RIEMANN INTEGRAL

Example: Use n equal partitions of $[0,1]$ to estimate the "area" under the curve $f(x) = x^2$ using

1. left corner of the intervals
2. right corner of the intervals
3. midpoint of the interval
4. line joining the left and right corners of the interval

Definitions:
P is a **partition** of $[a, b]$ iff it is a ordered set of the form $P = \{x_0, x_1, \cdots, x_n\}$ with $x_0 = a, x_n = b$ and $x_{k+1} > x_k$
P^* is a **refinement** of P iff $P^* \supseteq P$
P is a **common refinement** of P_1, P_2 iff $P = P_1 \cup P_2$
$\mathcal{P}[a, b]$ is the set of all partitions of $[a, b]$

Definition: Upper and Lower Riemann Sums $f: [a, b] \to \mathbb{R}$ is a bounded function, $\Delta x_k = x_{k+1} - x_k$
$U(P, f) = \sum_{k=0}^{n-1} M_k(f)\Delta x_k$ where $M_k(f) = \sup\{f(x) \,|x \in [x_k, x_{k+1}]\}$
$L(P, f) = \sum_{k=0}^{n-1} m_k(f)\Delta x_k$ where $m_k(f) = \inf\{f(x) \,|x \in [x_k, x_{k+1}]\}$

Definition: **Upper and Lower Riemann Integrals**
$U(f) = \inf\{U(P, f)| P \in \mathcal{P}[a, b]\}$
$L(f) = \sup\{L(P, f)| P \in \mathcal{P}[a, b]\}$

Definition:
f is **Riemann Integrable** on $[a, b]$ or $f \in \mathfrak{R}[a, b]$ iff $U(f) = L(f)$
Riemann Integral of f is the common value denoted by $\int_a^b f(x)dx$

Theorem: P^* is a refinement of P

1. $L(P, f) \leq L(P^*, f)$
2. $U(P^*, f) \leq U(P, f)$

Theorem: $L(f) \leq U(f)$

Theorem: $f \in \mathcal{R}[a, b]$ iff $\forall \varepsilon > 0 \, \exists P \in \mathcal{P}[a, b]; \; U(P, f) - L(P, f) < \varepsilon$

Theorem: If $f \in \mathcal{R}[a, b]$ and $P \in \mathcal{P}[a, b]$ such that $t_i \in [x_{i-1}, x_i]$ then
$\left|\sum_{i=1}^{n} f(t_i)\Delta x_i - \int_a^b f(x)dx\right| < U(P, f) - L(P, f)$

Theorem: $f \in \mathcal{C}[a, b] \Rightarrow f \in \mathcal{R}[a, b]$

Theorems: $f, g \in \mathcal{R}[a, b]$

1. $f + g \in \mathcal{R}[a, b]$ and $\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
2. $fg \in \mathcal{R}[a, b]$
3. $|f| \in \mathcal{R}[a, b]$ and $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$
4. $f \leq g \Rightarrow \int_a^b f(x)dx \leq \int_a^b g(x)dx$
5. $f \leq M \Rightarrow \int_a^b f(x)dx \leq M(b - a)$
6. $c \in [a, b] \Rightarrow f \in \mathcal{R}[a, c], f \in \mathcal{R}[c, b]$ and $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Definition: $f(x)$ is **Uniformly continuous** on $I \subset \mathbb{R}$
$\forall \varepsilon > 0, \exists \delta > 0, \forall x_1, x_2 \in I; \; |x_1 - x_2| < \delta \Longrightarrow |f(x_1) - f(x_2)| < \varepsilon$

Definition: $f(x)$ is **Lipschitz continuous** on $I \subset \mathbb{R}$
$\exists L > 0, \forall x_1, x_2 \in I; \; |f(x_1) - f(x_2)| \leq L|x_1 - x_2|$

Theorem: Lipschitz continuous ⇒ Uniformly continuous ⇒ Continuous
Example: Show that $\frac{1}{x}$ is not uniformly continuous on $(0,1]$ but x^2 is.

Theorem: Fundamental Theorem of Calculus
If $f \in \mathcal{R}[a,b]$ and there is a differentiable function F such that $F' = f$ then
$\int_a^b f(x)dx = F(b) - F(a)$

Theorem: Second Fundamental Theorem of Calculus
If $f \in \mathcal{R}[a,b]$ and $x \in [a,b]$ and $F(x) = \int_a^x f(x)dx$ then
1. F is continuous on $[a,b]$.
2. If f is continuous at a point $x_0 \in [a,b]$ then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Theorem: Integration by Parts
F, G differentiable on $[a,b]$, $F' = f \in \mathcal{R}[a,b]$and $G' = f \in \mathcal{R}[a,b]$ $then$
$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx$

Theorem: Change of Variable
g has continuous derivative g' on $[c,d]$. f is continous on $g([c,d])$and let $F(x) = \int_{g(c)}^x f(t)dt, x \in g([c,d])$. Then for each $x \in [c,d]$, $\int_c^x f(g(t))g'(t)dt$ exixts and has value $F(g(x))$.

