TH1: Consider the Fresnel Integral $S(x) = \int_0^x \sin(t^2)dt$ appearing in Optics.

a) Use Simpsons method to find $S(1)$ accurate to 0.001.

b) Let $T_n(t^2, 0)$ be the Taylor polynomial of $\sin(t^2)$ at 0 , taking the variable as t^2. We can also approximate $S(1)$ by integrating $T_n(t^2, 0)$. Use the Taylor series remainder to find the value of n so that the approximate answer is accurate to 0.001. Also find this approximate answer.

For the section below you can use calculator to integrate using the $\int$ key, make sure that you are in radian mode. Also note that if $F(x) = \int_a^x f(t)dt$ and f is a continuous function then F is differentiable and $F'(x) = f(x)$. We are interested in finding the real solutions to $S(x) = 1 - x$.

c) Let $f(x) = S(x) + x - 1$. Show that the interval $[0, 1]$ contains a solution to $f(x) = 0$. Use Newtons method to write five iterations starting from $x_0 = 0.5$. No need to check the conditions for convergence.

d) Let $g(x) = 1 - S(x)$. Show that we can find a solution to $g(x) = x$ using the Fixed Point Method on $[0, 1]$. Find an upper bound for the number of iterations to get the solution accurate to 0.001 starting from $x_0 = 0.5$. Also find this approximate solution.

TH2: Consider the Associated Laguerre ODE: $xy'' + (m + 1 - x)y' + ny = 0$

a) Find the solution using power series. When $n \in \mathbb{Z}^+$, the polynomial solutions is called Associated Laguerre Polynomials $L_n^m(x)$.

b) When $m = 0$ we get the Laguerre ODE: $xy'' + (1 - x)y' + ny = 0$. When $n \in \mathbb{Z}^+$, the polynomial solutions is called Associated Laguerre Polynomials $L_n(x) = L_n^0(x)$ and satisfies the Rodrigue's fromula $L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(e^{-x}x^n)$. Find the first five Laguerre polynomials.

c)For $f, g \in \mathcal{C}[0, \infty)$ with the Inner Product $< f, g >= \int_0^\infty e^{-x} f(x)g(x)dx$, Laguerre polynomials satisfies the Orthogonality Condition:
$\int_0^\infty e^{-x} L_m(x)L_n(x)dx = 0$ if $m \neq n$ and 1 if $m = n$.
Find the coefficients a_k of the Laguerre Series: $f(x) = \sum_{k=0}^\infty a_k L_k(x)$ and write the first 5 terms for $f(x) = \sin x$.

TH3: Use Laplace Transform and Complex Inverse Laplace Trasfrom to Solve the Wave equation

$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$

$u(0, t) = 0, u(\pi, t) = 0; t \geq 0$

$u(x, 0) = \sin x, \frac{\partial u}{\partial t}(x, 0) = x^2; 0 \leq x \leq \pi.$