# 1 Second Order Linear ODE: $y'' + p(x)y' + q(x)y = r(x)$

Solution $y = Au(x) + Bv(x) + z(x)$.
Where $u(x), v(x)$ are Linearly Independent
(i.e. $\forall A, B(\forall x(Au(x) + Bv(x) = 0)) \Rightarrow A = B = 0$) solutions to the Homogeneous Equation $y'' + p(x)y' + q(x)y = 0$. Moreover the solution space to the Homogeneous Equation forms a 2 Dimensional Vector Space Over $\mathbb{R}$ and $u, v$ are Basis Vectors which are not unique.
Any solution to the Homogeneous Equation can be written as a Linear Combination of the Basis Vector: i.e. Complimentary Solution $y_c = Au(x) + Bv(x)$.
The Particular Solution $y_p = z(x)$ is the constant-less solution to the Non Homogeneous Equation: $y'' + p(x)y' + q(x)y = r(x)$.
The constants $A, B$ can be uniquely determined by the Initial Conditions: $y(b), y'(b)$.

## 1.1 Wronskian Method

Wronskian $W(x) = \left|\begin{pmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{pmatrix}\right| = u(x)v'(x) - v(x)u'(x)$
$u, v$ linearly independent is equivalent to $\forall x(W(x) \neq 0)$.
Wronskian satisfies the 1st order homogeneous equation: $W'(x) + p(x)W(x) = 0$. So we can find $W(x)$ and if we know one solution $u(x)$ to the homogeneous equation we can find the other solution $v(x)$. This result is equivalent to the Reduction of Order method where we assume that $v(x) = A(x)u(x)$ and finding a 1st order ode satisfied by $A(x)$.
Also we can show that the particular solution can be written as $z(x) = A(x)u(x) + B(x)v(x)$. This method is called the Variation of Parameters and we can show that $A(x) = -\frac{v(x)r(x)}{W(x)}$ and $B(x) = \frac{u(x)r(x)}{W(x)}$.
So we can generate the full solution if we know only one solution $u(x)$ to the homogeneous equation.

## 1.2 Frobenius Method

Consider the 2nd order linear homogeneous ODE, $y'' + p(x)y' + q(x)y = 0$. Here we try to find at least one solution $u(x)$ as a power series of $(x - b)$. It is convenient to find the constants $A, B$ when the initial conditions $y(b), y'(b)$ are stated at $b$.
Let $p(x) = \sum_{k=0}^{\infty} p_{k-1}(x - b)^{k-1}$ and $q(x) = \sum_{k=0}^{\infty} p_{k-2}(x - b)^{k-2}$. We are looking for a solution of the form $y = \sum_{k=0}^{\infty} a_k(x - b)^{k+c}$. We need to find both $a_k$ and $c$. Any coefficients $a_k$ remain indeterminate will become the constants $A, B$. Here $c$ can provides powers of $(x - b)$ which are not integer and also help generate both solutions $u, v$ by the same procedure.
We have $y' = \sum_{k=0}^{\infty} a_k(k+c)(x-b)^{k+c-1}$ and $y'' = \sum_{k=0}^{\infty} a_k(k+c)(k+c-1)(x-b)^{k+c-2}$.
Substituting into the ODE we have,

$$\sum_{k=0}^{\infty} a_k(k + c)(k + c - 1)(x - b)^{k+c-2}$$
$$+ (\textstyle\sum_{i=0}^{\infty} p_{i-1}(x - b)^{i-1})(\sum_{j=0}^{\infty} a_j(j + c)(x - b)^{j+c-1})$$

$+ (\sum_{i=0}^{\infty} q_{i-2}(x-b)^{i-2})(\sum_{j=0}^{\infty} a_j(x-b)^{j+c}) = 0$

$\sum_{k=0}^{\infty} a_k(k+c)(k+c-1)(x-b)^{k+c-2}$
$+ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_j(j+c)p_{i-1}(x-b)^{i+j+c-2}$
$+ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_j q_{i-2}(x-b)^{i+j+c-2} = 0$