Theorem: Mean Value Theorem for Integrals
$f \in \mathcal{R}[a,b]$with $f([a,b]) = [m,M]$. Then $\exists\, c \in [m,M]$ such that $\int_a^b f(x)dx = c(b-a)$.
If also $f \in \mathcal{C}[a,b]$ then $\exists\, x_0 \in (a,b)$ such that $\int_a^b f(x)dx = f(x_0)(b-a)$.

Definition: Improper Integrals of the first kind
Suppose $\int_a^b f(x)dx$ exixts for each $b \geq a$.
If $\lim_{b\to\infty} \int_a^b f(x)dx$ exists and equal to $I \in \mathbb{R}$ we say that $\int_a^\infty f(x)dx$ converges and has value I
Otherwise we say that $\int_a^\infty f(x)dx$ diverges

Definition: Improper Integrals
$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx, f:[a,\infty) \to \mathbb{R}$
$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx, f:(-\infty,b] \to \mathbb{R}$
$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx,\ f:(-\infty,\infty) \to \mathbb{R}, c \in \mathbb{R}$
$\int_{a^+}^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx, f:(a,b] \to \mathbb{R}$
$\int_a^{b^-} f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx\,, f:[a,b) \to \mathbb{R}$
$\int_a^b f(x)dx = \int_a^{c^-} f(x)dx + \int_{c^+}^b f(x)dx, f:[a,c) \cup (c,b] \to \mathbb{R}, c \in (a,b)$

Example: Find $\int_{-1}^1 \frac{1}{x^2}dx$ if it exists

Example:
Prove that if f is bounded above and increasing, then $\lim_{x\to\infty} f(x)$ is existing and finite
Prove that $\int_a^\infty |f(x)|dx$ converges $\Longrightarrow \int_a^\infty f(x)dx$ converges
Prove that if $|f(x)| \leq Me^{ax}$, then the **Laplace Transform** of $f(x)$, $\overline{f}(s) = \int_0^\infty e^{-sx} f(x)dx$ exists for all $s > a$.

Theorem: Comparison Test
Assume that the proper integral $\int_a^b f(x)dx$ exists for each $b \geq a$ and suppose that $0 \leq f(x) \leq g(x)$ for all $x \geq a$, then $\int_a^\infty g(x)dx$ converges $\Longrightarrow \int_a^\infty f(x)dx$ converges

Theorem: Limit Comparison Test

Assume both proper integrals $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ exist for each $b \geq a$, where $f(x) \geq 0$ and $g(x) > 0$

If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \mathrm{c}$, then

1. $\mathrm{c} \neq 0, \infty \Rightarrow \int_a^\infty f(x)dx$ converges $\Leftrightarrow \int_a^\infty g(x)dx$ converges
2. $\mathrm{c} = 0$ and $\int_a^\infty g(x)dx$ converges $\Rightarrow \int_a^\infty f(x)dx$ converges
3. $\mathrm{c} = \infty$ and $\int_a^\infty g(x)dx$ diverges $\Rightarrow \int_a^\infty f(x)dx$ diverges

Note: There are similar comparison tests for other improper integrals

Example: **Gamma Function** is defined by $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$. Show that

1. $\Gamma(x)$ exists for all $x > 0$
2. $\Gamma(x) = (x-1)\Gamma(x-1)$
3. $\Gamma(n) = (n-1)!$ for integer $n \geq 1$
4. we can use 2. to define $\Gamma(x)$ for $x < 0$
5. $\Gamma(x)$ does not exist for $x = 0, -1, -2, -3, \ldots.$
6. Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and $\int_0^\infty e^{-t^2}dt = \sqrt{\pi}$ using $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$
7. Use the formula for the the n dimesional ball $V_n(r) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} r^n$ to find volumes of 2,3,4,5 dimesional balls
8. Use the fact that $\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x$ asmptotically as $t \to \infty$ to find $10!$ approximately
9. What is $-\Gamma'(1)$?. It is called the Euler Constant γ and no one knows if it is rational or irrational!

