In[1]:=
```
Table[{n, (0.5)^(n+1) / (n+1) - 10^(-6)}, {n, 1, 20}]
```

Out[1]= $\{\{1, 0.124999\}, \{2, 0.0416657\}, \{3, 0.015624\}, \{4, 0.006249\}, \{5, 0.00260317\}, \{6, 0.00111507\}, \{7, 0.000487281\}, \{8, 0.000216014\}, \{9, 0.0000966563\}, \{10, 0.0000433892\}, \{11, 0.0000193451\}, \{12, 8.39002\times10^{-6}\}, \{13, 3.35965\times10^{-6}\}, \{14, 1.03451\times10^{-6}\}, \{15, -4.63257\times10^{-8}\}, \{16, -5.51212\times10^{-7}\}, \{17, -7.88072\times10^{-7}\}, \{18, -8.99613\times10^{-7}\}, \{19, -9.52316\times10^{-7}\}, \{20, -9.77293\times10^{-7}\}\}$

In[2]:=
```
T[n_, x_] := Sum[(-1)^(k-1) (x)^k / k, {k, 1, n}]
```

In[3]:=
```
T[15, 1 / 2]
```

Out[3]= $\frac{59\,848\,147}{147\,603\,456}$

In[4]:=
```
N[%, 20]
```

Out[4]= 0.40546575684515137640

In[5]:=
```
N[Log[1 + 1 / 2], 20]
```

Out[5]= 0.40546510810816438198

In[6]:=
```
%% - %
```

Out[6]= $6.4873698699442\times10^{-7}$

In[7]:=
```
Table[{n, 2 (0.8)^n - 10^(-6)}, {n, 1, 70}]
```

Out[7]= $\{\{1, 1.6\}, \{2, 1.28\}, \{3, 1.024\}, \{4, 0.819199\}, \{5, 0.655359\}, \{6, 0.524287\}, \{7, 0.419429\}, \{8, 0.335543\}, \{9, 0.268434\}, \{10, 0.214747\}, \{11, 0.171798\}, \{12, 0.137438\}, \{13, 0.10995\}, \{14, 0.0879599\}, \{15, 0.0703677\}, \{16, 0.056294\}, \{17, 0.045035\}, \{18, 0.0360278\}, \{19, 0.028822\}, \{20, 0.0230574\}, \{21, 0.0184457\}, \{22, 0.0147564\}, \{23, 0.0118049\}, \{24, 0.00944373\}, \{25, 0.00755479\}, \{26, 0.00604363\}, \{27, 0.0048347\}, \{28, 0.00386756\}, \{29, 0.00309385\}, \{30, 0.00247488\}, \{31, 0.0019797\}, \{32, 0.00158356\}, \{33, 0.00126665\}, \{34, 0.00101312\}, \{35, 0.000810296\}, \{36, 0.000648037\}, \{37, 0.00051823\}, \{38, 0.000414384\}, \{39, 0.000331307\}, \{40, 0.000264846\}, \{41, 0.000211676\}, \{42, 0.000169141\}, \{43, 0.000135113\}, \{44, 0.00010789\}, \{45, 0.0000861123\}, \{46, 0.0000686898\}, \{47, 0.0000547519\}, \{48, 0.0000436015\}, \{49, 0.0000346812\}, \{50, 0.000027545\}, \{51, 0.000021836\}, \{52, 0.0000172688\}, \{53, 0.000013615\}, \{54, 0.000010692\}, \{55, 8.35361\times10^{-6}\}, \{56, 6.48289\times10^{-6}\}, \{57, 4.98631\times10^{-6}\}, \{58, 3.78905\times10^{-6}\}, \{59, 2.83124\times10^{-6}\}, \{60, 2.06499\times10^{-6}\}, \{61, 1.45199\times10^{-6}\}, \{62, 9.61594\times10^{-7}\}, \{63, 5.69275\times10^{-7}\}, \{64, 2.5542\times10^{-7}\}, \{65, 4.33628\times10^{-9}\}, \{66, -1.96531\times10^{-7}\}, \{67, -3.57225\times10^{-7}\}, \{68, -4.8578\times10^{-7}\}, \{69, -5.88624\times10^{-7}\}, \{70, -6.70899\times10^{-7}\}\}$

In[8]:=
```
T[66, -8 / 10]
```

Out[8]= $-\frac{16\,117\,216\,873\,691\,934\,804\,806\,620\,772\,027\,796\,322\,209\,555\,314\,759\,992\,463\,642\,416\,990\,260\,604}{10\,014\,190\,032\,827\,718\,657\,639\,678\,155\,081\,564\,003\,836\,547\,271\,930\,612\,623\,691\,558\,837\,890\,625}$

In[9]:=
```
N[%, 20]
```

Out[9]= -1.6094378897202630032

In[10]:=
```
N[Log[1 - 8 / 10], 20]
```

Out[10]= -1.6094379124341003746

In[11]:=
```
%% - %
```

Out[11]= $2.27138373714\times10^{-8}$

In[12]:= $\text{Integrate}\left[\frac{1}{1+t}\left(\frac{t-x}{t+1}\right)^n, \{t, x, 0\}\right]$

Out[12]= $\text{If}\left[\text{Re}[n] > -1 \,\&\&\, (x \notin \text{Reals} \,||\, \text{Re}[x] \geq -1), \frac{(-x)^{1+n}\,\text{Hypergeometric2F1}\left[1, 1, 2+n, \frac{x}{1+x}\right]}{1+n+x+n\,x}, \text{Integrate}\left[\frac{\left(\frac{t-x}{1+t}\right)^n}{1+t}, \{t, x, 0\}, \text{Assumptions} \to !(\text{Re}[n] > -1 \,\&\&\, (x \notin \text{Reals} \,||\, \text{Re}[x] \geq -1))\right]\right]$

In[13]:= $\text{R}[n\_, x\_] := \frac{(-x)^{1+n}\,\text{Hypergeometric2F1}\left[1, 1, 2+n, \frac{x}{1+x}\right]}{1+n+x+n\,x}$

In[14]:=
```
N[R[66, -8 / 10], 20]
```

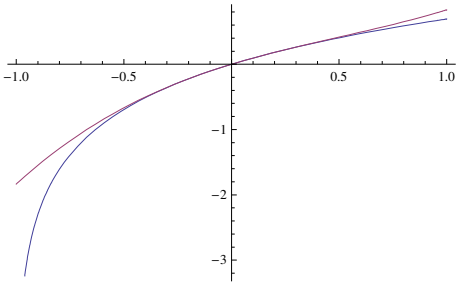
Out[14]= $2.2713837371389864372\times10^{-8}$

In[15]:= $\text{NIntegrate}\left[\frac{1}{1+t}\left(\frac{t+0.8}{t+1}\right)^{66}, \{t, -0.8, 0\}\right]$

Out[15]= 2.27138×10^{-8}

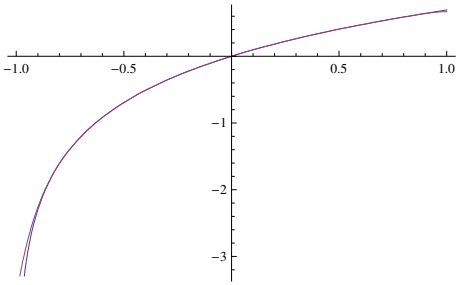
In[16]:=
```
Plot[{Log[1 + x], T[3, x]}, {x, -1, 1}]
```

Out[16]=



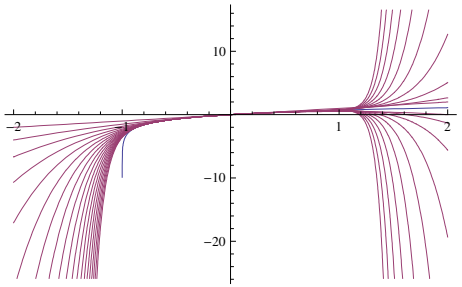
In[17]:=
```
Plot[{Log[1 + x], T[20, x]}, {x, -1, 1}]
```

Out[17]=



In[18]:=
```
Plot[{Log[1 + x], Table[T[n, x], {n, 1, 20}]}, {x, -2, 2}]
```

Out[18]=



In[1]:= `f[x_] := x - E^(-x)`

In[2]:= `f[0]`

Out[2]= -1

In[3]:= `f[1]`

Out[3]= $1 - \frac{1}{e}$

In[4]:= `N[%]`

Out[4]= 0.632121

In[5]:= `Table[{k, N[-(1/2)^(k+1) (1-0) + 10^(-6)]}, {k, 1, 20}]`

Out[5]= {{1, -0.249999}, {2, -0.124999}, {3, -0.062499}, {4, -0.031249},
{5, -0.015624}, {6, -0.0078115}, {7, -0.00390525}, {8, -0.00195213},
{9, -0.000975563}, {10, -0.000487281}, {11, -0.000243141}, {12, -0.00012107},
{13, -0.0000600352}, {14, -0.0000295176}, {15, -0.0000142588}, {16, -6.62939×10^{-6}},
{17, -2.8147×10^{-6}}, {18, -9.07349×10^{-7}}, {19, 4.63257×10^{-8}}, {20, 5.23163×10^{-7}}}

