Question 1

Assume that x and y coordinates of a moving particle is given by $x = x(t) = \int_0^t \cos(u^2)du$ and $y = y(t) = \int_0^t \sin(u^2)du$ where t is time. Show that as $t \to \infty$ the length of the path is infinite but the (x, y) coordinates of the particle remains finite (it is sufficient to show that one of the coordinates x or y remains finite).

Note: Length of a curve $(x(t), y(t))$ from $t = a$ to $t = b$ is given by $\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$.

Question 2

Derive an error formula for the Trapezoidal Rule and find $\int_0^1 \sin(x^2)dx$ to an accuracy of 0.01.

Question 3

Find the Lagrange Polynomial passing through the points $(1,1), (3,2), (4,3), (6,8)$.

Also find an upper bound for the error if the actual function is $f(x) = \frac{1}{\sqrt{5}}(\phi^x - (-\varphi)^x \cos \pi x)$.

Note: $\phi > 0$ (Golden Ratio) and $\varphi < 0$ are the roots of the equation $z^2 - z - 1 = 0$.

Question 4

Assume the convergence of the Fixed Point Method $x_{k+1} = g(x_k), k \geq 0$ for finding roots of $x = g(x)$. Show that we have $|x_k - x| \leq \frac{L^k}{1-L}|x_1 - x_0|$, where L is the Lipschitz constant of g .

Also use the above method to find the real root of $2x = \cos x$ accurate to 0.001.

Question 5

Use Taylor series of two variables to derive a sufficient conditions for $f \in C^2$ to have a local minimum at (a, b). Also find the critical points and classify them (maximum/minimum/saddle point) for the function$f(x, y) = x^3 - 12x + y^3 - 27y + 5$.

Note: Critical points are those points where ∇f is $\mathbf{0}$ or undefined.

Question 6

Use the Taylor series expansion for two variable functions to derive a formula to find a root of a set of simultaneous non-linear equations $\begin{matrix} f(x, y) = 0 \\ g(x, y) = 0 \end{matrix}$.

Use the above method to write two iterations to find a complex roots of $z^3 + 7z^2 + 16z - 1 = 0$ starting from $1 + i$.