   Prove that the **Beta function** $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$ exists for all $x, y > 0$. It can be shown that $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

MULTIVARIATE CALCULUS

Definition: **Function of two variables** $f: A \subseteq \mathbb{R}^2 \to \mathbb{R}$

Example: Draw the graphs of the following functions/surfaces

1. $f(x,y) = x^2 + y^2$
2. $f(x,y) = \sqrt{x^2 + y^2}$
3. $\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 1$

Definition: **Limit**

$\lim_{(x,y)\to(a,b)} f(x,y) = L \Leftrightarrow \forall\varepsilon > 0 \exists\delta > 0, 0 < d\big((x,y),(a,b)\big) < \delta \Rightarrow |f(x,y) - L| < \varepsilon$

Note: **Matric**

$0 < d\big((x,y),(a,b)\big) < \delta$ is a region around and excluding (a,b). Some options for the matric d are

1. $\sqrt{(x-a)^2 + (y-b)^2}$
2. $|x-a| + |y-b|$
3. $\max\{|x-a|, |y-b|\}$

We will use the first matric. One can show that they are equivalent, what is needed is a region around (a,b).

Example: Use the definition to show that $\lim_{(x,y)\to(1,2)} x^2y = 6$

Example: Investigate the existence of the limit, $\lim_{(x,y)\to(0,0)} f(x,y)$ for the following functions

1. $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$
2. $f(x,y) = \begin{cases} \frac{x^2y^2}{x^2y^2+(x-y)^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$
3. $f(x,y) = \begin{cases} x\sin\frac{1}{y} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

Theorem:

If $\lim_{(x,y)\to(a,b)} f(x,y) = L$, $\lim_{x\to a} f(x,y)$ exists for all y and $\lim_{y\to b} f(x,y)$ exists for all x then $\lim_{x\to a}\lim_{y\to b} f(x,y) = \lim_{y\to b}\lim_{x\to a} f(x,y) = L$.

Theorem:

If $\lim_{x\to a} f(x,y)$ exists for all y and $\lim_{y\to b} f(x,y)$ exists for all x uniformly then $\lim_{x\to a}\lim_{y\to b} f(x,y) = \lim_{y\to b}\lim_{x\to a} f(x,y) = \lim_{(x,y)\to(a,b)} f(x,y)$.

Theorem:

If $\lim_{(x,y)\to(a,b)} f(x,y) = L$ and $g(x)$ is a continuous function s.t. $g(a) = b$, then $\lim_{x\to a} f\big(x, g(x)\big) = L$

Example:

Use the above theorem to prove that $\lim_{(x,y)\to(0,0)} f(x,y)$ is not existing for $f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$.

Prove by definition that if $\lim_{(x,y)\to(0,0)} f(x,y)$ along $y = x$ and $y = 2x$ are different, then the limit is not existing.

Definition: **Continuity** of f $(f \in C)$ at (a,b)

$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Definition: **Partial derivatives**

$f_x(a,b) = f_1(a,b) = \frac{\partial f}{\partial x}(a,b) = \lim_{x\to a}\frac{f(x,b)-f(a,b)}{x-a} = \lim_{\Delta x\to 0}\frac{f(a+\Delta x,b)-f(a,b)}{\Delta x}$

$f_y(a,b) = f_2(a,b) = \frac{\partial f}{\partial y}(a,b) = \lim_{y\to b}\frac{f(a,y)-f(a,b)}{y-b} = \lim_{\Delta y\to 0}\frac{f(a,b+\Delta y)-f(a,b)}{\Delta y}$

Definition: $f \in C^1 \Leftrightarrow f_x \in C$ and $f_y \in C$

Theorem: **Mean Value**

1. f_x and f_y exists
2. $\mathbb{D} = \{(x,y)|(x-a)^2 + (y-b)^2 < \delta^2\} \subset A$
3. $\Delta x^2 + \Delta y^2 < \delta^2$

Then

1. $f(a+\Delta x, b+\Delta y) - f(a,b) = \Delta x f_x(a+\theta\Delta x, b) + \Delta y f_y(a+\Delta x, b+\alpha\Delta y)$
2. $0 < \theta, \alpha < 1$

Definition: **Differentiability** of f $(f \in \mathcal{D})$ at (a,b)

1. f_x and f_y exists at (a,b)
2. $f(a+\Delta x, b+\Delta y) - f(a,b) = \Delta x f_x(a,b) + \Delta y f_y(a,b) + \Delta x\phi(\Delta x,\Delta y) + \Delta y\psi(\Delta x,\Delta y)$ for all $\Delta x^2 + \Delta y^2 < \delta^2$
3. $\lim_{(\Delta x,\Delta y)\to(0,0)} \phi(\Delta x,\Delta y) = \lim\limits_{(\Delta x,\Delta y)\to(0,0)} \psi(\Delta x,\Delta y) = 0$

Theorem: $f \in \mathcal{C}^1 \Rightarrow f \in \mathcal{D} \Rightarrow f \in \mathcal{C}$

