Note 1.
$\mathcal{B}$=*set of Bounded functions*
$\mathcal{C}$=*set of Continuous functions*
$\mathcal{D}$=*set of Differentiable functions*
$\mathcal{D}^n$=*set of n times Differentiable functions*
$\mathcal{C}^n$=*set of n times Continuously Differentiable functions*
$\mathcal{R}$=*set of Riemann Integrable functions*

1 Taylor Series with Remainder

Theorem 1. *Taylor series of $f \in \mathcal{D}^{n+1}$ at a.*
$f(x) = T_n(x,a) + R_n(x,a)$
Taylor Polynomial $T_n(x,a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$
Lagrange Remainder $R_n(x,a) = \frac{f^{(n+1)}(\zeta)}{(n+1)!}(x-a)^{n+1}$
where ζ between x and a

Example 1. *Let $f(x) = \ln(1+x)$. Show that*
1. $T_n(x,0) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} x^k$
2. Find the range of convergence of x
3. Show that $R_n(x,0) \to 0$ as $n \to \infty$ for $x \geq 0$
4. Find the value of $\ln(1.5)$ *accurate to* 0.000001

Theorem 2. *Integral form of the Remainder. $f \in \mathcal{C}^{n+1}$*
$R_n(x,a) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt$

Theorem 3. *Some Theorems on Riemann Integrals*
If $f \in \mathcal{C}[a,b]$ then $f \in \mathcal{R}[a,b]$
If $f \in \mathcal{R}[a,b]$ and $F(x) = \int_a^x f(t)dt$ then $F \in \mathcal{C}[a,b]$
If $f \in \mathcal{C}[a,b]$ and $F(x) = \int_a^x f(t)dt$ then $F \in \mathcal{D}[a,b]$ and $F' = f$

Theorem 4. *Generalized Mean Value Theorem for Riemann Integrals*
Let $f \in \mathcal{C}[a,b]$ and $g \in \mathcal{R}[a,b]$ and g does not change sign on $[a,b]$.
Then there exists $\zeta \in (a,b)$ such that $\int_a^b f(t)g(t)dt = f(\zeta)\int_a^b g(t)dt$

Theorem 5. *Other forms of Remainders. $f \in \mathcal{C}^{n+1}$*
$R_n(x,a) = \frac{f^{(n+1)}(\zeta)}{n!(p+1)}(x-\zeta)^{n-p}(x-a)^{p+1}, 0 \leq p \leq n$
$p = n$, *Lagrange Remainder* $R_n(x,a) = \frac{f^{(n+1)}(\zeta)}{(n+1)!}(x-a)^{n+1}$
$p = 0$, *Cauchy Remainder* $R_n(x,a) = \frac{f^{(n+1)}(\zeta)}{n!}(x-\zeta)^n(x-a)$
where ζ between x and a

Example 2.
1. Show that $R_n(x,0) \to 0$ as $n \to \infty$ for $x < 0$
2.Find the value of $\ln(0.2)$ *accurate to* 0.000001

Example 3.
1. Find the value of $\ln(0.02)$ *accurate to* 10^{-6} *using the Taylor series expansion at* 0 *for* $x < 0$.
2. Find the values of $\ln 2, \ln 5, \ln 500, \ln(0.2), \ln(0.02)$ *accurate to* 10^{-6} *using the Taylor series expansion at* 0 *for* $x > 0$.
3. Find the Taylor series expansions for $e^x, \sin x, \tan^{-1} x, \tan x$ *at* 0 *with remainder.*
4. Find the range of convergence.
5. Show that the remainder $R_n(x, a) \to 0$ *as* $n \to \infty$ *within the range of convergence.*
6. Find the values of $e, \sin 1, \tan^{-1} 1$ *accurate to the 10th decimal place.*
7. Find the values of $e^4, \sin 4, \tan^{-1} 4$ *accurate to the 6th decimal place using a suitable Taylor series expansion.*
8. Deduce the values of 7. from 6. whenever it is possible.
9. Show that e is irrational.

Example 4. *Consider the function* $f(x) = x^{1/10}$
1. Write down the nth degree Taylor Polynomial near $c > 0$.
2. Show that the remainder satisfies $|R_n(x, c)| < \begin{cases} \frac{x^{1/10}}{10(n+1)}\left(\frac{x-c}{c}\right)^{n+1} & x > c > 0 \\ \frac{c^{1/10}}{10(n+1)}\left(\frac{c-x}{c}\right)^{n+1} & c > x > 0 \end{cases}$
3.Show that the value of $1000^{1/10}$ *accurate to 3 decimal places is 1.995.*
4.Find the value of $1025^{1/10}$ *accurate to 10 decimal places.*