Letting $i+j=k$ and writing $i=k-j$ we arrive at
$\sum_{k=0}^{\infty} a_k(k+c)(k+c-1)(x-b)^{k+c-2}$
$+ \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} a_j(j+c)p_{k-j-1}\right)(x-b)^{k+c-2}$
$+ \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} a_j q_{k-j-2}\right)(x-b)^{k+c-2} = 0$

$\sum_{k=0}^{\infty}\left(a_k(k+c)(k+c-1) + \sum_{j=0}^{k} a_j((j+c)p_{k-j-1} + q_{k-j-2})\right)(x-b)^{k+c-2} = 0$

So we want for all $k \geq 0$
$0 = a_k(k+c)(k+c-1) + \sum_{j=0}^{k} a_j((j+c)p_{k-j-1} + q_{k-j-2})$
$= a_k((k+c)(k+c-1+p_{-1}) + q_{-2}) + \sum_{j=0}^{k-1} a_j((j+c)p_{k-j-1} + q_{k-j-2})$
$= a_k((k+c)^2 + (-1+p_{-1})(k+c) + q_{-2}) + \sum_{j=0}^{k-1} a_j((j+c)p_{k-j-1} + q_{k-j-2})$

For $k=0$ we have $a_0(c^2 + (-1+p_{-1})c + q_{-2}) = 0$.
To keep $a_0$ as a constant we need $c$ to satisfy the Indicial Equation:
$c^2 + (-1+p_{-1})c + q_{-2} = 0$. Let the roots be $\alpha, \beta$.
Note that the quadratic equation associated with the coefficient of $a_k$ is $(k+c)^2 + (-1+p_{-1})(k+c) + q_{-2}$ which has roots $\alpha - k$ and $\beta - k$. The two quadratic equations can have a common root, which occurs when $\beta - k = \alpha$ or when $\alpha - k = \beta$. If $\alpha < \beta$ this occurs when $\alpha + k = \beta$. So this happens when the roots of the indicial equation is differ by an integer and when we put $c = \alpha$ which is the smaller root. We can think about 3 cases

Case 1: $\alpha \neq \beta$ and are not differ by an integer
In this case $a_k$ can be expressed by $a_0$ only.
We can have two independent solutions by substituting $c = \alpha, \beta$.

Case 2: $\alpha \neq \beta$ and are differ by an integer $k = n$.
In this case $a_n$ can be expressed by $a_0$ only for the larger root $c = \beta$ and we get one solution.
But we cannot express $a_n$ by $a_0$ for the smaller root $c = \alpha$. So if the equations does not force $a_0 = 0$, then all coefficients are expressible by $a_0$ and $a_n$. Which means you end up with two solutions followed by two constants $a_0, a_n$ if the second one is not merely a shift of index of the first one.
But if the equations force $a_0 = 0$ you end up expressing all coefficients by $a_n$ which means you get one solution which can still be a shift of index to the $c = \alpha$ case.

Case 3: $\alpha = \beta$
In this case coefficient of $a_k$ will never be 0. So close to $c = \alpha$ we can force $a_k((k+c)^2 + (-1+p_{-1})(k+c) + q_{-2}) + \sum_{j=0}^{k-1} a_j((j+c)p_{k-j-1} + q_{k-j-2}) = 0$ for all $k \geq 1$ and all coefficients can be expressible by $a_0$ and $c$. We end up with a different differential equation
$y'' + p(x)y' + q(x)y = a_0(c-\alpha)^2 x^{c-2}$ with $y = y(x,c)$ be a solution.
Note that $y = y(x,\alpha)$ is a solution. We will show that $y_c(x,\alpha)$ is the second independent solution.
In this context $\frac{d}{dx} = \frac{\partial}{\partial x}$ and differentiating both sides partially wrt $c$ and with $y \in \mathcal{C}^3$ we have: $\frac{\partial}{\partial c}\frac{\partial^2 y}{\partial x^2} + p(x)\frac{\partial}{\partial c}\frac{\partial y}{\partial x} + q(x)\frac{\partial y}{\partial c} = \frac{\partial^2}{\partial x^2}\left(\frac{\partial y}{\partial c}\right) + p(x)\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial c}\right) + q(x)\left(\frac{\partial y}{\partial c}\right) = 2a_0(c-\alpha)x^{c-2} + a_0(c-\alpha)^2 x^{c-2}\log x$
Now putting $c = \alpha$ we have: $\frac{\partial^2}{\partial x^2}\left(\frac{\partial y}{\partial c}|_{c=\alpha}\right) + p(x)\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial c}|_{c=\alpha}\right) + q(x)\left(\frac{\partial y}{\partial c}|_{c=\alpha}\right) = 0$.
This means $\frac{\partial y}{\partial c}|_{c=\alpha} = y_c(x,\alpha)$ is the second solution.