Example: Let $f(x,y) = g\left(\sqrt{x^2+y^2}\right), g(x) = x\sin\frac{1}{x}, g(0) = 0$. Show that $f \in \mathcal{D}$ but $f \notin \mathcal{C}^1$

Definition: Higher order derivatives

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$
$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$
$$f_{yx} = \left(f_y\right)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$
$$f_{yy} = \left(f_y\right)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} \text{ and so on}$$

Note:

1. We write $f \in \mathcal{C}^2$ to mean $f_{xx}, f_{xy}, f_{yx}, f_{yy} \in \mathcal{C}$
2. In a similar manner we write $f \in \mathcal{C}^n$ to mean that all the n th order partial derivatives are continuous. There are 2^n of them.
3. There are $\binom{n}{m} = {}^nC_m = \frac{n!}{m!(n-m)!}$, n th order partial derivatives that contains x, m times.

Example: Let

$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases}.$$

Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

Theorem: $f \in \mathcal{C}^2 \Rightarrow f_{xy} = f_{yx}$

Example: If $u = u(x,y) \in \mathcal{C}^2$ then prove that the **Laplace operator** $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ becomes $\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$ when $x = r\cos\theta, y = r\sin\theta$.

Theorem: Chain rule

1. $f = f(x,y), y = y(t), x = x(t)$ all in $\mathcal{C}^1$. Then
   $$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$
2. $f = f(x,y), y = y(u,v), x = x(u,v)$ all in $\mathcal{C}^1$. Then
   $$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \text{ and } \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

Note: The above may be written as

$$\frac{\partial f}{\partial t} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \frac{\partial f}{\partial(x,y)}\frac{\partial(x,y)}{\partial t} \text{ and } \frac{\partial f}{\partial(u,v)} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{\partial f}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)}$$

The determinant , $\det\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ is called the **Jacobian** or J

With $\underline{x} = \binom{x}{y}$ and $\underline{u} = \binom{u}{v}$, the above may also be written as

$(f \circ \underline{x})'(t) = (f' \circ \underline{x})(t)\underline{x}'(t)$ and $(f \circ \underline{x})'(\underline{u}) = (f' \circ \underline{x})(\underline{u})\underline{x}'(\underline{u})$

We also see that $\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} = \frac{\partial f}{\partial(x,y)} = f'(\underline{x})$ is acting as the true first derivative of $f = f(x,y)$. Therefore it is also called $\nabla f = \text{grad} f$ or the **Gradient** of f.

Example: Assume all functions are C^1

Show that if $x = x(u,v), y = y(u,v), u = u(r,s), v = v(r,s)$ then $\frac{\partial(x,y)}{\partial(r,s)} = \frac{\partial(x,y)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(r,s)}$.

Show that if $u = f(x,y), v = g(x,y)$ then a functional relation of the form $h(u,v) = 0$ exists iff det $\frac{\partial(u,v)}{\partial(x,y)} \equiv 0$.

Definition: **Directional Derivative** of f in the direction of the unit vector $\underline{u} = (u,v)$ at (a,b).

$D_{\underline{u}} f(a,b) = \lim_{\Delta t \to 0} \frac{f(a+u\Delta t, b+v\Delta t) - f(a,b)}{\Delta t}$

Theorem: $f \in C^1, \nabla f(a,b) \neq \underline{0}$

1. $D_{\underline{u}} f(a,b) = \frac{\partial f}{\partial x}(a,b)u + \frac{\partial f}{\partial y}(a,b)v = \nabla f(a,b) \cdot \underline{u}$
2. $\max_{\underline{u}} D_{\underline{u}} f(a,b) = D_{\widehat{\nabla f(a,b)}} f(a,b) = \|\nabla f(a,b)\|$
3. $\min_{\underline{u}} D_{\underline{u}} f(a,b) = D_{-\widehat{\nabla f(a,b)}} f(a,b) = -\|\nabla f(a,b)\|$

Theorem: **Normal vector** to a surface at (a,b)

$\underline{n}(a,b) = (f_x(a,b), f_x(a,b), -1) = (\nabla f(a,b), -1)$

Proof: Let $\underline{r} = \underline{r}(t) = (x(t), y(t), z(t)) \in C^1$ be a curve on the surface of $z = f(x,y) \in C^1$ and $\underline{r}(t_0) = (x(t_0), y(t_0), z(t_0)) = (a, b, f(a,b))$.

Note that $\underline{r}'(t_0) = (x'(t_0), y'(t_0), z'(t_0))$ is the tangent vector to the curve at (a,b).