Casio 1. *CASIO fx-991ES*
Formula:$(ALPHAX + 1)2x^{\square}ALPHAX - 10x^{\square}6CALCX?10 =$
Sum: SHIFT $\sum_{\square}^{\square}\square$*:* $\sum((-1)x^{\square}(ALPHAX-1)(1\div 2)x^{\square}ALPHAX\div ALPHAX, 1, 10)$

Mathematica 1.
Formula: $f[n\_] := (1 + n)2^n - 10^6; Table[\{n, f[n]\}, \{n, 1, 50\}]$
Sum: $Sum[(-1)\Lambda(k - 1)(1/2)\Lambda k/k, \{k, 1, 10\}]$
Taylor Series: $Series[Log[1 + x], \{x, 0, 20\}]$

2 Interpolation

2.1 Interpolating Polynomial

Theorem 6. *Lagrange Method of finding the Interpolating Polynomial* $p(x)$ *of* $f(x)$ *for the points* $x_k, 0 \le k \le n$
$w_j(x) = \prod_{\substack{i=0\\i\neq j}}^{n}(x - x_i)$
$w(x) = \prod_{i=0}^{n}(x - x_i)$
$\ell_j(x) = \frac{w_j(x)}{w_j(x_j)} = \prod_{\substack{i=0\\i\neq j}}^{n}\left(\frac{x-x_i}{x_j-x_i}\right) = \frac{w(x)}{(x-x_j)w'(x_j)}$ *and* $\ell_j(x_k) = 0$ *if* $j \neq k$ *and 1 if* $j = k$.
$p(x) = \sum_{k=0}^{n} f(x_k)\ell_k(x)$ *and* $p(x_j) = f(x_j), 0 \le j \le n$
$f(x) = p(x) + \frac{f^{(n+1)}(\zeta)}{(n+1)!}w(x)$ *with* $\zeta \in (x_0, x_n)$ *when* $f \in \mathcal{C}^{(n+1)}$

Example 5. *Consider the data set* $\{(2,1),(3,2),(4,3),(6,4)\}$
1. Find the Interpolating polynomial directly by matrix inversion
2. Use the Lagrange Method to find the Interpolating Polynomial
3. Assume that the data set is generated by the function $f(x) = 4\sin^2(\frac{\pi x}{12})$. *Find an upper bound for the error.*
4. Use a suitable numerical method to find the maximum of $w(x) = (x-2)(x-3)(x-4)(x-6)$ *on* $[2,6]$ *and redo the calculation in 3.*
6. For the same function on $[0,6]$, *find the number of points required to make the error* ≤ 0.001 *and find the Interpolating Polynomial.*

Example 6.
1. Consider the data set $\{(1,1),(2,1),(3,2),(4,3)\}$. *Find the Lagrange polynomial. Assume that the above data are obtained from the Fibonacci sequence with* $F(n) = \frac{1}{\sqrt{5}}(\phi^n - \psi^n)$ *where* $\phi > 0$ *and* $\psi < 0$ *are the roots of* $y^2 - y - 1 = 0$.
If the continuous version is $f(x) = \Re F(x) = \frac{1}{\sqrt{5}}(\phi^x - (-\psi)^x \cos \pi x)$, *find the maximum error.*
2. One method of finding the Interpolating polynomial is to use the Newtons divided differences. For x_0, x_1, x_2 *we define* $f[x_0,x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0}$ *and* $f[x_0,x_1,x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$ *and so on and the Lagrange polynomial is given by* $p(x) = f(x_0) + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1)$. *See why the formula is working and use it to find the Lagrange polynomial for* $\{(1,1),(2,1),(3,2),(5,5)\}$.
3. Yet another way of finding the Lagrange polynomial is to define it as the iterative process $p(x) = A_1 + P_1(x)(x-x_0)$ *and* $P_1(x) = A_2 + P_2(x)(x-x_1)$ *and so on. Use this to find the find the Lagrange polynomial for* $\{(1,1),(2,1),(3,2),(5,5)\}$. *Mathematica seems to use this method.*
4. Consider the data set $\{(1,1),(2,1),(3,2)\}$. *Find a polynomial (Hermite polynomial) that goes though the above points and satisfying* $p'(1) = 0, p'(2) = 1, p'(3) = 0$.
5.For the data set $\{(x_k,y_k)\}, k = 0,1,,n$ *a natural cubic spline is a twice differentiable piecewise cubic polynomial* $p(x)$ *which satisfies* $p(x_k) = y_k$ *with* $p''(x_0) = p''(x_n) = 0$. *Let* $p(x) = \sum_{k=0}^{n-1} p_k(x)$ *where* $p_k(x)$ *is the part of* $p(x)$ *on* $[x_k, x_{k+1}]$ *which is 0 elsewhere. Assume that* $p_k(x) = a_k(x-x_k)^3 + b_k(x-x_k)^2 + c_k(x-x_k) + d_k$ *and that* $x_{k+1} - x_k = h$ *is a constant. With* $s_k = p''(x_k)$ *derive the formula* $s_{k+2} + 4s_{k+1} + s_k = \frac{6}{h^2}h^2(y_{k+2} - 2y_{k+1} + y_k), k = 0,1,,n-2$.
Also write the system of equations in matrix form that must be solved find s_k *for* $k = 1,,n-1$. *The last scores of the Sri Lankan cricket team are* $\{291, 245, 183, 386, 135, 181, 216, 236, 217\}$. *Predict their score in the next match.*

