Q1. Derive a formula for approximating $\int_a^b f(x)dx$ by approximating $f(x)$ on each interval by a line joining $(x_k, f(x_k))$ and $(x_{k+1}, f(x_{k+1}))$. Hence approximate $\int_0^1 \sin(x^2)dx$.

Q2. Suppose we want to approximate $\int_a^b f(x)dx$. We divide $[a,b]$ into even number of intervals. Then on each interval with 3 points, we find a quadratic polynomial that passes through the function values and integrate that polynomial. Derive a formula for the above Simpson's method and approximate $\int_0^1 \sin(x^2)dx$.

Q3. Let $f, g \in C[a,b]$ and g does not change sign on $[a,b]$. Show that $\exists c \in (a,b)$ such that $\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$.

Q4. Find e up to and error of 0.01. Show that e is irrational.

Q5. Find and error formula for the Trapezoidal rule and approximate $\int_0^1 \sin(x^2)dx$ to an accuracy of 0.01.

Q6. Find the Lagrange Polynomial passing through the points $(1,1),(3,2),(4,3),(6,8)$.
Also find the maximum error if $f(x) = \frac{1}{\sqrt{5}}(\phi^x - (-\varphi)^x \cos \pi x)$. Here $\phi > 0$(Golden Ratio) and $\varphi < 0$ are the roots of the equation $z^2 - z - 1 = 0$.

Q7. Find the time period T of a pendulum of length ℓ which is oscillating at an angle α if the acceleration due to gravity is g. Use the Taylor series approximation of $\frac{1}{\sqrt{1-x}}$ to write the answer as an infinite series.
What happens when $\alpha \to 0$?.

Q8. Derive an error formula for the Simpson's rule $\int_{x_0}^{x_2} f(x)dx \approx \frac{h}{3}(f(x_0) + 4f(x_1) + f(x_2))$ using Taylor series. What is the error is $f(x)$ is a cubic polynomial?.

Q9(20151028). Show that the Gamma function given by $\Gamma(x) = \int_{0^+}^{\infty} e^{-t}t^{x-1}dt$ exists if and only if $x > 0$.

Q10(20151111).
Prove that $f \in \mathcal{R}[a,b] \Rightarrow |f| \in \mathcal{R}[a,b]$.
Let $f \in \mathcal{R}[a,b]$ for all $b > a$. Consider the integral of $f(x) - |f(x)|$ and show that if $\int_a^{\infty} |f(x)|\, dx$ is converging then $\int_a^{\infty} f(x)\, dx$ is converging.
Deduce that the integral $\int_0^{\infty} \frac{\sin x}{x} dx$ is converging.

Q11. Find the roots of $x^3 + 7x^2 + 16x - 1 = 0$ analytically or using the Bisection method.

Q12. Write a code for the Bisection method in any programming language of your choice.
Write the Taylor series of f at x_n. Assume that $f(x_n + h) \approx 0$ and derive a method to find roots of $f(x) = 0$.
Use the above Newton's method to solve $x^3 + 7x^2 + 16x - 1 = 0$.

Q13. Use the Fixed Point method to find the real root of $2x = \cos x$ accurate to 0.001.
Compare with the fixed point method and find a condition for convergence of the Newton's method. What is the rate of convergence?.

Q14. Use Taylor series of two variables to find sufficient conditions for $f \in C^2$ to have a local minimum/maximum/saddle point at (a,b).
Find them for the function $f(x,y) = x^3 - 12x + y^3 - 27y + 5$

P1. Consider fitting of data (x_n, y_n): $n = 1,2,\cdots,m$ by a least Square Line $ax + b$. We do this by minimizing the sum of square error $f(a,b) = \sum_{n=1}^{m}(y_n - ax_n - b)^2$ as a function of (a, b).
Find the critical points of f and confirm that it is corresponding to a global minimum.

Find the least square line for the following data.

x_n	1	2	3	4
y_n	1	3	2	4

P2: Consider the numerical minimization of the function $f(a,b)$ using the Steepest Descend Method. Here we start at a point (a_0, b_0) and follow the direction of the minimum slope of f at (a_0, b_0) until we get the minimum of f at (a_1, b_1) along this selected direction. Then we repeat the process at (a_1, b_1) and so on. Show that such consecutive minimum slope directions are perpendicular.

The function for P1 is $f(a,b) = 30 - 58a + 30a^2 - 20b + 20ab + 4b^2$. Write the first two steps of the Steepest Descend Method starting from $(0,0)$.

P3. Use the Taylor series expansion for two variable functions and derive a formula to find a root of a set of simultaneous non-linear equations $\begin{matrix} f(x,y) = 0 \\ g(x,y) = 0 \end{matrix}$.
Use matrices and compare the formula with the one variable Newton's method.
Use the above method to find the complex roots of $z^3 + 7z^2 + 16z - 1 = 0$.

P4. It is known that the function $f = f(x)$ minimizing the integral $I(f) = \int_a^b L(x, f, f')dx$ satisfies $\frac{\partial L}{\partial f} - \frac{d}{dx}\left(\frac{\partial L}{\partial f'}\right) = 0$. Here $L(x, f, f')$ is any C^1 function of x, f, f'.
Show that the shortest length between two points is the straight line joining those two points.
Note: Length of a curve described by $f = f(x)$ from $x = a$ to $x = b$ is given by $\int_a^b \sqrt{1 + (f'(x))^2}dx$.

P5. If $f, g \in C^1$ and $\nabla g \neq \mathbf{0}$ then show that the maxima/minima of $f(x,y)$ subjected to $g(x,y) = 0$ are included in the set of solutions of $\nabla f(x,y) = \lambda \nabla g(x,y)$ and $g(x,y) = 0$(Lagrange Multipliers).
Find the absolute maximum/minimum of $f(x,y) = x^4 + y^4 - x^2 - y^2 + 1$ on the closed disk $(x-1)^2 + y^2 \leq 4$.

M1. Use the error formula for the Simpson's rule to find the number of divisions needed to find the value of $\int_0^1 \sin(u^2)du$ accurate to 0.001.

M2. Find the above integral to the given accuracy.

M3. Let $f \in \mathcal{R}[a,b]$ for all $b > a$ and assume that this implies $|f| \in \mathcal{R}[a,b]$ for all $b > a$. Consider the integral of $|f(x)| - f(x)$ and show that if $\int_a^\infty |f(x)|\, dx$ is converging then $\int_a^\infty f(x)\, dx$ is converging.

M4. Deduce that the integral $\int_1^\infty \frac{\sin x}{\sqrt{x}} dx$ is converging.

M5. Assume that x and y coordinates of a moving particle is given by $x = x(t) = \int_0^t \cos(u^2)du$ and $y = y(t) = \int_0^t \sin(u^2)du$ where t is time. Show that as $t \to \infty$ the length of the path is infinite.
Note: Length of a curve $(x(t), y(t))$ from $t = a$ to $t = b$ is given by $\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$.

M6. Show that the (x, y) coordinates of the particle remains finite. Use the results of M4.