, b_j)$.*

Theorem 23. *Complex Inversion Formula for Laplace Transform*
Let the Laplace Transform $F(s)$ be valid for $\operatorname{Re} s > a$. Then the inverse Laplace transform $f(t)$ is given by
$f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(z) e^{zt} dz =$ Sum of Residues of $F(z)e^{zt}$ for $\operatorname{Re} z \leq a$.

Theorem 24. *Maximum Modulus Principle*
Let $f \in \mathcal{A}$ on D and not a constant. Then the maximum of $|f(z)|$ occurs on the boundary of D.
Similar properties hold for Harmonic functions.

Theorem 25. *Louville's Theorem*
Let $f \in \mathcal{A}$ and bounded on $\mathbb{C}$ then f is a constant.
Iff $f \in \mathcal{A}$ on $\mathbb{C}$ then f is called Entire.

Theorem 26. *Fundamental Theorem of Algebra*
Every polynomial of degree larger than zero has a root.

Theorem 27. *Two curves mapped by an analytic function with a non-zero derivative preserves the angle between them.*
Such mappings are called Conformal.