Now $\underline{n}(a,b) \cdot \underline{r}'(t_0) = f_x(a,b)x'(t_0) + f_y(a,b)y'(t_0) - z'(t_0) = \frac{df}{dt}(t_0) - z'(t_0) = 0$

Ie $\underline{n}(a,b) = (f_x(a,b), f_y(a,b), -1) = (\nabla f(a,b), -1)$ is a vector perpendicular to the surface $z = f(x,y)$ at (a,b).

Theorem: Equation of the **tangent plane** to the surface $z = f(x,y) \in C^1$ at (a,b).

$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) = \nabla f(a,b)\binom{x-a}{y-b} = \nabla f(a,b)\left(\binom{x}{y} - \binom{a}{b}\right)$

Example: Let $f(x,y) = x^4 + y^4 - x^2 - y^2 + 1$. At the point $(1,2)$ find

1. Direction in which the function increases most rapidly
2. Directional derivative in that direction
3. Equation of the tangent plane.

Theorem: **Taylor's expansion** for one variable $f: I \in \mathbb{R} \to \mathbb{R}$

If $f \in C^{n+1}$ and $a, a+h \in I$

then $f(a+h) = \sum_{m=0}^{n} \frac{1}{k!}\frac{d^m f}{dx^m}(a)h^m + \frac{1}{(n+1)!}\frac{d^{n+1} f}{dx^{n+1}}(c)h^{n+1}$

where c is between a and $a+h$.

Note: We can also write the above as

If $f \in C^{n+1}$ and $a + th \in I$ for all $t \in [0,1]$

Then $f(a+h) = \sum_{m=0}^{n} \frac{1}{k!}\left(h\frac{d}{dx}\right)^m f(a) + \frac{1}{(n+1)!}\left(h\frac{d}{dx}\right)^{n+1} f(c)$

for some $c = a + \theta h$ with $\theta \in (0,1)$.

We agree to use the notation $\left(h\frac{d}{dx}\right)^m f(a) \equiv h^m \frac{d^m f}{dx^m}(a)$

Note: The first two terms are the equation of the tangent line.

Proof: Use generalized mean value theorem on
$F(t) = \sum_{m=0}^{n} \frac{1}{m!} f^{(m)}(t)(x-t)^m$ and $G(t) = (x-t)^{n+1}$

Example: When $n = 1$
$f(a+h) = f(a) + \frac{1}{1!} f'(a)h + \frac{1}{2!} f''(c)h^2$

Example: Write the Taylor's expansion for $f(x) = e^x$ at $a = 0$.

Example: Derive the second derivative test to find the extrema of $f(x)$. What to do when $f''(a) = 0$?

Theorem: Taylor's for two variables $f: A \subset \mathbb{R}^2 \to \mathbb{R}$
$f \in C^{n+1}$ and $(a+th, b+tk) \in A$ for all $t \in [0,1]$
Then $f(a+h, b+k) = \sum_{m=0}^{n} \frac{1}{k!} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^m f(a,b) + \frac{1}{(m+1)!} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{m+1} f(\boldsymbol{c})$
for some $\boldsymbol{c} = (a+\theta h, b+\theta k)$ with $\theta \in (0,1)$.

Proof: Use Taylor'r expansion for $F(t) = f(a+th, b+tk)$

Example: When $n = 1$

$$
\begin{aligned}
&f(a+h, b+k)\\
&= \sum_{m=0}^{1} \frac{1}{k!} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^m f(a,b) + \frac{1}{(1+1)!} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{1+1} f(a+\theta h, b+\theta k)\\
&= f(a,b) + \frac{1}{1!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(\boldsymbol{c})\\
&= f(a,b) + \frac{1}{1!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b) + \frac{1}{2!}\left(h^2\frac{\partial^2}{\partial x^2} + 2hk\frac{\partial^2}{\partial x \partial y} + h^2\frac{\partial^2}{\partial y^2}\right) f(\boldsymbol{c})\\
&= f(a,b) + f_x(a,b)h + f_y(a,b)k + \frac{1}{2!}\left(f_{xx}(\boldsymbol{c})h^2 + 2f_{xy}(\boldsymbol{c})hk + f_{yy}(\boldsymbol{c})k^2\right)\\
&= f(a,b) + \begin{pmatrix} f_x(a,b) & f_y(a,b) \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} h & k \end{pmatrix} \begin{pmatrix} f_{xx}(\boldsymbol{c}) & f_{xy}(\boldsymbol{c}) \\ f_{yx}(\boldsymbol{c}) & f_{yy}(\boldsymbol{c}) \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix}\\
&= f(a,b) + \nabla f(a,b) \begin{pmatrix} h \\ k \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} h & k \end{pmatrix} Hf(\boldsymbol{c}) \begin{pmatrix} h \\ k \end{pmatrix}\\
&= f(a,b) + \frac{1}{1!} f'(a,b) \begin{pmatrix} h \\ k \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} h & k \end{pmatrix} f''(\boldsymbol{c}) \begin{pmatrix} h \\ k \end{pmatrix}
\end{aligned}
$$

Note: The first two terms are the equation of the tangent plane.