Mathematica 2. $InterpolatingPolynomial[\{\{2,1\},\{3,2\},\{4,3\},\{6,4\}\},x]$

2.2 Least Square Polynomial

Theorem 7. *Least Square Line $y = ax + b$ for the points $(x_k, y_k), 1 \le k \le n$ That minimizes $E(a,b) = \sum_{k=1}^{n}(ax_k + b - y_k)^2$ is given by*

$$\begin{pmatrix} \sum_{k=1}^{n} x_k^2 & \sum_{k=1}^{n} x_k \\ \sum_{k=1}^{n} x_k & \sum 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{n} x_k y_k \\ \sum_{k=1}^{n} y_k \end{pmatrix}$$

This can also be written as $X^T X \begin{pmatrix} a \\ b \end{pmatrix} = X^T Y$ where

$X^T = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{pmatrix}$ *and* $Y^T = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix}$

Theorem 8. *Properties of the Least Square Line*
With $\overline{x} = \frac{1}{n}\sum_{k=1}^{n} x_k$, $\overline{y} = \frac{1}{n}\sum_{k=1}^{n} y_k$, $s_{xx} = \frac{1}{n}\sum_{k=1}^{n}(x_k - \overline{x})^2 = \frac{1}{n}\sum_{k=1}^{n} x_k^2 - \overline{x}^2$, and $s_{xy} = \frac{1}{n}\sum_{k=1}^{n}(x_k - \overline{x})(y_k - \overline{y}) = \frac{1}{n}\sum_{k=1}^{n} x_k y_k - \overline{x}\,\overline{y}$
We have $a = \frac{s_{xy}}{s_{xx}}$
Also $(\overline{x}, \overline{y})$ is on the Least square line and therefore $\overline{y} = a\overline{x} + b$ or $b = a\overline{x} - \overline{y}$
If $HE = \begin{pmatrix} E_{aa} & E_{ab} \\ E_{ba} & E_{bb} \end{pmatrix}$ is the Hessian of $E(a,b)$ we have $HE = 2X^T X$ and
$\det HE = E_{aa}E_{bb} - (E_{ab})^2 = 4n^2 s_{xx} > 0$ *when x_k are different and*
$\mathrm{tr}HE = E_{aa} + E_{bb} = 2\sum x_k^2 + 2n > 0$ *so (a,b) is a global minimum.*

Example 7. *Let $A = \{(2,1),(3,2),(4,3),(6,4)\}$*
1. Find the Lest Square Line for A.
2. Show that the least square parabola $y = ax^2 + bx + c$ for the data set $(x_k, y_k), k = 1, 2, \cdots, n$ is given by

$$\begin{pmatrix} \sum x_k^4 & \sum x_k^3 & \sum x_k^2 \\ \sum x_k^3 & \sum x_k^2 & \sum x_k \\ \sum x_k^2 & \sum x_k & \sum 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \sum x_k^2 y_k \\ \sum x_k y_k \\ \sum y_k \end{pmatrix}$$

3. Find the Least Square Parabola for A

Example 8. *Let $A = \{(1,1),(2,1),(3,2),(4,3)\}$*
1. Find the least square polynomials of degree 0,1,2,3,4 for A if it is possible.
2. Calculate the exact error before and after finding the coefficients in each case.
3. Show that we have a unique solution when each x_k is different.
4. Fit a least square function of the form $y = ax + bx^3 + cx^4$ for A.
5. Fit a least square function of the form $y = ae^x + b\sin x + c\cos x$ for A.
6. Find the best combination of functions out of $\{1, x, x^2, e^x, \sin x, \cos x, \log x\}$ if we are looking for a combination of 3 functions.
7. Show that the Correlation Coefficient given by $r = \frac{s_{xy}}{\sqrt{s_{xx}s_{yy}}}$ is a measure of the linearity of data in the case of the least square line.
8. Suppose we have a 3D date set $B = \{(1,1,1),(2,1,2),(3,2,3),(4,3,4)\}$. Propose a Lagrange-type two variable polynomial and a least square plane.