In[6]:= `N[6 Log[10] / Log[2] - 1]`

Out[6]= 18.9316

In[7]:=
```
a = 0; b = 1; For[k = 0, k ≤ 19, k++, {x = (a + b) / 2,
  Print[N[{k, a, b, x, f[x], Abs[b - a] / 2}, 10]], If[f[x] == 0, k = 20, If[f[x] > 0, b = x, a = x]]}]
```

{0, 0, 1.000000000, 0.5000000000, -0.1065306597, 0.5000000000}

{1.000000000, 0.5000000000, 1.000000000, 0.7500000000, 0.2776334473, 0.2500000000}

{2.000000000, 0.5000000000, 0.7500000000, 0.6250000000, 0.08973857148, 0.1250000000}

{3.000000000, 0.5000000000, 0.6250000000, 0.5625000000, -0.007282824731, 0.06250000000}

{4.000000000, 0.5625000000, 0.6250000000, 0.5937500000, 0.04149754984, 0.03125000000}

{5.000000000, 0.5625000000, 0.5937500000, 0.5781250000, 0.01717583919, 0.01562500000}

{6.000000000, 0.5625000000, 0.5781250000, 0.5703125000, 0.004963760389, 0.007812500000}

{7.000000000, 0.5625000000, 0.5703125000, 0.5664062500, -0.001155202015, 0.003906250000}

{8.000000000, 0.5664062500, 0.5703125000, 0.5683593750, 0.001905359613, 0.001953125000}

{9.000000000, 0.5664062500, 0.5683593750, 0.5673828125, 0.0003753491691, 0.0009765625000}

{10.00000000, 0.5664062500, 0.5673828125, 0.5668945313, -0.0003898587974, 0.0004882812500}

{11.00000000, 0.5668945313, 0.5673828125, 0.5671386719, $-7.237911847\times10^{-6}$, 0.0002441406250}

{12.00000000, 0.5671386719, 0.5673828125, 0.5672607422, 0.0001840598537, 0.0001220703125}

{13.00000000, 0.5671386719, 0.5672607422, 0.5671997070, 0.00008841202725, 0.00006103515625}

{14.00000000, 0.5671386719, 0.5671997070, 0.5671691895, 0.00004058732179, 0.00003051757813}

{15.00000000, 0.5671386719, 0.5671691895, 0.5671539307, 0.00001667477100, 0.00001525878906}

{16.00000000, 0.5671386719, 0.5671539307, 0.5671463013, 4.718446081×10^{-6}, 7.629394531×10^{-6}}

{17.00000000, 0.5671386719, 0.5671463013, 0.5671424866, $-1.259728757\times10^{-6}$, 3.814697266×10^{-6}}

{18.00000000, 0.5671424866, 0.5671463013, 0.5671443939, 1.729359694×10^{-6}, 1.907348633×10^{-6}}

{19.00000000, 0.5671424866, 0.5671443939, 0.5671434402, 2.348157265×10^{-7}, 9.536743164×10^{-7}}

In[8]:= `N[x, 20]`

Out[8]= 0.56714344024658203125

In[9]:= `FindRoot[f[y] == 0, {y, 1/2}, WorkingPrecision → 20]`

Out[9]= {y → 0.56714329040978387300}

In[10]:= `%[[1]][[2]] - N[x, 20]`

Out[10]= $-1.498367981583\times10^{-7}$

In[1]:= `f[x_] := x - E^(-x)`

In[2]:= `f[0.3]`

Out[2]= -0.440818

In[3]:= `f[1]`

Out[3]= $1 - \frac{1}{e}$

In[4]:= `N[%]`

Out[4]= 0.632121

In[5]:= `g[x_] := E^(-x)`

In[6]:= `N[Abs[g'[1]]]`

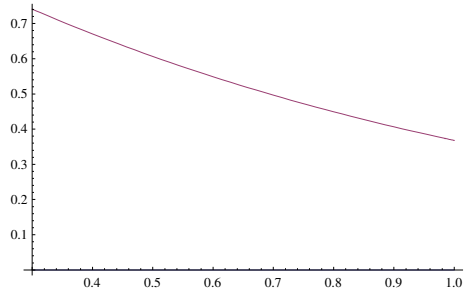
Out[6]= 0.367879

In[7]:= `N[Abs[g'[0.3]]]`

Out[7]= 0.740818

In[8]:= `Plot[{0, Abs[g'[x]]}, {x, 0.3, 1}]`

Out[8]=



In[9]:= `L = Abs[g'[0.3]]`

Out[9]= 0.740818

In[12]:= `N[Log[10^(-6) (1 - L) / Abs[0.5 - g[0.5]]] / Log[L]]`

Out[12]= 43.088

In[13]:= `x = 1/2; For[k = 0, k ≤ 44, k++, {Print[N[{k, x, Abs[x - g[x]]}, 20]], x = g[x]}]`

{0, 0.50000000000000000000, 0.10653065971263342360}

{1.0000000000000000000, 0.60653065971263342360, 0.06129144782002836818 4}

{2.0000000000000000000, 0.54523921189260505542, 0.03446388298546315444 3}

{3.0000000000000000000, 0.57970309487806820986, 0.01963846693916634584 7}

{4.0000000000000000000, 0.56006462793890186402, 0.01110752103831329135 0}

{5.0000000000000000000, 0.57117214897721515537, 0.00630920199689171401 33}

{6.0000000000000000000, 0.56486294698032344135, 0.00357510058974281424 59}

{7.0000000000000000000, 0.56843804757006625560, 0.00202859482314544218 98}

{8.0000000000000000000, 0.56640945274692081341, 0.00115018151532161380 62}

{9.0000000000000000000, 0.56755963426224242722, 0.00065242132677103481 030}

{10.0000000000000000000, 0.56690721293547139240, 0.00036998303530710160 875}

{11.0000000000000000000, 0.56727719597077849401, 0.00020984411705033422 211}

{12.0000000000000000000, 0.56706735185372815979, 0.00011900823390999387 117}

{13.0000000000000000000, 0.56718636008763815366, 0.00006749583065240481 0286}

{14.0000000000000000000, 0.56711886425698574885, 0.00003827945065880079 4741}

{15.000000000000000000, 0.56715714370764454965, 0.000021710048371329448542}

{16.000000000000000000, 0.56713543365927322020, 0.000012312671351699430873}

{17.000000000000000000, 0.56714774633062491963, $6.9830608182713188866 \times 10^{-6}$}

{18.000000000000000000, 0.56714076326980664831, $3.9603922702478266151 \times 10^{-6}$}

{19.000000000000000000, 0.56714472366207689614, $2.2461111319596106240 \times 10^{-6}$}

{20.000000000000000000, 0.56714247755094493653, $1.2738664628566168581 \times 10^{-6}$}

{21.000000000000000000, 0.56714375141740779314, $7.2246494438729632933 \times 10^{-7}$}

{22.000000000000000000, 0.56714302895246340585, $4.0974110488355872071 \times 10^{-7}$}

{23.000000000000000000, 0.56714343869356828941, $2.3238193158954556634 \times 10^{-7}$}

{24.000000000000000000, 0.56714320631163669986, $1.3179384908384030462 \times 10^{-7}$}

{25.000000000000000000, 0.56714333810548578370, $7.4745998585648272877 \times 10^{-8}$}

{26.000000000000000000, 0.56714326335948719805, $4.2391691145232799503 \times 10^{-8}$}

{27.000000000000000000, 0.56714330575117834329, $2.4042163342896300180 \times 10^{-8}$}

{28.000000000000000000, 0.56714328170901500039, $1.3635351581586064503 \times 10^{-8}$}

{29.000000000000000000, 0.56714329534436658197, $7.7331981764373032538 \times 10^{-9}$}

{30.000000000000000000, 0.56714328761116840554, $4.3858314544915963971 \times 10^{-9}$}

{31.000000000000000000, 0.56714329199699986003, $2.4873948837897066942 \times 10^{-9}$}

{32.000000000000000000, 0.56714328950960497624, $1.4107093184563514195 \times 10^{-9}$}

{33.000000000000000000, 0.56714329092031429470, $8.0007432483695266398 \times 10^{-10}$}

{34.000000000000000000, 0.56714329012023996986, $4.5375678511027853684 \times 10^{-10}$}

{35.000000000000000000, 0.56714329057399675497, $2.5734511616933525811 \times 10^{-10}$}

{36.000000000000000000, 0.56714329031665163880, $1.4595155594997770823 \times 10^{-10}$}

{37.000000000000000000, 0.56714329046260319475, $8.2775445683566486386 \times 10^{-11}$}

{38.000000000000000000, 0.56714329037982774907, $4.6945538629577576176 \times 10^{-11}$}

{39.000000000000000000, 0.56714329042677328770, $2.6624847248610860829 \times 10^{-11}$}

{40.000000000000000000, 0.56714329040014844045, $1.5100103475179521618 \times 10^{-11}$}

{41.000000000000000000, 0.56714329041524854392, $8.5639223704593851553 \times 10^{-12}$}

{42.000000000000000000, 0.56714329040668462155, $4.8569711119905473984 \times 10^{-12}$}

{43.000000000000000000, 0.56714329041154159266, $2.7545985778814336245 \times 10^{-12}$}