### 1.3 Radius of Convergence

Power series are functions of the form $f(x) = \sum_{k=0}^{\infty} a_k(x-b)^k$. They converge Absolutely and Uniformly for $|x-b| < R$ and diverge for $|x-b| > R$ for some $R \geq 0$. $R$ is called the radius of convergence and we can use the Root test to find it: $R = \frac{1}{\lim_{k\to\infty}|a_k|^{1/k}}$. Let $f, g$ are power series with radii of convergence $R_f, R_g$.
$f+g$ is a power series $(f+g)(x) = \sum_{k=0}^{\infty} A_k(x-b)^k$ where $A_k = a_k + b_k$.
$fg$ is a power series defined as the Cauchy Product $(fg)(x) = \sum_{k=0}^{\infty} A_k(x-b)^k$ where $A_k = \sum_{j=0}^{k} a_j b_{k-j}$. Both the sum and product power series converges with a radius of convergence $\min(R_f, R_g)$.
With $p(x) = \sum_{k=0}^{\infty} p_{k-1}(x-b)^{k-1}$ and $q(x) = \sum_{k=0}^{\infty} p_{k-2}(x-b)^{k-2}$ note that $(x-b)p(x)$ and $(x-b)^2 q(x)$ have power series and have radius of convergence $R_p$ and $R_q$ respectively. Note that radii of convergence of $y, y', y''$ are all equal and be $R$. Since we are looking for solutions to $y'' + p(x)y' + q(x)y = 0$ or $y'' = -p(x)y' - q(x)y$ we have: $R = \min(\min(R_p, R), \min(R_q, R))$. $R = \min(R_p, R_q)$ satisfies this equation.

### 1.4 Legendre Equation

Legendre ODE $(1-x^2)y'' - 2xy' + n(n+1)y = 0$
Solution: $y = AP_n(x) + BQ_n(x)$
$P_n(x)$:Legendre functions of the first kind(Legendre polynomials when $n \in \mathbb{Z}^+$)
$Q_n(x)$:Legendre functions of the second kind
$P_n(x)$ satisfies the Rodrigue's formula: $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left((x^2-1)^n\right)$
For $f, g \in \mathcal{C}[-1,1]$ with the Inner Product $< f, g >= \int_{-1}^{1} f(x)g(x)dx$, Legendre polynomials satisfies the Orthogonality Condition: $\int_{-1}^{1} P_m(x)P_n(x)dx = 0$ if $m \neq n$ and $\frac{2}{2n+1}$ if $m = n$.
This allows us to write the Legendre Series: $f(x) = \sum_{k=0}^{\infty} a_k P_k(x)$ where $a_k = \frac{2k+1}{2}\int_{-1}^{1} f(x)P_k(x)dx$.

### 1.5 Associated Legendre Equation

Associated Legendre ODE $(1-x^2)y'' - 2xy' + \left(n(n+1) - \frac{m^2}{1-x^2}\right) y = 0$
Solution: $y = AP_n^m(x) + BQ_n^m(x)$
$P_n^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x)$:Associated Legendre functions of the first kind(Associated Legendre polynomials when $n \in \mathbb{Z}^+$)
$Q_n(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} Q_n(x)$:Associated Legendre functions of the second kind.
Also see the Mathematica file: Legendre.nb

### 1.6 Gamma Function and Pochhammer Symbol

Gamma Function: $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, x > 0$.
Gamma function satisfies use the formula $\Gamma(x+1) = x\Gamma(x)$ which implies $n! = \Gamma(n+1)$ for a none negative integer $n$.
We define the Gamma function for $x < 0$ by $\Gamma(x) = \frac{\Gamma(x+1)}{x}$.
Since $\Gamma(x) \to \infty$ as $x \to 0^+$ we have $|\Gamma(x)| \to \infty$ as $x \to -n$ where $n \in \mathbb{Z}^+$.
Pochhammer symbol: $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$. Note that $(1)_n = n!$ for none negative integer $n$. We can show that $(\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$.