Casio 2. *CASIO fx-991ES*
MODE 3:STAT 3:$\_ + cx^2$ 2 = 3 = 4 = 6 = REPLAY UP 1 = 2 = 3 = 4 = SHIFT STAT(1) 1:Type, 2: Data, 3:Edit, 4:Sum(1:$\sum x^2$, 2:$\sum x$, 4:$\sum y$, 5:$\sum xy$, 6:$\sum x^3$, 7:$\sum x^2y$, 8:$\sum x^4$), 7:Reg(1:A, 2:B, 3:C)

Mathematica 3. $Fit[\{\{2,1\},\{3,2\},\{4,3\},\{6,4\}\},\{x^2,x,1\},x]$

2.3 Two Variable Taylor Series and Local Minima

Theorem 9. *Taylor series of* $f : \mathbb{R} \to \mathbb{R}, f \in \mathcal{C}^{n+1}$ *at* a.
$D = h\frac{d}{dx}, D^k f = D(D^{k-1} f)$
$f(a+h) = \sum_{k=0}^{n} \frac{1}{k!} D^k f(a) + \frac{1}{(n+1)!} D^{n+1} f(\zeta), \zeta = a + \eta h, 0 \le \eta \le 1$

Theorem 10. *A Sufficient condition for a Local Minimum.*
$f : \mathbb{R} \to \mathbb{R}, f \in \mathcal{C}^2$. $f'(a) = 0$.
$f''(a) > 0$
$\Rightarrow f''(\zeta) > 0$, *for sufficiently small* h
$\Rightarrow f(a+h) - f(a) = \frac{1}{2}h^2 f''(\zeta) = \frac{1}{2}D^2 f(\zeta) > 0$, *for sufficiently small* h
$\Rightarrow a$ *is a local minimum of* f.

Theorem 11. *Taylor series of* $f : \mathbb{R}^2 \to \mathbb{R}, f \in \mathcal{C}^{n+1}$ *at* $\boldsymbol{a} = (a, b)$. $\boldsymbol{h} = (h, \ell)$.
$D = h\frac{\partial}{\partial x} + \ell\frac{\partial}{\partial y}, D^k f = D(D^{k-1} f)$
$f(\boldsymbol{a}+\boldsymbol{h}) = \sum_{k=0}^{n} \frac{1}{k!} D^k f(\boldsymbol{a}) + \frac{1}{(n+1)!} D^{n+1} f(\boldsymbol{\zeta}), \boldsymbol{\zeta} = \boldsymbol{a} + \eta \boldsymbol{h}, 0 \le \eta \le 1$

Theorem 12. $f : \mathbb{R}^2 \to \mathbb{R}, f \in \mathcal{C}^2, \boldsymbol{b} \in \mathbb{R}^2$
$Df(\boldsymbol{b}) = \left(h\frac{\partial}{\partial x} + \ell\frac{\partial}{\partial y}\right) f(\boldsymbol{b}) = hf_x(\boldsymbol{b}) + \ell f_y(\boldsymbol{b}) = \nabla f(\boldsymbol{b})\boldsymbol{h}^T$
where $\nabla f(\boldsymbol{b}) = (f_x(\boldsymbol{b}), f_y(\boldsymbol{b}))$ *is the Gradient*
$D^2 f(\boldsymbol{b}) = \left(h\frac{\partial}{\partial x} + \ell\frac{\partial}{\partial y}\right)^2 f(\boldsymbol{b}) = h^2 f_{xx}(\boldsymbol{b}) + 2h\ell f_{xy}(\boldsymbol{b}) + \ell^2 f_{yy}(\boldsymbol{b}) = \boldsymbol{h}Hf(\boldsymbol{b})\boldsymbol{h}^T$
$= \ell^2 f_{xx}(\boldsymbol{b}) \left(\left(\frac{h}{\ell} + \frac{f_{xy}(\boldsymbol{b})}{f_{xx}(\boldsymbol{b})} \right)^2 + \frac{detHf(\boldsymbol{b})}{(f_{xx}(\boldsymbol{b}))^2} \right)$, *if* $f_{xx}(\boldsymbol{b}) \neq 0$.
Here $Hf(\boldsymbol{b}) = \begin{pmatrix} f_{xx}(\boldsymbol{b}) & f_{xy}(\boldsymbol{b}) \\ f_{yx}(\boldsymbol{b}) & f_{yy}(\boldsymbol{b}) \end{pmatrix}$ *is the Hessian.*
Also $\det Hf(\boldsymbol{b}) = f_{xx}(\boldsymbol{b}) f_{yy}(\boldsymbol{b}) - (f_{xy}(\boldsymbol{b}))^2$ *is the Determinant and*
$\mathrm{tr}Hf(\boldsymbol{b}) = f_{xx}(\boldsymbol{b}) + f_{yy}(\boldsymbol{b})$ *is the Trace*