{44.000000000000000000, 0.56714329040878699409, $1.5622521012171932570 \times 10^{-12}$}

In[14]:= `N[x, 20]`

Out[14]= 0.56714329041034924619

In[15]:= `FindRoot[f[y] == 0, {y, 1 / 2}, WorkingPrecision → 20]`

Out[15]= {y → 0.56714329040978387300}

In[16]:= `%[[1]][[2]] - N[x, 20]`

Out[16]= $-5.653732 \times 10^{-13}$

In[1]:= `f[x_] := x - E^(-x)`

In[2]:= `f[0]`

Out[2]= -1

In[3]:= `f[1]`

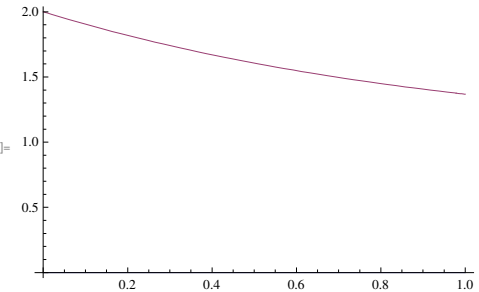
Out[3]= $1 - \frac{1}{e}$

In[4]:= `f'[x]`

Out[4]= $1 + e^{-x}$

In[5]:= `Plot[{0, f'[x]}, {x, 0, 1}]`

Out[5]=



In[6]:= `b = 1 / f'[1]`

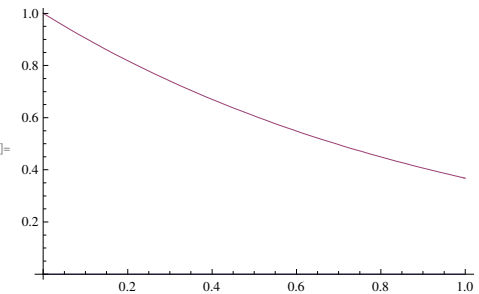
Out[6]= $\frac{1}{1 + \frac{1}{e}}$

In[7]:= `f''[x]`

Out[7]= $-e^{-x}$

In[8]:= `Plot[{0, Abs[f''[x]]}, {x, 0, 1}]`

Out[8]=



In[10]:= `g = Abs[f''[0]]`

Out[10]= 1

In[11]:= `x = 1 / 2`

Out[11]= $\frac{1}{2}$

In[12]:= `a = Abs[f[x] / f'[x]]`

Out[12]= $\frac{-\frac{1}{2} + \frac{1}{\sqrt{e}}}{1 + \frac{1}{\sqrt{e}}}$

In[13]:= **q = a b g**

Out[13]= $$\frac{-\frac{1}{2}+\frac{1}{\sqrt{e}}}{\left(1+\frac{1}{e}\right)\left(1+\frac{1}{\sqrt{e}}\right)}$$

In[14]:= **N[q]**

Out[14]= 0.0484772

In[15]:= **% < 1 / 2**

Out[15]= True

In[16]:= **{x - 2 a, x + 2 a}**

Out[16]= $$\left\{\frac{1}{2}-\frac{2\left(-\frac{1}{2}+\frac{1}{\sqrt{e}}\right)}{1+\frac{1}{\sqrt{e}}},\ \frac{1}{2}+\frac{2\left(-\frac{1}{2}+\frac{1}{\sqrt{e}}\right)}{1+\frac{1}{\sqrt{e}}}\right\}$$

In[17]:= **N[%]**

Out[17]= {0.367378, 0.632622}

In[18]:= **Table[N[{k, -2 a q^(2^k - 1) + 10^(-6)}], {k, 1, 10}]**

Out[18]= $\{\{1., -0.00642815\}, \{2., -0.0000141088\}, \{3., 9.99917\times10^{-7}\}, \{4., 1.\times10^{-6}\}, \{5., 1.\times10^{-6}\}, \{6., 1.\times10^{-6}\}, \{7., 1.\times10^{-6}\}, \{8., 1.\times10^{-6}\}, \{9., 1.\times10^{-6}\}, \{10., 1.\times10^{-6}\}\}$

In[19]:= **Log[Log[10^(-6) / (2 a)] / Log[q] + 1] / Log[2]**

Out[19]= $$\frac{\mathrm{Log}\left[1+\frac{\mathrm{Log}\left[\frac{1+\frac{1}{\sqrt{e}}}{2000000\left(-\frac{1}{2}+\frac{1}{\sqrt{e}}\right)}\right]}{\mathrm{Log}\left[\frac{-\frac{1}{2}+\frac{1}{\sqrt{e}}}{\left(1+\frac{1}{e}\right)\left(1+\frac{1}{\sqrt{e}}\right)}\right]}\right]}{\mathrm{Log}[2]}$$

In[20]:= **N[%]**

Out[20]= 2.29193

In[21]:= **x**

Out[21]= $\frac{1}{2}$

In[22]:= **For[k = 1, k ≤ 3, k++, x = x - f[x] / f'[x]]**

In[23]:= **A = N[x, 30]**

Out[23]= 0.567143290409781028699576649415

In[24]:= **Clear[x]**

In[25]:= **B = FindRoot[f[x] == 0, {x, 1}, WorkingPrecision → 30]**

Out[25]= {x → 0.567143290409783872999968662210}

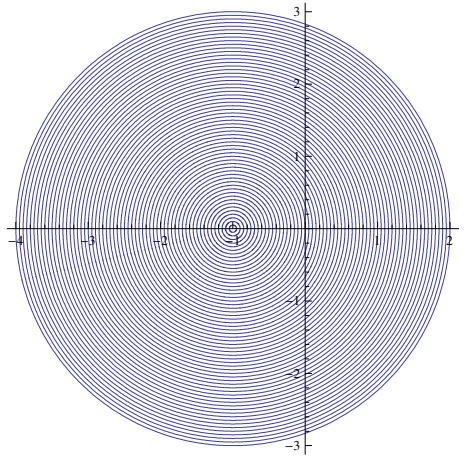
In[26]:= **A - B[[1]][[2]]**

Out[26]= $-2.84430039201280\times10^{-15}$


```
% We want to find the complex roots of z^5 - z - 1 = 0,
      using the iterative method z = (1 + z)^(1 / 5) = g (z).
            To use the Banach Fixed Point Theorem we want to see
      1) g : A → A where A is a closed subset of C
   2) g is a contraction (Lipschitz continuous with L < 1)

Lets try a circular region A, around z = -1. To do this we let 1 + z = R E^(i t),
so that z = R E^(I t) - 1, and let R goes from 0 to 3, gives us a disk aroud z = -1.
```

In[1]:= **A =**
ParametricPlot[Table[{Re[R E^(I t) - 1], Im[R E^(I t) - 1]}, {R, 0, 3, 0.05}], {t, 0, 2 Pi}]

Out[1]=




```
% Is this region a contraction for g?
```

In[2]:= **g[z_] := (1 + z)^(1 / 5)**

In[3]:= **g'[z]**

Out[3]= $$\frac{1}{5\,(1+z)^{4/5}}$$

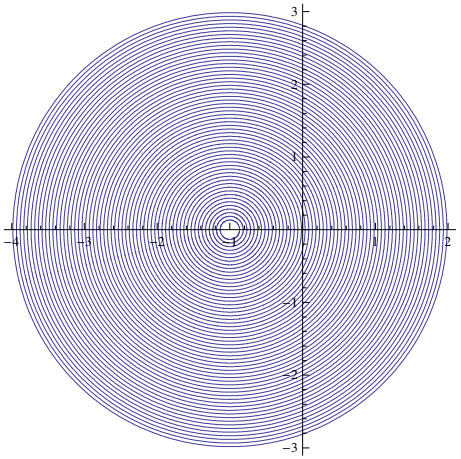
```
% Now g' (z) =
 1 / 5 (1 + z)^(-4 / 5) = 1 / 5 (R E^(I t))^(-4 / 5) = 1 / (5 R^(4 / 5)) E^(-4 I t / 5),
So that the complex Mod (this is the Matric now), |g' (z)| = 1/(5 R^(4/5)),
this is less than 1 means,
R^(4 / 5) > 1 / 5 or R > (1 / 5)^(5 / 4),
so we get rid of radii less than this ie less than
```

In[4]:= **N[(1 / 5)^(5 / 4)]**

Out[4]= 0.133748

In[5]:=
```
A = ParametricPlot[
  Table[{Re[R E^(I t) - 1], Im[R E^(I t) - 1]}, {R, (1 / 5)^(5 / 4), 3, 0.05}], {t, 0, 2 Pi}]
```

Out[5]=



```
% But how actually g ' decides the convergence in
 the complex case ?. What does the MVT looks like in complex ?
  Complex line integral over a curve c (with end point z1 and z2) of g ' is
```

g (z2) - g (z1) = $\int_{z1}^{z2} g'\, dz$ provided that g is analytic and the curve c is smooth,

(the integral will remain the same for any smooth curve with the same end points,
that can be collapsed to c, ie homeomorphic to c) . Now for a straight line c

$\left| g(z2) - g(z1) \right| <= \int_{z1}^{z2} \left| g' \right| \left| dz \right| \le \text{Max}\{ |g'| \} \int_{z1}^{z2} \left| dz \right|$

= Max{|g'|}Lenth of c=Max{|g'|}|z2-z1|.