### 1.7 Bessel Equation

Bessel ODE $x^2y'' + xy' + (x^2 - \nu^2)y = 0$
Solution: $y = AJ_\nu(x) + BY_\nu(x)$
$J_\nu(x) = \sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma(k+1+\nu)} \left(\frac{x}{2}\right)^{2k+\nu}$: Bessel Function of the First Kind
If $\nu \notin \mathbb{Z}$ the solution is $y = AJ_\nu(x) + BJ_{-\nu}(x)$.
When $\nu = n \in \mathbb{Z}$ we have $J_n(x) = (-1)^n J_{-n}(x)$, so $J_{-n}(x)$ is not the second independent solution. Note that $\cos n\pi = (-1)^n$ for $n \in \mathbb{Z}$. Motivated by this we define a second solution for $\nu \notin \mathbb{Z}$:
$Y_\nu(x) = \frac{J_\nu(x)\cos(\pi\nu) - J_{-\nu}(x)}{\sin(\pi\nu)}, \nu \in \mathbb{Z}$: Bessel Function of the second Kind
When $\nu = n \in \mathbb{Z}$ we use the L'Hopital Rule and define:
$Y_n(x) = \frac{1}{\pi}\left(\frac{\partial}{\partial\nu} J_\nu(x)|_{\nu=n} - (-1)^n \frac{\partial}{\partial\nu} J_{-\nu}(x)|_{\nu=n}\right)$.
Let $x = \lambda_k, k \geq 0$ be roots of $J_\nu(ax) = 0$. For $f, g \in \mathcal{C}^1[0,a]$ with the Inner Product $< f, g >= \int_0^a xf(x)g(x)dx$, Bessel Functions satisfies the Orthogonality Condition: $\int_0^a xJ_\nu(\lambda_m x)J_\nu(\lambda_n x)dx = 0$ if $m \neq n$ and $\frac{a^2}{2}\left(J'_\nu(\lambda_n a)\right)^2$ when $m = n$.
This allows us to write the Bessel Series: $f(x) = \sum_{k=0}^\infty a_k J_\nu(\lambda_k x)$ where $a_k = \frac{2}{a^2(J'_\nu(\lambda_k a))^2} \int_0^a xf(x)J_\nu(\lambda_k x)dx$.
Also see the Mathematica file: Bessel.nb

**Definition 1.** *$L^p$ spaces and norms*
$\|f\|_p = \left(\int_{-\infty}^{\infty} |f(t)|^p dt\right)^{\frac{1}{p}}, 1 \leq p < \infty,\ L^p(\mathbb{R}) = \{f : \|f\|_p < \infty\}$
$\|f\|_\infty = \sup\{|f(x)| : x \in \mathbb{R}\},\ L^\infty(\mathbb{R}) = \{f : \|f\|_\infty < \infty\}$

**Definition 2.** *Schwartz spaces and norms*
$\|f\|_{m,n} = \sup\{|x^m f^n(x)| : x \in \mathbb{R}\}$
$\mathcal{S} = \{f \in \mathcal{C}^\infty : \forall m, n \in \mathbb{N}, \|f\|_{m,n} < \infty\}$
*Note that if $f \in \mathcal{S}$ then $x^m f^n(x) \to 0$ as $x \to \pm\infty$ for all $m, n \in \mathbb{N}$*

**Theorem 1.**
$L^2 \subset L^1 \subset L^\infty$
$\mathcal{S} \subset L^p, 1 \leq p \leq \infty$

**Definition 3.**
*Fourier Transform of $f$: $\mathcal{F}(f)(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$*
*Inverse Fourier Transform of $F$: $\mathcal{F}^{-1}(F)(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega t}d\omega$*