Theorem 13. *A Sufficient condition for a Local Minimum.*
$f : \mathbb{R}^2 \to \mathbb{R}, f \in \mathcal{C}^2$. $\nabla f(\boldsymbol{a}) = \boldsymbol{0}$.
$\mathrm{tr}Hf(\boldsymbol{a}) > 0$ *and* $\det Hf(\boldsymbol{a}) > 0$
$\Leftrightarrow f_{xx}(\boldsymbol{a}) > 0$ *and* $\det Hf(\boldsymbol{a}) > 0$
$\Rightarrow f_{xx}(\boldsymbol{\zeta}) > 0$ *and* $\det Hf(\boldsymbol{\zeta}) > 0$, *for sufficiently small* $\boldsymbol{h}$
$\Rightarrow f(\boldsymbol{a}+\boldsymbol{h}) - f(\boldsymbol{a}) = \frac{1}{2}D^2 f(\boldsymbol{\zeta}) > 0$, *for sufficiently small* $\boldsymbol{h}$
$\Rightarrow \boldsymbol{a}$ *is a local minimum of* f.

Theorem 14. *Chain Rule and Equal Mixed Partial Derivatives*
$f = f(x,y) \in \mathcal{C}^1 (\Leftrightarrow f_x, f_y \in \mathcal{C}); x = x(t), y = y(t) \in \mathcal{C}^1 \Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$
$f = f(x,y) \in \mathcal{C}^2 (\Leftrightarrow f_{xx}, f_{xy}, f_{yx}, f_{yy} \in \mathcal{C}) \Rightarrow f_{xy}, f_{yx} \in \mathcal{C} \Rightarrow f_{xy} = f_{yx}$

3 Numerical Integration

3.1 Trapezoidal Rule

Theorem 15. *Trapezoidal Rule*
$f \in \mathcal{C}^2[a,b], h = \frac{b-a}{n}, x_0 = a, x_n = b, x_k = x_0 + kh, 0 \le k \le n, \zeta \in (a,b)$
$\int_a^b f(x)dx = \frac{h}{2}\left[f(x_0) + 2\sum_{k=1}^{n-1} f(x_k) + f(x_n)\right] - \frac{(b-a)^3}{12n^2} f''(\zeta)$

Example 9. *For each of the following integrals, use the Trapezoidal rule to find the number of divisions needed to find its value accurate to 0.001 and find the integral to that accuracy.*
1. $\int_0^1 e^{-\frac{x^2}{2}} dx$ 2. $\int_0^1 \sin(x^2)dx$ 3. $\int_0^{\frac{\pi}{2}} \sqrt{2-\cos^2 x}dx$ 4. $\int_2^{10} \frac{x}{\log x}dx$

3.2 Simpson's Rule

Theorem 16. *Simpson's Rule(see Proofs.pdf for the proof for the error)*
$f \in \mathcal{C}^4[a,b], n \text{ is even}, h = \frac{b-a}{n}, x_0 = a, x_n = b, x_k = x_0 + kh, 0 \le k \le n, \zeta \in (a,b)$
$\int_a^b f(x)dx = \frac{h}{3}\left[f(x_0) + 4\sum_{\substack{k=1 \\ kodd}}^{n-1} f(x_k) + 2\sum_{\substack{k=2 \\ keven}}^{n-2} f(x_k) + f(x_n)\right] - \frac{(b-a)^5}{180n^4} f^{(4)}(\zeta)$

Example 10.
1. Do the same thing in Example 9 using the Simpson's rule.
2. Show directly that cubic polynomials are integrated exactly by the Simpson's rule.
3. Derive a numerical integration rule and its error that uses the function value at the mid point (Mid Point Rule), left end point, right end point of each interval.
4. Use Mid Point Rule rule to do the same question in Example 9.
5. Use integration by parts to prove the error formula for the Trapezoidal rule.
6. Use the remainder in the Lagrange Polynomial(see Theroem 6) to derive the error formula for the Trapezoidal rule.
7. Use Taylor series to derive approximate formulas for the remainder in both Trapezoidal and Simpson's rule.
8. Use a variant of the Lagrange Polynomial remainder(see Proofs.pdf) to derive the error formula for the Simpson's rule.

Example 11. *One method of doing numerical integration is Gaussian Quadrature. Note that both the Trapezoidal and the Simpsons rules looks like* $\int_a^b f(x)dx \approx \sum_k w_k f(x_k)$ *and we knew* x_k *and found* w_k*. In this method we find both* x_k *and* w_k *so that the integral and the sum are equal for a given* n *degree polynomial* $p(x)$*. It is achieved by forcing both sides equal for each power of* x^j *for* $j = 0, 1, 2, , n$*. What is the degree of the polynomial we need to use if we want 3 points and the corresponding 3 weights? Find them for* $[a,b] = [-1,1]$ *and use it to approximate* $\int_0^1 \sin(x^2)dx$*. Casio fx-991ES uses a variant of this method.*

Casio 3. $\int(f(X), a, b, m)$ *and the default variable is X and* $n = 2^m$ *for the Simpson's method in model fx-991MS*

Mathematica 4. $NIntegrate[f(x), \{x, a, b\}, Method -> TrapezoidalRule]$

4 Numerical solutions of non-linear equations of one variable

4.1 Bisection Method

Theorem 17. *Intermediate Value Theorem*
Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Let c be between $f(a)$ and $f(b)$. Then there exists $x \in (a, b)$ such that $f(x) = c$.