There will be a maximum (since g is analytic) and,
L = Max { | g ' |} is a Lipchitz constant with the requirement for a contraction is ,
L < 1 (ie .R > (1 / 5) ^ (5 / 4)).
A suffficient requirement here is to able to join any two points by a
straight line which lies entirely within the set. Such sets are called Convex,
here the set A is clearly not convex due to its hole at the center.

But now we can directly work with the given function for Lipschitz continuity as follows
| g (z2) - g (z1) | = |(1+z1)^(1/5)-(1+z2)^(1/5)|
Notice that
(x - y) = (x^(1 / 5) - y^(1 / 5))
(x^(4 / 5) + x^(3 / 5) y^(1 / 5) + x^(2 / 5) y^(2 / 5) + x^(1 / 5) y^(3 / 5) + y^(4 / 5))
and that,
| x - y | >= | x^(1 / 5) - y^(1 / 5) | (| x^(4 / 5) | + | x^(3 / 5) || y^(1 / 5) |
+ | x^(2 / 5) || y^(2 / 5) | + | x^(1 / 5) || y^(3 / 5) | + | y^(4 / 5) |)

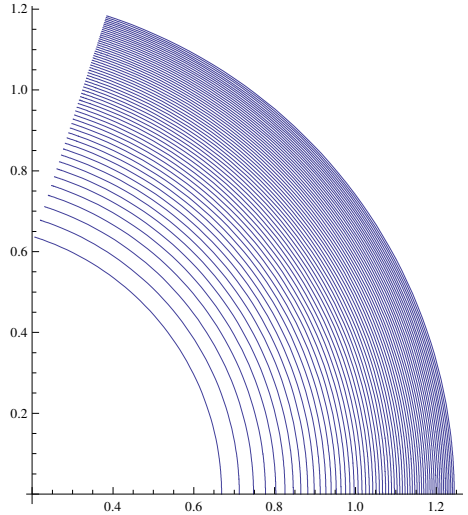
>= | x^(1 / 5) - y^(1 / 5) | 5 R^(4 / 5) for | x |, | y | > R

So with x = 1 + z2 and y = 1 + z1,
we get | g (z2) - g (z1) | ≤ 1 / (5 R^(4 / 5)) | z2 - z1 | giving us the same region A with,
R > (1 / 5) ^ (5 / 4) for g to be a contraction (without assuming the convexity of A).

% Now the image of g (z) = (1 + z) ^ (1 / 5) = (R E^(I t)) ^ (1 / 5) = R^(1 / 5) E^(I t / 5)

In[6]:=
```
B = ParametricPlot[Table[{Re[R^(1 / 5) E^(I t / 5)], Im[R^(1 / 5) E^(I t / 5)]},
  {R, (1 / 5)^(5 / 4), 3, 0.05}], {t, 0, 2 Pi}]
```

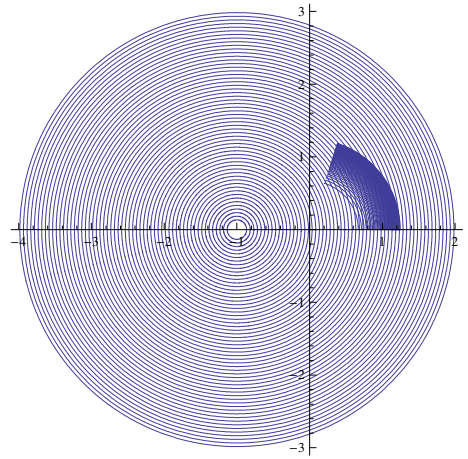
Out[6]=



% To see if g : A → A, we plot them together

In[7]:=
```
Show[{A, B}]
```

Out[7]=



% Lets see what the roots actually are

In[8]:=
```
Roots[z^5 - z - 1 == 0, z]
```

Out[8]=
```
z == Root[-1 - #1 + #1^5 &, 1] || z == Root[-1 - #1 + #1^5 &, 2] ||
 z == Root[-1 - #1 + #1^5 &, 3] || z == Root[-1 - #1 + #1^5 &, 4] || z == Root[-1 - #1 + #1^5 &, 5]
```

In[9]:= **N[%]**

Out[9]= z == 1.1673 || z == -0.764884 - 0.352472 i ||
z == -0.764884 + 0.352472 i || z == 0.181232 - 1.08395 i || z == 0.181232 + 1.08395 i

In[10]:= **T = Table[{Re[%[[n]][[2]]], Im[%[[n]][[2]]]}, {n, 1, 5}]**

Out[10]= {{1.1673, 0}, {-0.764884, -0.352472},
{-0.764884, 0.352472}, {0.181232, -1.08395}, {0.181232, 1.08395}}

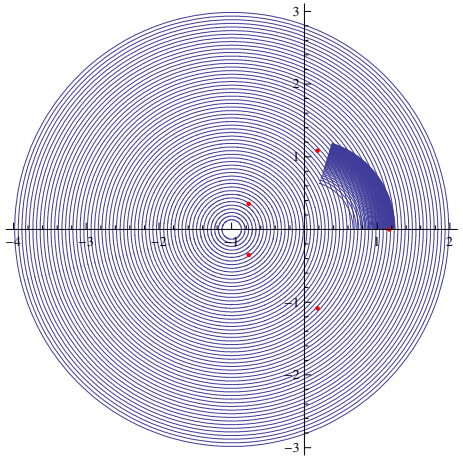
% Lets see where the roots actually located

In[11]:= **P = Graphics[{Red, Point[T]}];**

% Now everyting together

In[12]:= **Show[{A, B, P}]**

Out[12]=



% We can see that we can get the 1 st root by this g. So lets do the iterations.

In[13]:= **z = 1 + 2 I; S = {};**
For[k = 0, k ≤ 5, k++, {Print[N[z]], S = Append[S, {Re[z], Im[z]}], z = N[g[z], 10]}]

1. + 2. i

1.21599 + 0.192593 i

1.17321 + 0.0203438 i

1.16795 + 0.00218662 i

1.16737 + 0.000235485 i

1.16731 + 0.0000253657 i

% S is the point set

In[14]:= **S**

Out[14]= {{1, 2}, {1.2159869826, 0.1925934177},
{1.1732078739, 0.0203438340}, {1.1679474384, 0.0021866166},
{1.1673733782, 0.0002354852}, {1.1673114550, 0.0000253657}}

% These are the points in 2 D

In[15]:= **Q = Graphics[Point[S]];**

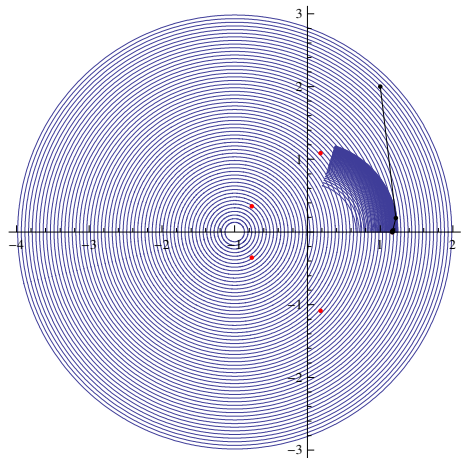
% This is the line connected version

In[16]:= **U = Graphics[Line[S]];**

% See all of them in the same picture

In[17]:= **Show[{A, B, P, Q, U}]**

Out[17]=



% It is clear that we cannot get the other roots by using this g. This is because our root taking is not general (ie there are other branches). We know E^(I t) = Cos[t] + I Sin[t], so that E^(I 2 n Pi) = 1 where n is an integer. Now

g[z_] := (1 + z)^(1 / 5) = ((1 + z) .1)^(1 / 5) = (1 + z)^(1 / 5) E^(I 2 n Pi / 5)

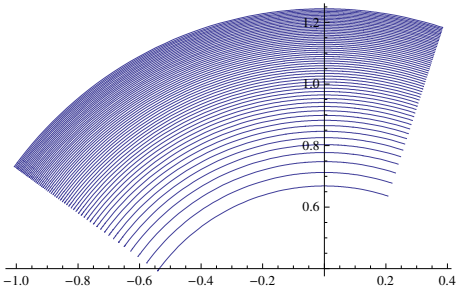
Lets define a new g with n = 1

In[18]:= **g[z_] := (1 + z)^(1 / 5) E^(I 2 Pi / 5)**

% Now the image of g (z) with 1 + z = R E^(I t) is

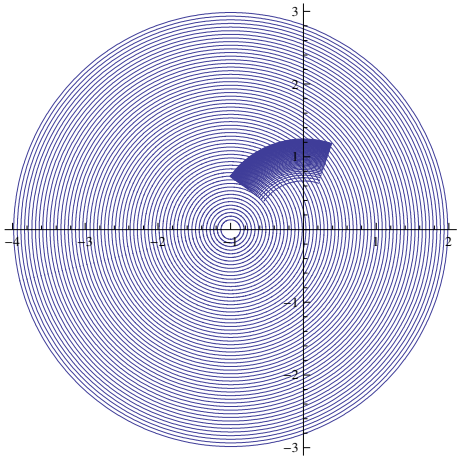
In[19]:=
```
B = ParametricPlot[
  Table[{Re[R^(1 / 5) E^(I t / 5) E^(I 2 Pi / 5)], Im[R^(1 / 5) E^(I t / 5) E^(I 2 Pi / 5)]},
   {R, (1 / 5)^(5 / 4), 3, 0.05}], {t, 0, 2 Pi}]
```