**Theorem 2.**
*If $f \in L^1(\mathbb{R}) \Rightarrow \mathcal{F}(f)(\omega)$ exists, continuous and $\to 0$ as $\omega \to \pm\infty$*
*If $f \in \mathcal{S}$ then $\mathcal{D}(f) = f' \in \mathcal{S}$*
*If $f \in \mathcal{S}$ then $\mathcal{F}(f) \in \mathcal{S}$ and $\mathcal{F}^{-1}\mathcal{F}(f) = f$ so $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is a bijection*
*If $F \in \mathcal{S}$ then $\mathcal{F}^{-1}(F) \in \mathcal{S}$ and $\mathcal{F}\mathcal{F}^{-1}(F) = F$ so $\mathcal{F}^{-1} : \mathcal{S} \to \mathcal{S}$ is a bijection*

**Definition 4.**
*Inner Product, $< f, g >= \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$. So $\|f\|_2 = \sqrt{< f, f >}$*
*$(f, g) = \int_{-\infty}^{\infty} f(x)g(x)dx$. So $(f, g) =< f, \overline{g} >$*
*Convolution, $(f * g)(t) = \int_{-\infty}^{\infty} f(x)g(t - x)dx$*

**Theorem 3.** *$f, g \in \mathcal{S}$*
$\int_{-\infty}^{\infty} f'(x)g(x)dx = -\int_{-\infty}^{\infty} f(x)g'(x)dx$, *or* $(f', g) = -(f, g')$
$(\mathcal{I}f, g) = (\mathcal{F}f)(0)(\mathcal{F}g)(0) - (f, \mathcal{I}g),\ (\mathcal{I}f)(x) = \int_{-\infty}^{x} f(t)dt$
$\mathcal{F}(e^{iat}f(t))(\omega) = i\omega\mathcal{F}(f)(\omega - a)$
$\mathcal{F}(f')(\omega) = i\omega\mathcal{F}(f)(\omega)$
$\mathcal{F}(f'')(\omega) = -\omega^2\mathcal{F}(f)(\omega)$
$\int_{-\infty}^{\infty} \mathcal{F}f(x)g(x)dx = \int_{-\infty}^{\infty} f(x)\mathcal{F}g(x)dx$, *or* $(\mathcal{F}f, g) = (f, \mathcal{F}g)$
$\int_{-\infty}^{\infty} \mathcal{F}^{-1}f(x)g(x)dx = \int_{-\infty}^{\infty} f(x)\mathcal{F}^{-1}g(x)dx$, *or* $(\mathcal{F}^{-1}f, g) = (f, \mathcal{F}^{-1}g)$
$\int_{-\infty}^{\infty} |f(x)|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathcal{F}f(x)|^2dx$ *or* $\|f\|_2 = \frac{1}{\sqrt{2\pi}}\|\mathcal{F}f\|_2$

**Note 1.** *We are going to take $f : \mathbb{R} \to \mathbb{C}$ in $\mathcal{S}$ and going to convert it into a function $f : \mathcal{S} \to \mathbb{C}$ by $f(g) = (f, g) = \int_{-\infty}^{\infty} f(x)g(x)dx$ for $g \in \mathcal{S}$. We note that*
*1.$f$ is continuous.*
*Equivalently if $g_n \to g$ in $\mathcal{S}$ then $f(g_n) \to f(g)$ i.e. $(f, g_n) \to (f, g)$*
*2.$f$ is linear.*
*Equivalently if $g, h \in \mathcal{S}$ and $a, b \in \mathbb{C}$ then $f(ag + bh) = af(g) + bf(h)$ i.e. $(f, ag + bh) = a(f, g) + b(f, h)$*