Algorithm 1. *Bisection Method*
1. Find a_0, b_0 such that $f(a_0)f(b_0) < 0$.
2. $k = 0$.
3. $x_k = \frac{a_k+b_k}{2}$
4. If $f(x_k) = 0$ then stop and return x_k
5. If $f(x_k)f(a_k) > 0$ then $a_{k+1} = x_k$ and $b_{k+1} = b_k$
else $b_{k+1} = x_k$ and $a_{k+1} = b_k$
6. If $|b_k - a_k| < \epsilon$ then stop and return x_k
else $k \leftarrow k + 1$ and goto 3

Theorem 18. *Convergence of the Bisection Method*
1. $f : [a, b] \to \mathbb{R}$
2. f is continuous (ie.$f \in \mathcal{C}$).
3. $a_0, b_0 \in [a, b]$ and we select a_k, b_k, x_k for $k \geq 0$ according to the above algorithm.
Then
1. $\lim_{k\to\infty} x_k = \lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = z \in [a, b]$ is a root of f.
2. $|x_k - z| \leq \frac{1}{2}|b_k - a_k| = \left(\frac{1}{2}\right)^{k+1} |b_0 - a_0|$

Example 12. *Consider real roots of $x - e^{-x} = 0$ and $x^5 - x - 1 = 0$*
1.Find an intervals that contains the real roots.
2.Find the no of iterations needed to find roots to an accuracy of 10^{-6}.
3.Do the iterations and find the roots.

Casio 4. *Bisection Method*
ALPHA X ALPHA = (ALPHA A + ALPHA B) ÷ 2 ALPHA : ALPHA X-e$^\square$ - ALPHA X CALC

Mathematica 5. *Bisection Method*
Algorithm
f[x_]:=x-E^(-x);a = 0; b = 1; For[k = 0, k <= 19, k++, {x = (a + b)/2, Print[N[k, a, b, x, f[x], Abs[a - b]/2, 10]], If[f[x] == 0, k = 20, If[f[x] > 0, b = x, a = x]]}]

Builtin function
FindRoot[f[x] == 0, {x, 0}], by iterations starting $x_0 = 0$

4.2 Some Real Analysis and Topology

Definition 1. *Types of Continuity of* $f : A \to \mathbb{R}$
1. f *is continuous at* $y \in A$ *iff* $\forall \epsilon > 0, \exists \delta > 0, \forall x; |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$
2. f *is continuous on* A*(i.e.* $f \in \mathcal{C}(A)$*) iff*
$\forall y \in A, \forall \epsilon > 0, \exists \delta > 0, \forall x \in A; |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$
or equivalently $\forall \epsilon > 0, \forall y \in A, \exists \delta > 0, \forall x \in A; |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$
3. f *is uniformly continuous on* A*(i.e.* $f \in \mathcal{UC}(A)$*) iff*
$\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in A; |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$
4. f *is Lipchitz continuous on* A*(i.e.* $f \in \mathcal{LC}(A)$*) iff*
$\exists L > 0, \forall x, y \in A; |f(x) - f(y)| \le L|x - y|$. L *is called the Lipchitz costant.*

Theorem 19. $f : A \to \mathbb{R}$
1. $f \in \mathcal{LC}(A) \Rightarrow f \in \mathcal{UC}(A) \Rightarrow f \in \mathcal{C}(A)$
2. $f \in \mathcal{C}(A) \Rightarrow f \in \mathcal{UC}(A)$ *when* A *is closed*
3. $f \in \mathcal{C}^1(A) \Rightarrow f \in \mathcal{UC}(A)$ *when* A *is closed with* $L = \max\{|f'(x)| : x \in A\}$ *using Mean Value and Extreme Value theorems on* f'.

Definition 2. *Cauchy Sequence and Completeness. Let* $u : \mathbb{N} \to A$ *be a sequence.*
1. u *is a Cauchy sequence on* A *iff* $\forall \epsilon > 0, \exists N > 0, \forall n, m > 0; m, n > N \Rightarrow |u(m) - u(n)| < \epsilon$
2. A *is Complete iff Every Cauchy sequence on* A *is converging to a point of* A.

Theorem 20.
1. All converging sequences are Cauchy.
2. $\mathbb{R}^n$ *is complete.*
3. A closed subset of a complete space is complete.
4. A complete space is closed.