Out[19]=



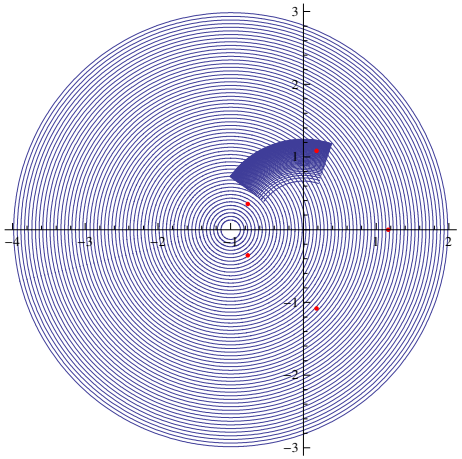
In[20]:=
```
Show[{A, B}]
```

Out[20]=



In[21]:=
```
Show[{A, B, P}]
```

Out[21]=




```
% We can see that now we can get the 2 nd root by this g. So lets do the iterations
```

In[22]:=
```
z = 1 + 2 I;  S = {};
For[k = 0, k ≤ 5, k++, {Print[N[z]], S = Append[S, {Re[z], Im[z]}], z = N[g[z], 10]}]
```

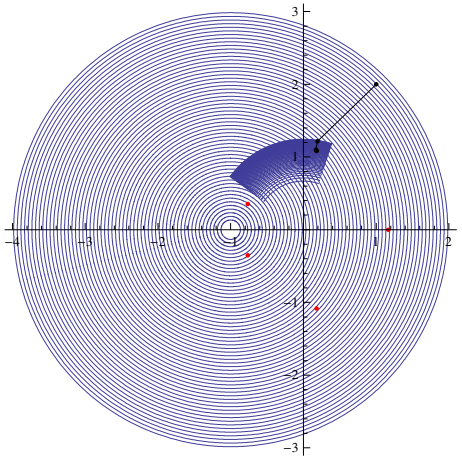
```
1. + 2. i

0.192593 + 1.21599 i

0.17188 + 1.09902 i

0.178956 + 1.0848 i

0.180915 + 1.08385 i

0.181207 + 1.08392 i
```

In[23]:=
```
Q = Graphics[Point[S]];
```

In[24]:=
```
U = Graphics[Line[S]];
```

In[25]:=
```
Show[{A, B, P, Q, U}]
```

Out[25]=

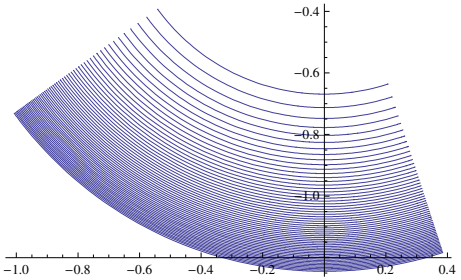


```
% Now to find the 5 th root,  lets define a new g with n = 3
```

In[26]:=
```
g[z_] := (1 + z) ^ (1 / 5) E^(I 6 Pi / 5)
```

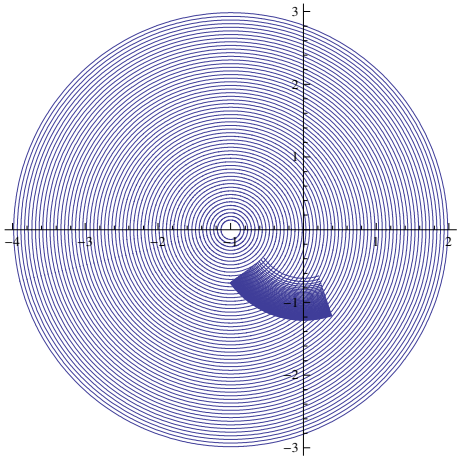
In[27]:=
```
B = ParametricPlot[
  Table[{Re[R^(1 / 5) E^(I t / 5) E^(I 6 Pi / 5)], Im[R^(1 / 5) E^(I t / 5) E^(I 6 Pi / 5)]},
   {R, (1 / 5) ^ (5 / 4), 3, 0.05}], {t, 0, 2 Pi}]
```

Out[27]=



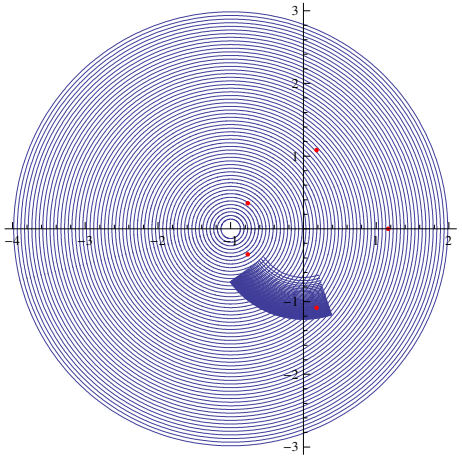
In[28]:=
```
Show[{A, B}]
```

Out[28]=



In[29]:=
```
Show[{A, B, P}]
```

Out[29]=



In[30]:=
```
z = 1 + 2 I;  S = {};
For[k = 0, k ≤ 5, k++, {Print[N[z]], S = Append[S, {Re[z], Im[z]}], z = N[g[z], 10]}]
```

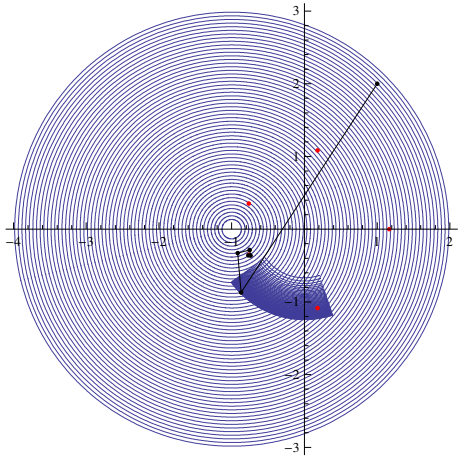
```
1. + 2. i
-0.870551 - 0.870551 i
-0.917779 - 0.32846 i
-0.752724 - 0.286094 i
-0.738875 - 0.363051 i
-0.770659 - 0.361744 i
```

In[31]:= `Q = Graphics[Point[S]];`

In[32]:= `U = Graphics[Line[S]];`

In[33]:= `Show[{A, B, P, Q, U}]`

Out[33]=



```
% We have a serious problem,
iterations goes outside the region and we don ' t get the root we want. How did that happen?

This happens because of the way Mathematica defines Arg,
it goes from - Pi to Pi,  not from 0 to 2 Pi. We were good
 as long as we were on the upper half plane. Now redefine Arg as arg
```

In[34]:= `arg[z_] := 2 Pi (1 - Sign[Arg[z]]) / 2 + Arg[z]`

```
% Redefine g
```

In[35]:= `g[z_] := Abs[(1 + z)]^(1 / 5) E^(I arg[1 + z] / 5) E^(I 6 Pi / 5)`

```
% Lets do the iterations now
```

In[36]:= `z = 1 + 2 I;  S = {};`
`For[k = 0, k ≤ 5, k++, {Print[N[z]], S = Append[S, {Re[z], Im[z]}], z = N[g[z], 10]}]`

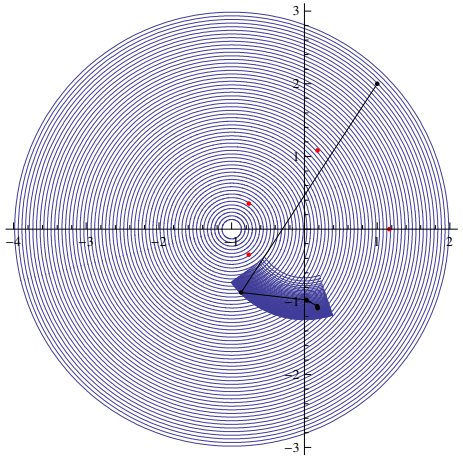

```
1. + 2. i
-0.870551 - 0.870551 i
0.0287747 - 0.97436 i
0.173477 - 1.05806 i
0.182598 - 1.08049 i
0.181673 - 1.08369 i
```