*Now consider the four operators defined for $f \in \mathcal{S}$:*
*Differential operator $\mathcal{D} : (\mathcal{D}f)(x) = f'(x)$*
*Fourier Transform $\mathcal{F}$ :*
*Inverse Fourier Transform $\mathcal{F}^{-1}$ :*
*Integral operator $\mathcal{I} : (\mathcal{I}f)(x) = \int_{-\infty}^{x} f(t)dt$*
*It can be shown that all the above operations except the last one land on $\mathcal{S}$. In other words if $f \in \mathcal{S}$ then $\mathcal{D}f, \mathcal{F}f, \mathcal{F}^{-1}f \in \mathcal{S}$.*
*We also have noted that all the following are true*
*3. $(\mathcal{D}f)(g) = -f(\mathcal{D}g)$ i.e. $(\mathcal{D}f, g) = -(f, \mathcal{D}g)$*
*4. $(\mathcal{F}f)(g) = f(\mathcal{F}g)$ i.e. $(\mathcal{F}f, g) = (f, \mathcal{F}g)$*
*5. $(\mathcal{F}^{-1}f)(g) = f(\mathcal{F}^{-1}g)$ i.e. $(\mathcal{F}^{-1}f, g) = (f, \mathcal{F}^{-1}g)$*
*6. $(\mathcal{I}f)(g) = (\mathcal{F}f)(0)(\mathcal{F}g)(0) - f(\mathcal{I}g)$ i.e. $(\mathcal{I}f, g) = (\mathcal{F}f)(0)(\mathcal{F}g)(0) - (f, \mathcal{I}g)$*
*Now we are going to extend the class of functions $\mathcal{S}$ to include "functions" which are not cooked out of $f \in \mathcal{S}$ as in the above process.*

**Definition 5.** *a Tempered Distribution $f \in \mathcal{T}$ is a function $f : \mathcal{S} \to \mathbb{C}$ .*
*Satisfying all the above properties 1 and 2: Continuous and Linear.*
*We see form the above discussion that $\mathcal{S} \subset \mathcal{T}$.*
*For $g \in \mathcal{S}$, we will still write $f(g) = (f, g)$ as necessary although there is no $(f, g)$ realized as an integral*
*We can use the formulas 3 through 6 to define the operations $\mathcal{D}, \mathcal{F}, \mathcal{F}^{-1}, \mathcal{I}$ on $\mathcal{T}$.*
*It can be also shown now that all the 4 operations land on $\mathcal{T}$. In other words if $f \in \mathcal{T}$ then $\mathcal{D}f, \mathcal{F}f, \mathcal{F}^{-1}f, \mathcal{I}f \in \mathcal{T}$ so $\mathcal{D}, \mathcal{F}, \mathcal{F}^{-1}, \mathcal{I} : \mathcal{T} \to \mathcal{T}$.*

**Definition 6.**
*Delta function $\delta \in \mathcal{T}$ and $\delta(f) = (\delta, f) = f(0)$*
*Heaviside function $H(t) = 1$ if $t \geq 0$ and $0$ otherwise*
*Rectangle Function $\Pi(t) = 1$ if $-\frac{1}{2} \leq t \leq \frac{1}{2}$ and $0$ otherwise.*
*Sinc Function $sincx = \frac{\sin x}{x}$*

**Theorem 4.**
$\mathcal{F}\delta = 1$
$\mathcal{F}1 = 2\pi\delta$
$\mathcal{D}H = \delta$
$\mathcal{I}\delta = H$
$\mathcal{F}(e^{iat})(\omega) = 2\pi\delta(\omega - a)$
$\mathcal{F}(\sin at)(\omega) = \frac{\pi}{i}\left(\delta(\omega - a) - \delta(\omega + a)\right)$
$\mathcal{F}(\cos at)(\omega) = \pi\left(\delta(\omega - a) + \delta(\omega + a)\right)$
$\mathcal{F}(\Pi)(\omega) = sinc(\omega/2)$
$\mathcal{F}(f * g)(\omega) = \mathcal{F}(f)(\omega)\mathcal{F}(g)(\omega)$

**Example 1.** *Use Fourier Transform to Solve the Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.*
*$u(x, 0) = f(x)$, $u(x, y)$ bounded as $y \to \infty$.*
*Compute the answer for $f(x) = x$ for $0 < x < 1$ and $0$ otherwise.*
*Do the same question with the given boundary condition replaced by $\frac{\partial u}{\partial x}(x, 0) = f(x)$*