Note 2. *Topology on Matric Spaces*
1. Spaces with a distance measuring function(metric) is called a matric space. Cauchy sequence u *must be defined on such spaces* A*, i.e* $u : \mathbb{N} \to A$.
2. Matric is a function $d : A \times A \to \mathbb{R}$ *satisfying the properties below for* $x, y, z \in A$:
$d(x, y) \ge 0$, $d(x, y) = 0$ *iff* $x = y$, $d(x, y) = d(y, x)$, $d(x, y) \le d(x, z) + d(z, y)$
3. In $A = \mathbb{R}$ *we can use absolute value* $d(x, y) = |x - y|$.
4. In $A = \mathbb{R}^2$ *three examples are* $d((x, y), (a, b)) = \sqrt{(x - a)^2 + (y - b)^2}$,
$d((x, y), (a, b)) = |x - a| + |y - b|$, $d((x, y), (a, b)) = \max\{|x - a|, |y - b|\}$
5. In $A = \mathbb{C}$ *we can use complex modulus* $d(z, w) = |z - w|$*(see Note 3).*
6. In a matric spaces a Ball with center x *and radius* r *is* $B(x, r) = \{y | d(x, y) < r\}$.
7. Closure Points $\overline{A}$ *of a set* A *are those points such that balls with them as center will contain a point in* A*. ie.* $x \in \overline{A} \Leftrightarrow \forall r > 0 \exists y \in A; y \in B(x, r)$
8. $\overline{A} \supset A$.
9. A set is Closed iff it Closure points are points of the set itself (ie.$\overline{A} \subset A$ *or equivalently* $\overline{A} = A$*).*
10. A set is Open iff it's complement A^c *is closed.*
11. A Neighbourhood of x *is an open set containing* x.

4.3 Fixed Point(Iterative) Method

Definition 3.
1. g is a Contraction iff it is Lipchitz continuous with Lipchitz constant $L < 1$.
2. z *id a Fixed Point of* g *iff* $z = g(z)$

Theorem 21. *Global Convergence of the Fixed Point method(Banach Fixed Point Theorem)*
1.$g : [a, b] \to [a, b]$
2.g *is a contraction with Lipschitz constant L*
3.$x_0 \in [a, b]$ *and* $x_{k+1} = g(x_k), k \geq 0$
Then
1.$\lim_{k\to\infty} x_k = z \in [a, b]$ *is a unique fixed point of* g
2.$|x_k - z| \leq \frac{L^k}{1-L}|x_1 - x_0|$

Theorem 22. *Local convergence of the Fixed Point method*
Let $z = g(z)$ *be a fixed point. If* $g \in \mathcal{C}^1$ *with* $|g'(z)| < 1$*, then there exists a neighbourhood of* z *such that the fixed point method is converging.*

Algorithm 2. *Fixed Point Method*
1. Select x_0
2. $k = 0$.
3. $x_{k+1} = g(x_k)$
4. If $|x_{k+1} - x_k| < \epsilon$ *then stop and return* x_k *else goto 3*

Example 13. *Consider real roots of* $x - e^{-x} = 0$ *and* $x^5 - x - 1 = 0$
1.Write them as $x = g(x)$ *to be solved by the fixed point method, is the method converging?*
2.Find the no of iterations needed to find that root to an accuracy of 10^{-6}.
3.Do the iterations and find the root.

Example 14. *Let* $T_n(x) = \sum_{k=1}^{n} \frac{(-x)^k}{k!}$ *be the n th degree Taylor polynomial of* e^{-x} *at* $x = 0$ *and* $\lim_{x\to\infty} T_n(x) = e^{-x}$*. Solve* $x = T_2(x)$ *and find an approximate solution to* $x = e^{-x}$*. Also find a n for which the difference in the solutions to* $x = T_n(x)$ *and* $x = e^{-x}$ *is less than 0.001. Assume that one real solution to* $x = T_n(x)$ *remain in* $[0.5, 0.6]$ *for all* $n \geq 2$.

Casio 5. *Fixed Point Method*
ALPHA X ALPHA = $e^{\square}$ *-ALPHA X CALC =*

Mathematica 6. *Fixed Point Method*
Algorithm
g[x_]:=E^(-x);x = 0; For[k = 0, k <= 19, k++, {x = g[x], Print[N[{k, x, Abs[x - g[x]]}, 10]]}]
Builtin function
FindRoot[f[x] == 0, {x, 0}], by iterations starting $x_0 = 0$

Note 3. *See the note AllRoots.pdf on finding the complex roots of* $x^5 - x - 1 = 0$ *using the Fixed Point method.*

4.4 Newton's Method

Definition 4. *Newton's method for finding roots of* $f(x) = 0$
$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$