In[37]:= `Q = Graphics[Point[S]];`

In[38]:= `U = Graphics[Line[S]];`

In[39]:= `Show[{A, B, P, Q, U}]`

Out[39]=



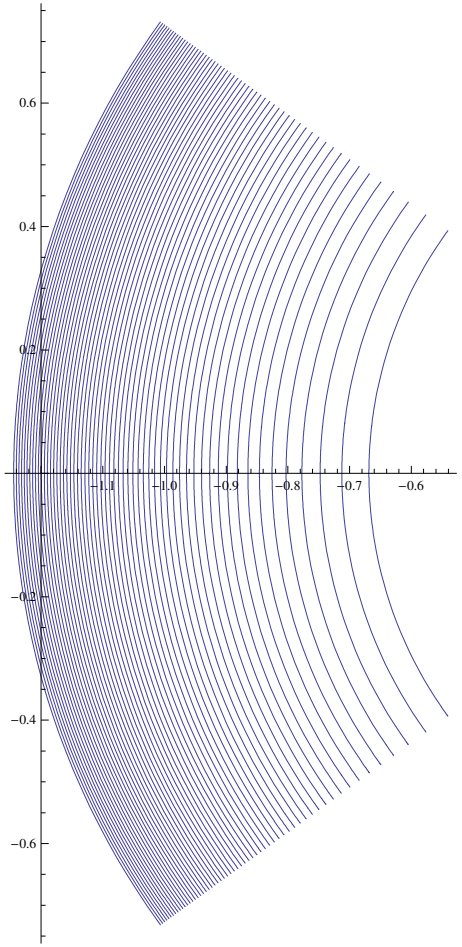
```
% To get the 3 rd root Lets define new g with n = 2
```

In[40]:= `g[z_] := Abs[(1 + z)]^(1 / 5) E^(I arg[1 + z] / 5) E^(I 4 Pi / 5)`

```
% Now the image of g (z) with 1 + z = R E^(I t)
```

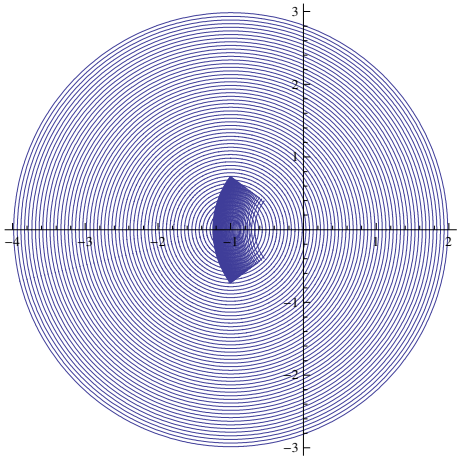
In[41]:=
```
B = ParametricPlot[
  Table[{Re[R^(1 / 5) E^(I t / 5) E^(I 4 Pi / 5)], Im[R^(1 / 5) E^(I t / 5) E^(I 4 Pi / 5)]},
   {R, (1 / 5)^(5 / 4), 3, 0.05}], {t, 0, 2 Pi}]
```

Out[41]=



In[42]:=
```
Show[{A, B}]
```

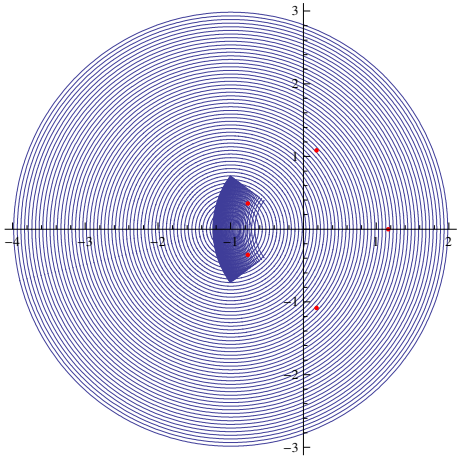
Out[42]=




```
% This time however, the image of g is not within A (there is ahole in A at the center),
violating the requirements of the Banach Fixed Point Theorem. So things can go wrong.

% Lets plot with roots
```

In[43]:=
```
Show[{A, B, P}]
```

Out[43]=




```
% We can see that there are two roots,  violating uniqueness. Lets try iterations anyway
```

In[44]:=
```
z = 1 + 2 I;  S = {};
For[k = 0, k ≤ 5, k++, {Print[N[z]], S = Append[S, {Re[z], Im[z]}], z = N[g[z], 10]}]
```

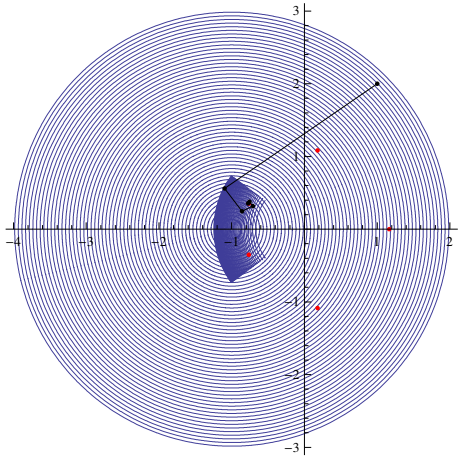
```
1. + 2. i
-1.09696 + 0.558928 i
-0.858087 + 0.246567 i
-0.710552 + 0.316129 i
-0.755481 + 0.376594 i
-0.774697 + 0.354669 i
```

In[45]:= `Q = Graphics[Point[S]];`

In[46]:= `U = Graphics[Line[S]];`

In[47]:= `Show[{A, B, P, Q, U}]`

Out[47]=



```
% We will never get the 4 th root if we start anywhere on the upper half
  plane. But inspired by symmetry,  lets start with z = 1 - 2 I to get the other root.
```

In[48]:=
```
z = 1 - 2 I;  S = {};
For[k = 0, k ≤ 5, k++, {Print[N[z]], S = Append[S, {Re[z], Im[z]}], z = N[g[z], 10]}]
```

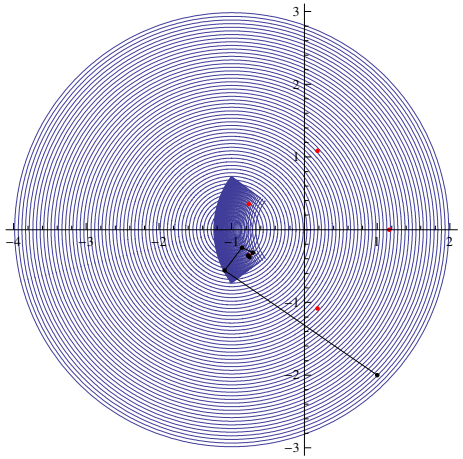
```
1. - 2. i
-1.09696 - 0.558928 i
-0.858087 - 0.246567 i
-0.710552 - 0.316129 i
-0.755481 - 0.376594 i
-0.774697 - 0.354669 i
```

In[49]:= `Q = Graphics[Point[S]];`

In[50]:= `U = Graphics[Line[S]];`

In[51]:= `Show[{A, B, P, Q, U}]`

Out[51]=



```
% We had all those trouble and guesswork since we failed to meet requirements of the Banach
  Fixed Point Theorem. Now we try to change the region A to account for thse requirements. I
  have figured out that it can be done by reducing the radious to 2 and angle less than (see why?)
```

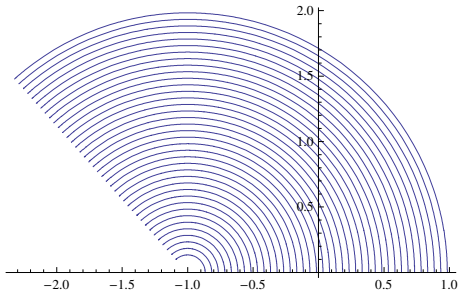
In[52]:= `N[5 (Pi - ArcSin[(1 / 5) ^ (5 / 4)] - 4 Pi / 5)]`

Out[52]= `2.47084`

```
% Taking angles upto 2.3 the new domain is
```

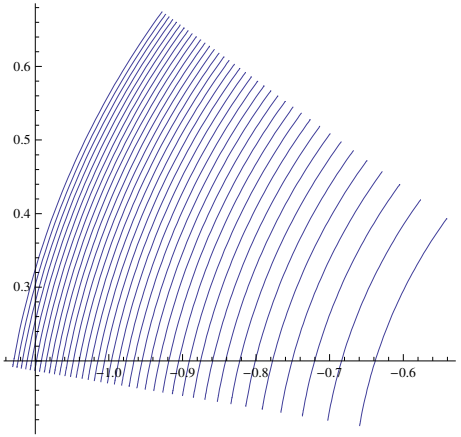
In[53]:=
```
A = ParametricPlot[
  Table[{Re[R E^(I t) - 1], Im[R E^(I t)]}, {R, (1 / 5) ^ (5 / 4), 2, 0.05}], {t, 0, 2.3}]
```

Out[53]=



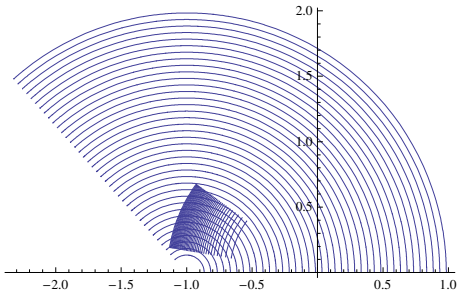
In[54]:=
```
B = ParametricPlot[
  Table[{Re[R^(1 / 5) E^(I t / 5) E^(I 4 Pi / 5)], Im[R^(1 / 5) E^(I t / 5) E^(I 4 Pi / 5)]},
   {R, (1 / 5)^(5 / 4), 2, 0.05}], {t, 0, 2.3}]
```