Definition: $f'' = Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$: **Hessian** of f
$\det Hf = f_{xx} f_{yy} - {f_{xy}}^2$: determinant
$\operatorname{tr} Hf = f_{xx} + f_{yy}$: **trace**

Note: $\det Hf > 0$ and $f_{xx} > 0 (< 0) \Longrightarrow f_{yy} > 0 (< 0) \Longrightarrow \operatorname{tr} Hf > 0 (< 0)$

Example:
Write the Taylor's expansion for $f(x,y) = e^{xy}$ and $f(x,y) = \sin(\sin x + xe^y)$ at $(a,b) = (0,0)$.
Get the same answer by applying multiple one variable Taylor series expansions at 0.

Definition: (a,b) is a **critical point** of $f \in C^1 \Leftrightarrow \nabla f(a,b) = \mathbf{0}$ or f is not defined

Definition:

1. f has a **relative maximum** at $(a,b) \Leftrightarrow f(a,b) \geq f(a+h, b+k)$ in a neighbourhood of (a,b)
2. f has a **relative minimum** at $(a,b) \Leftrightarrow f(a,b) \leq f(a+h, b+k)$ in a neighbourhood of (a,b)
3. f has a **saddle point** at $(a,b) \Leftrightarrow f$ is both above and below its tangent plane at (a,b).

Theorem: $f \in C^1$and (a,b) is a relative maximum/minimum/saddle point of $f \Rightarrow \nabla f(a,b) = \mathbf{0}$

Theorem: $f \in C^2$ and $\nabla f(a,b) = \mathbf{0}$ then

1. $\det Hf(a,b) > 0$ and $\mathrm{tr} Hf(a,b) > 0$ then (a,b) is a relative mimimum
2. $\det Hf(a,b) > 0$ and $\mathrm{tr} Hf(a,b) < 0$ then (a,b) is a relative maximum
3. $\det Hf(a,b) < 0$ then (a,b) is a saddle point
4. $\det Hf(a,b) = 0$ inconclusive(why?)

Example: Find the critical points and determine the nature of them (relative maxima/minima/saddle points).
$f(x,y) = x^3 - 12x + y^3 - 27y + 5$
$f(x,y) = x^4 + y^4 - x^2 - y^2 + 1$
$f(x,y) = x^4 + y^4$
$f(x,y) = x^3 + 3xy^2 - 75x - 9y^2$

Example: Propose a method to determine the nature of critical points when $\det Hf = 0$.

Theorem: Lagrange Multipliers
If $f, g \in C^1$ and $\nabla g \neq \mathbf{0}$ then the maxima/minima of $f(x,y)$ subjected to $g(x,y) = 0$ are included in the set of solutions of $\nabla f(x,y) = \lambda \nabla g(x,y)$ and $g(x,y) = 0$.

Example:
Find the shortest distance from the point $(1,0)$ to the parabola $y^2 = 4x$.
Find the directions of the axes of the ellipse $5x^2 - 6xy + 5y^2 - 4x - 4y - 4 = 0$.
Find the absolute maximum/minimum of $f(x,y) = x^4 + y^4 - x^2 - y^2 + 1$ on the closed disk $(x-1)^2 + y^2 \leq 4$.

ORDINARY DIFFERENTIAL EQUATIONS

Definition: 1st Order Ordinary Differential Equation
$\frac{dy}{dx} = f(x, y)$

Definition/Theorem: **Variable separable** 1st order ODE
$f(x, y) = \frac{g(x)}{h(y)}$
$\int h(y)dy = \int g(x)dx$

Definition/Theorem: Homogeneous 1st order ODE
$f(x, Vx) = g(V)$
$\frac{dy}{dx} = V + x\frac{dV}{dx} = g(V) \Rightarrow \frac{dV}{dx} = \frac{g(V)-V}{x}$: varaibale separable

Definition/Theorem: Linear 1st order ODE
$f(x, y) = Q(x) - P(x)y$
Integrating Factor: $I(x) = e^{\int P(x)dx}$
$\frac{dy}{dx} + P(x)y = Q(x) \Rightarrow I(x)\frac{dy}{dx} + I(x)P(x)y = Q(x)I(x)$
$\Rightarrow \frac{d}{dx}(I(x)y) = I(x)Q(x) \Rightarrow y = \frac{1}{I(x)}\int I(x)Q(x)dx$