Theorem 23. *Local convergence of the Newton's method(taking it as an Fixed Point method).*
Let z *be a root of* f*. If* $f \in \mathcal{C}^2, f'(z) \neq 0$ *and* $|f(z)f''(z)| < (f'(z))^2$ *then there exists a neighbourhood of* z *such that the Newton's method is converging.*

Theorem 24. *Global convergence of the Newton's method(Newton-Kantorovich Theorem, see Proofs.pdf for the proof)*
1.$f : [a, b] \to \mathbb{R}$
2.$f' \neq 0$ *and there exists* $\beta > 0$ *such that* $\frac{1}{|f'(x)|} \leq \beta$
3.f' *is Lipschitz continuous with constant* γ
4.$x_0 \in [a, b]$ *and* $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, k \geq 0$
5.$\left|\frac{f(x_0)}{f'(x_0)}\right| = \alpha$
6.$q = \alpha\beta\gamma < \frac{1}{2}$
7.$[x_0 - 2\alpha, x_0 + 2\alpha] \subset [a, b]$
Then
1. $\lim_{k\to\infty} x_k = z \in [x_0 - 2\alpha, x_0 + 2\alpha]$ *is a unique root of* f
2. $|x_k - z| \leq 2\alpha q^{2^k-1}$

Theorem 25. *Local convergence of the Newton's Method*
Let z *be a root of* f*. If* $f \in \mathcal{C}^2$ *and* $f'(z) \neq 0$ *then there exists a neighbourhood of* z *where the Newton's method is converging.*

Note 4. *See that now we don't need* $|f(z)f''(z)| < (f'(z))^2$ *which was a requirement for local convergence when we analyzed the Newton's method as an Fixed Point method.*

Example 15. *Consider the Newton's method of finding the real roots of* $x - e^{-x}$ *and* $x^5 - x - 1 = 0$
1. Treat the method as an fixed point method and find the no of iterations needed to calculate the root to an accuracy of 10^{-6} *and find the root.*
2. Use the error formula above for the Newton's method and find the no of iterations needed to calculate the root to an accuracy of 10^{-6} *and find the root.*
3.Use more terms in the Taylor series(instead of 2 terms used in the Newton's method) and propose a possibly faster method to find the root.
4. If f *was not differentiable, propose a method which uses the secant(instead of the tangent) joining two successive points.*
5. Try to find complex roots using the Newton's method(see Note 3).

Example 16.
1. Do Example 5 for $\sin x = 2x$ *and* $x = e^{-x}$
2. Try to solve $x^m = 0$ *for* $m \in \mathbb{R}$*. What is going wrong/right?.*
3. Show that the sequence $x_{k+1} = \frac{x_k}{2} + \frac{a}{2x_k}$ *converges to* $\sqrt{a}$*, provided we select* x_0 *on a suitable range. What is such a range?*
4. Suppose we want to solve $tan^{-1}x = 0$ *by the Newton's method. Find the value* z *such that the Newton's method is converging for* $0 < x_0 < z$*, diverging for* $x_0 > z$ *and enters into a cycle for* $x_0 = z$*.*
5. Your CASIO calculator can integrate, $\int_a^b f(x)dx$ *is evaluated as* $\int(f(x), a, b)$*. Find the* z *value for which* $P(x < z) = 0.8$ *when* $X \sim N(0, 1)$*, ie when* X *is Normally distributed with mean 0 and standard deviation 1 which is having a PDF* $p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$*.*
6. Find the height of the circular sector with arc length $2x$ *and chord length* x*.*
7. What is the height if the shape is a parabola?

Example 17. *An fixed point method of finding solutions to a non-linear equation* $f(x) = 0$ *is said to have a convergence of order* p *iff* $|x_{k+1} - z| \le r|x_k - z|^p$ *where* x_k *is the* k*th iteration,* z *is the solution and* r *is a constant. Show that* $p = 1$ *for the fixed point method and* $p = 2$ *for the Newtons method.*

Example 18. *Find the global maximums of*
1. $w'(x)$ *on* $[2, 6]$*.* $w(x) = (x - 2)(x - 3)(x - 4)(x - 6)$
2. $f''(x)$ *on* $[0, 1]$*.* $f(x) = \sin(x^2)$
3. $f^{(4)}(x)$ *on* $[0, 1]$*.* $f(x) = \sin(x^2)$

Casio 6. *Solving cubic* $x^3 + 2x^2 + 3x + 4 = 0$ *with roots* x_1, x_2, x_3
MODE 5:EQN 4:$ax^3 + bx^2 + cx + d$ $1 = 2 = 3 = 4 == x1 = x2 = x3$

Mathematica 7.
$NRoots[x^5 - x - 1 == 0, x]$
$NSolve[f[x] == 0, x]$
$FindRoot[f[x] == 0, \{x, x0\}]$