Out[54]=



In[55]:=
```
Show[{A, B}]
```

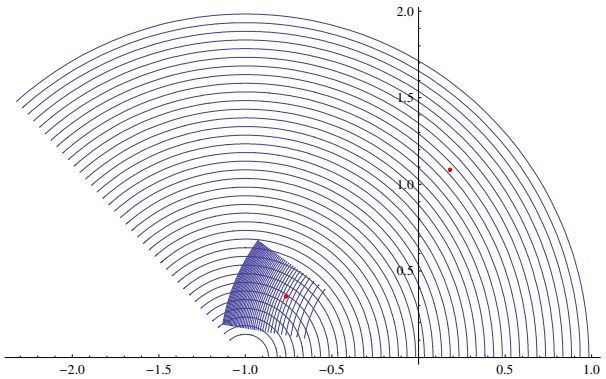
Out[55]=




```
% Now we meet both requirements of the theorem
```


In[56]:=
```
Show[{A, B, P}]
```

Out[56]=




```
% We can see that there is a unique root in the overlapping regions,
we must be able to get it. Try iterations
```

In[57]:=
```
z = -0.5 + 1.5 I;  S = {};
For[k = 0, k ≤ 5, k++, {Print[N[z]], S = Append[S, {Re[z], Im[z]}], z = N[g[z], 10]}]
```

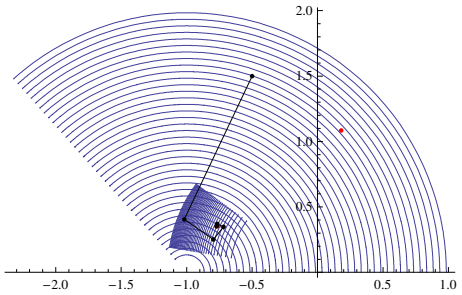
```
-0.5 + 1.5 i
-1.01838 + 0.404996 i
-0.796244 + 0.250753 i
-0.718113 + 0.347448 i
-0.766478 + 0.370704 i
-0.771869 + 0.350813 i
```

In[58]:=
```
Q = Graphics[Point[S]];
```

In[59]:=
```
U = Graphics[Line[S]];
```

In[60]:=
```
Show[{A, B, P, Q, U}]
```

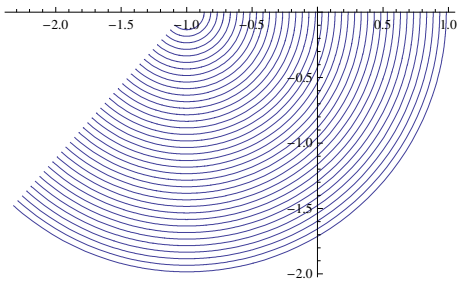
Out[60]=




```
% Now we try to get the 4 th root acoording to the Banach Fixed Point Theorem. We
 find a domain on the lower half plane which is a mirror image of the previous one
```

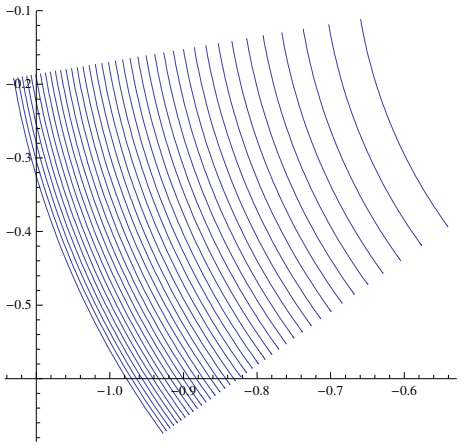
In[61]:=
```
A = ParametricPlot[Table[{Re[R E^(I t) - 1], Im[R E^(I t)]},
    {R, (1 / 5)^(5 / 4), 2, 0.05}], {t, 2 Pi - 2.3, 2 Pi}]
```

Out[61]=



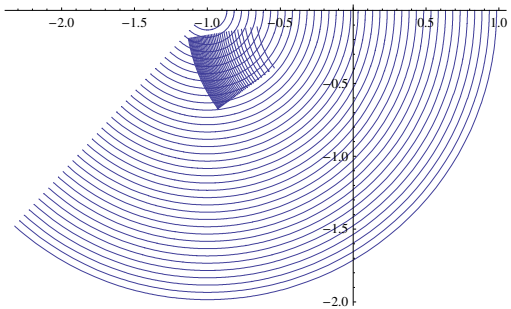
In[62]:=
```
B = ParametricPlot[
  Table[{Re[R^(1 / 5) E^(I t / 5) E^(I 4 Pi / 5)], Im[R^(1 / 5) E^(I t / 5) E^(I 4 Pi / 5)]},
   {R, (1 / 5)^(5 / 4), 2, 0.05}], {t, 2 Pi - 2.3, 2 Pi}]
```

Out[62]=



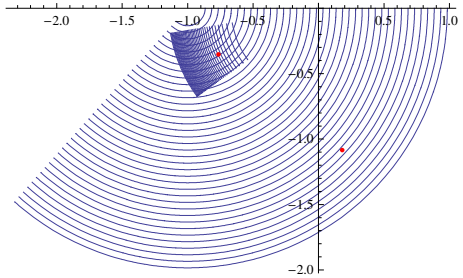
In[63]:=
```
Show[{A, B}]
```

Out[63]=



In[64]:=
```
Show[{A, B, P}]
```

Out[64]=



In[65]:=
```
z = -0.5 - 1.5 I;  S = {};
For[k = 0, k ≤ 5, k++, {Print[N[z]], S = Append[S, {Re[z], Im[z]}], z = N[g[z], 10]}]
```

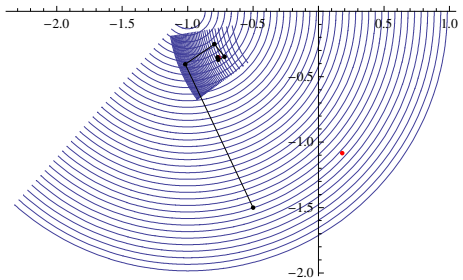
```
-0.5 - 1.5 i

-1.01838 - 0.404996 i

-0.796244 - 0.250753 i

-0.718113 - 0.347448 i

-0.766478 - 0.370704 i

-0.771869 - 0.350813 i
```

In[66]:=
```
Q = Graphics[Point[S]];
```

In[67]:=
```
U = Graphics[Line[S]];
```

In[68]:=
```
Show[{A, B, P, Q, U}]
```

Out[68]=



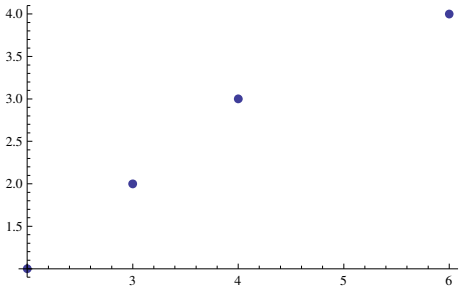

```
% Finally we found all the roots and saw
 Banach Fixed Point Theorem in action !  We are happy now !
```

In[1]:= `A = {{2, 1}, {3, 2}, {4, 3}, {6, 4}}`

Out[1]= {{2, 1}, {3, 2}, {4, 3}, {6, 4}}

In[2]:= `a = ListPlot[A, PlotStyle → PointSize[0.02]]`

Out[2]=



In[3]:= `InterpolatingPolynomial[A, x]`

Out[3]= $1 + \left(1 + \frac{1}{24}(4 - x)(-3 + x)\right)(-2 + x)$

In[4]:= `Expand[%]`

Out[4]= $-\frac{x}{12} + \frac{3x^2}{8} - \frac{x^3}{24}$

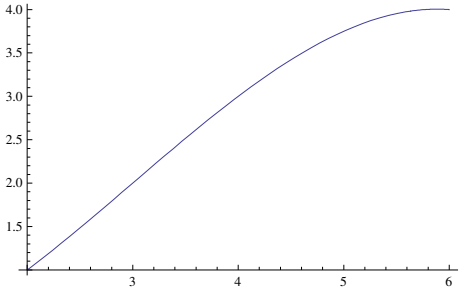
In[5]:= $p[x\_] := -\frac{x}{12} + \frac{3x^2}{8} - \frac{x^3}{24}$