Definition/Theorem: **Bernoulli** 1st order ODE
$f(x, y) = Q(x)y^n - P(x)y$
$z = y^{1-n} \Rightarrow \frac{dz}{dx} = (1-n)y^{-n} \Rightarrow \frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x)$: Linear

Example: Solve the first order ODEs
$\frac{dy}{dx} = ye^x, \frac{dy}{dx} = \frac{x^2+y^2}{xy}, \frac{dy}{dx} - \frac{y}{x} = \ln x$

Definition/Theorem: Exact ODE
$M(x, y) + N(x, y)\frac{dy}{dx} = 0$ with $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.
$M, N \in C^1 \Longrightarrow \exists f$ such that $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$.
So $\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$ or $f = c$ is the solution

Definition/Theorem: Reducible to Exact ODE
Let $M(x, y) + N(x, y)\frac{dy}{dx} = 0$ with $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.
If $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)/N = g(x)$ is a function of x alone, define $I(x) = \exp(\int g(x)dx)$.
With $I(x)M(x, y) + I(x)N(x, y)\frac{dy}{dx} = 0$ we have $\frac{\partial NI}{\partial x} = I\frac{\partial N}{\partial x} + NIg(x) = I\frac{\partial M}{\partial y} = \frac{\partial MI}{\partial y}$ so new ODE is exact.
If $\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)/M = h(y)$ is a function of y alone, define $J(y) = \exp(\int h(y)dy)$.
With $J(y)M(x, y) + J(y)N(x, y)\frac{dy}{dx} = 0$ we have $\frac{\partial MJ}{\partial y} = J\frac{\partial M}{\partial y} + MJh(y) = J\frac{\partial N}{\partial x} = \frac{\partial NJ}{\partial x}$ so new ODE is exact.

Example: Solve $(3x^2 + 6xy^2) + (6x^2y + 4y^3)\frac{dy}{dx} = 0$, $(x^3 + y^3) - xy^2\frac{dy}{dx} = 0$, $y - (2x + y)\frac{dy}{dx} = 0$.

Theorem: Cauchy-Peano
Let $f\colon R \subset \mathbb{R}^2 \to \mathbb{R}$ be continuous.
Then the initial value problem $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$ has a solution in R in a neighborhood of (x_0, y_0).

Theorem: Picard -Lindelof
Let $f\colon R \subset \mathbb{R}^2 \to \mathbb{R}$ be continuous. Also let f be Lipschitz continuous($\mathcal{LC}$) in y uniformly in x.
Then the initial value problem $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$ has a unique solution in R in a neighborhood of (x_0, y_0).

Question: Investigate the nature of solutions of $\frac{dy}{dx} = \frac{3}{2}y^{1/3}, y(0) = 0$ according to the above theorems.
Theorem: $f \in \mathcal{D} \Rightarrow (f' \in \mathcal{B} \Leftrightarrow f \in \mathcal{LC})$, $\mathcal{B}$: Bounded

Definition: nth Order Ordinary Differential Equation
$F\left(\frac{d^n y}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \cdots, \frac{dy}{dx}, y, x\right) = 0$

Definition: 2 nd Order Ordinary Differential Equation
$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$

Definition: **2nd Order Linear Ordinary Differential Equation**
$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$

Definition: 2nd Order Linear Ordinary Differential Equation with constant coefficients
$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = r(x)$
If $\alpha^2 + p\alpha + q = (\alpha - a)(\alpha - b) = \alpha(\alpha - b) - a(\alpha - b) = \alpha(\alpha - a) - b(\alpha - a)$
$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = \frac{d}{dx}\left(\frac{dy}{dx} - ay\right) - b\left(\frac{dy}{dx} - ay\right) = \frac{dz}{dx} - bz = r(x) \text{ and } \frac{dy}{dx} - ay = z\text{: Linear 1st ordere ODEs}$

Example: Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \sin x$ as two Linear 1st order ODEs

Definition/Theorem:
$x^2\frac{d^2y}{dx^2} + xp\frac{dy}{dx} + qy = r(x)$
$x = e^z \Rightarrow x\frac{dy}{dx} = \frac{dy}{dz} \Rightarrow x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{d^2y}{dz^2}\frac{1}{x} \Rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$
$\frac{d^2y}{dz^2} - \frac{dy}{dx} + p\frac{dy}{dz} + qy = r(e^z) \Rightarrow \frac{d^2y}{dz^2} + (p-1)\frac{dy}{dz} + qy = r(e^z)\text{: 2nd Order Linear}$

Definition: The solutions to $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$ (**homogeneous equation**)can be obtained by substituting
$y = ce^{\alpha x} \Rightarrow \alpha^2 + p\alpha + q = (\alpha - a)(\alpha - b) = 0,$ **charasteristic equation**
1. $a, b \in \mathbb{R}, a \neq b \Rightarrow y = ce^{ax} + de^{bx}$
2. $a, b \in \mathbb{R}, a = b \Rightarrow y = ce^{ax} + dxe^{bx}$
3. $a, b \in \mathbb{C} \Rightarrow a = a_1 + ia_2 = \bar{b} \Rightarrow y = ce^{a_1 x}\sin a_2 x + ce^{a_1 x}\cos a_2 x$

Definition/Theorem: The solutions to $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x), y(x_0) = y_0, y'(x_0) = y'_0$
1. Exists and unique on an interval (c, d) where $x_0 \in (c, d) \subseteq (a, b)$ and $p(x), q(x), r(x)$ continuous on (a, b).
2. The solution can be expressed as $y = y_c + y_p$
3. $y_c = au(x) + bv(x)$ (**complimentary/fundamental solution**) is the solution when $r(x) \equiv 0$ (homogeneous equation)and u, v (**fundamental set of solutions**)are linearly independent $\forall(a, b)[\forall x(au(x) + bv(x) = 0) \Rightarrow (a, b) = (0,0)]$.
4. y_p (**particular solution**) is a solution when $r(x) \not\equiv 0$.

Example: Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \sin x$ by separately finding y_c and y_p

Definition: **Wronskian**
$W(u, v)(x) = uv' - vu' = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$

Theorem: If u, v are solutions to the homogeneous equation, the Wronskian satisfies
1. $W' + p(x)W = 0$
2. $W(u, v)(x) = c\exp(-\int p(x)dx) = W(u, v)(x_0)\exp\left(-\int_{x_0}^{x} p(t)dt\right)$
3. $\forall x, W(u, v)(x) \neq 0$ or $\forall x, W(u, v)(x) = 0$
4. $\exists x, W(u, v)(x) \neq 0 \Leftrightarrow u, v$ are linearly independent

Example: If e^{ax} is a solution to $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + a^2y = 0$ find the other independent solution.

Theorem: If u, v are fundamental solutions to the homogeneous equation, then the particular solution is given by

$$y_p(x) = c(x)u(x) + d(x)v(x)$$

$$\begin{pmatrix} u & v \\ u' & v' \end{pmatrix}\begin{pmatrix} c' \\ d' \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix};\ c' = \frac{-rv}{W} = \frac{W_1}{W}, W_1 = \begin{vmatrix} 0 & v \\ r & v' \end{vmatrix};\ d' = \frac{ru}{W} = \frac{W_2}{W}, W_2 = \begin{vmatrix} u & 0 \\ u' & r \end{vmatrix}$$

$$y_p(x) = \int_{x_0}^{x} \frac{v(x)u(t) - u(x)v(t)}{W(u,v)(t)} r(t)dt$$

Example: Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \sin x$ using the Wronskian.

Definition: **Legendre ODE**: $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$, n is an integer. The fundamental solutions are

1. **Legendre polynomial**s given by $P_n(x) = \frac{d^n}{dx^n}[(x^2-1)^n]$: bounded solution as $x \to \pm 1$.
2. Legendre functions of the second kind $Q_n(x)$: unbounded solution as $x \to \pm 1$.

Example: Let $n = 1$. Show that $P_1(x) = x$ and find $Q_1(x)$. Hence solve the ODE with RHS=x and $y(0) = 0, y'(0) = 1$.

Definition: Laguerre ODE: $x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + ny = 0$, n is an integer. The fundamental solutions are

1. **Laguerre polynomial**s given by $L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}[e^{-x}x^n]$: bounded solution as $x \to 0$.
2. Laguerre functions of the second kind $M_n(x)$: unbounded solution as $x \to 0$.

Example: Let $n = 1$. Show that $L_1(x) = 1 - x$ and find $M_1(x)$. Hence solve the ODE with RHS=x and $y(1) = 0, y'(1) = 1$.

Definition: Bessel Equation: $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$, n is an integer. The fundamental solutions are

1. **Bessel function** of the first kind $\mathrm{J_n}(x)$: bounded solution at $x = 0$
2. Bessel function of the second kind $\mathrm{K_n}(x)$: unbounded solution at $x = 0$

Definition: Airy Equation: $\frac{d^2y}{dx^2} - xy = 0$. The fundamental solutions are

1. **Airy function** of the first kind $\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right)dt$: bounded solution as $x \to \infty$
2. Airy function of the second kind $\mathrm{Bi}(x)$: unbounded solution as $x \to \infty$