In[6]:= `p[{2, 3, 4, 6}]`

Out[6]= {1, 2, 3, 4}

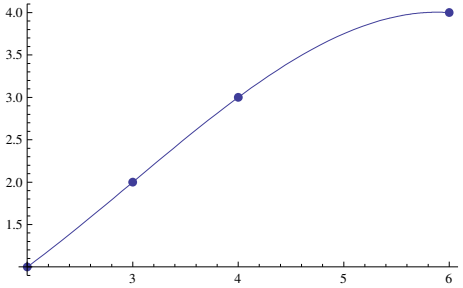
In[7]:= `b = Plot[p[x], {x, 2, 6}]`

Out[7]=



In[8]:= `Show[a, b, PlotRange → All]`

Out[8]=



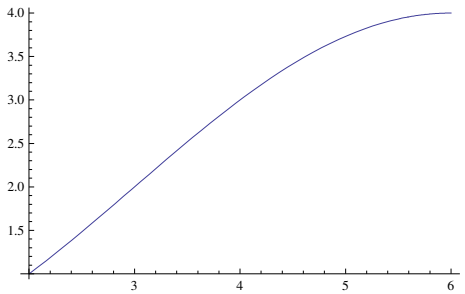
In[9]:= `f[x_] := 4 Sin[Pi x / 12]^2`

In[10]:= `f[{2, 3, 4, 6}]`

Out[10]= {1, 2, 3, 4}

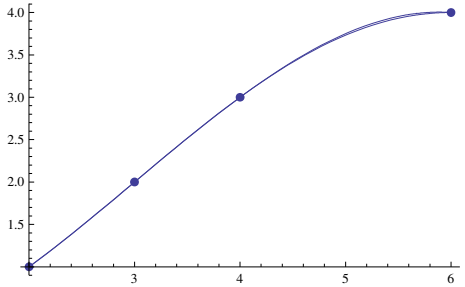
In[11]:= `c = Plot[f[x], {x, 2, 6}]`

Out[11]=



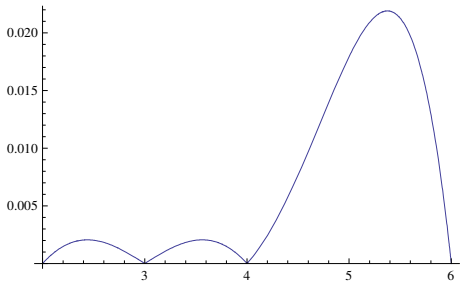
In[12]:= `Show[a, b, c, PlotRange → All]`

Out[12]=



In[13]:= `Plot[Abs[f[x] - p[x]], {x, 2, 6}]`

Out[13]=



In[14]:= `f''''[x]`

Out[14]= $-\frac{1}{648}\pi^4 \mathrm{Cos}\left[\frac{\pi x}{12}\right]^2 + \frac{1}{648}\pi^4 \mathrm{Sin}\left[\frac{\pi x}{12}\right]^2$

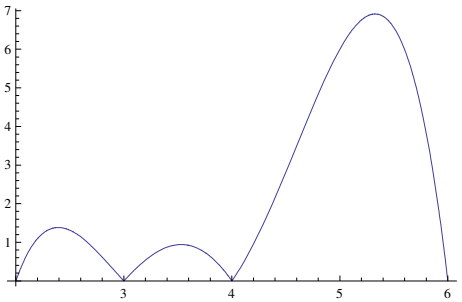
In[15]:= `TrigReduce[%]`

Out[15]= $-\frac{1}{648}\pi^4 \mathrm{Cos}\left[\frac{\pi x}{6}\right]$

In[16]:= `W[x_] := (x - 2) (x - 3) (x - 4) (x - 6)`

In[17]:= `Plot[Abs[W[x]], {x, 2, 6}]`

Out[17]=



In[18]:= `W'[x]`

Out[18]= $(-6+x)\,(-4+x)\,(-3+x)+(-6+x)\,(-4+x)\,(-2+x)+(-6+x)\,(-3+x)\,(-2+x)+(-4+x)\,(-3+x)\,(-2+x)$

In[19]:= `Solve[% == 0, x]`

Out[19]=

$$\left\{\left\{x\to\frac{15}{4}+\frac{7\,5^{2/3}}{4\left(3\left(27+8\,i\sqrt{69}\right)\right)^{1/3}}+\frac{\left(5\left(27+8\,i\sqrt{69}\right)\right)^{1/3}}{4\,3^{2/3}}\right\},\right.$$

$$\left\{x\to\frac{15}{4}-\frac{7\,5^{2/3}\left(1+i\sqrt{3}\right)}{8\left(3\left(27+8\,i\sqrt{69}\right)\right)^{1/3}}-\frac{\left(1-i\sqrt{3}\right)\left(5\left(27+8\,i\sqrt{69}\right)\right)^{1/3}}{8\,3^{2/3}}\right\},$$

$$\left.\left\{x\to\frac{15}{4}-\frac{7\,5^{2/3}\left(1-i\sqrt{3}\right)}{8\left(3\left(27+8\,i\sqrt{69}\right)\right)^{1/3}}-\frac{\left(1+i\sqrt{3}\right)\left(5\left(27+8\,i\sqrt{69}\right)\right)^{1/3}}{8\,3^{2/3}}\right\}\right\}$$

In[20]:= `N[%, 20]`

Out[20]= $\{\{x\to 5.3263454633578327541+0.\times10^{-20}\,i\},$
$\{x\to 2.3927479811269487269+0.\times10^{-20}\,i\},\ \{x\to 3.5309065555152185190+0.\times10^{-20}\,i\}\}$

In[21]:= `W[x] /. %`

Out[21]= $\{-6.9140967887662470 36+0.\times10^{-19}\,i,$
$-1.3827491294425262 42+0.\times10^{-19}\,i,\ 0.941377168208773279+0.\times10^{-19}\,i\}$

In[22]:= `W[{2, 6}]`

Out[22]= {0, 0}

In[23]:= `Maximize[{Abs[W[x]], 2 ≤ x ≤ 6}, x]`

Out[23]= $\{-(-6+\text{Root}[-180+160\,\#1-45\,\#1^2+4\,\#1^3\,\&,3])\,(-4+\text{Root}[-180+160\,\#1-45\,\#1^2+4\,\#1^3\,\&,3])$
$(-3+\text{Root}[-180+160\,\#1-45\,\#1^2+4\,\#1^3\,\&,3])\,(-2+\text{Root}[-180+160\,\#1-45\,\#1^2+4\,\#1^3\,\&,3]),$
$\{x\to\text{Root}[-180+160\,\#1-45\,\#1^2+4\,\#1^3\,\&,3]\}\}$

In[24]:= `N[%, 20]`

Out[24]= {6.9140967887662470361, {x → 5.3263454633578327541}}

In[25]:= $\frac{1}{648}\pi^4$ `( 6.9140967887662470361094495442097563417 6`20.) / 4!`

Out[25]= 0.04330606246880370867 9

In[26]:= `FindMaximum[{Abs[f[x] - p[x]], 2 ≤ x ≤ 6}, {x, 5}]`

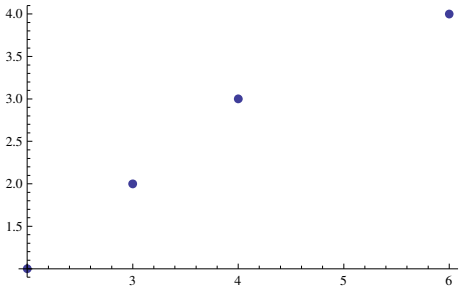
Out[26]= {0.0219112, {x → 5.37469}}

In[1]:= `T = {{2, 1}, {3, 2}, {4, 3}, {6, 4}}`

Out[1]= {{2, 1}, {3, 2}, {4, 3}, {6, 4}}

In[2]:= `q1 = ListPlot[T, PlotStyle → PointSize[0.02]]`

Out[2]=



In[3]:= $X[u\_,v\_] := \sum_{k=1}^{4} T[[k]][[1]]\text{^}u\,T[[k]][[2]]\text{^}v$

In[4]:= `A = {{X[2, 0], X[1, 0]}, {X[1, 0], X[0, 0]}}`

Out[4]= {{65, 15}, {15, 4}}

In[5]:= `A // MatrixForm`

Out[5]//MatrixForm=

$$\begin{pmatrix}65 & 15\\15 & 4\end{pmatrix}$$

In[6]:= `B = {{X[1, 1]}, {X[0, 1]}}`

Out[6]= {{44}, {10}}

In[7]:= `B // MatrixForm`

Out[7]//MatrixForm=

$$\begin{pmatrix}44\\10\end{pmatrix}$$

In[8]:= `Inverse[A].B`

Out[8]= $\left\{\left\{\frac{26}{35}\right\},\left\{-\frac{2}{7}\right\}\right\}$

In[9]:= `N[%]`

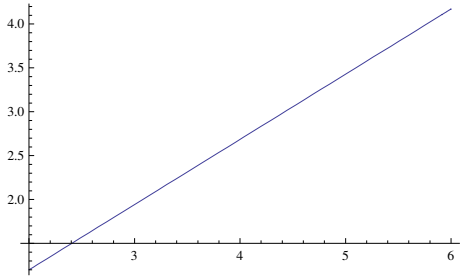
Out[9]= {{0.742857}, {-0.285714}}

In[10]:= `p = Fit[T, {1, x}, x]`

Out[10]= $-0.285714+0.742857\,x$

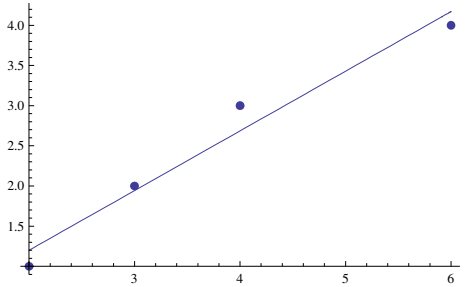
In[11]:= `q2 = Plot[p, {x, 2, 6}]`

Out[11]=



In[12]:= **Show[q1, q2]**

Out[12]=



In[13]:= **A = {{X[4, 0], X[3, 0], X[2, 0]}, {X[3, 0], X[2, 0], X[1, 0]}, {X[2, 0], X[1, 0], X[0, 0]}}**

Out[13]= {{1649, 315, 65}, {315, 65, 15}, {65, 15, 4}}

In[14]:= **A // MatrixForm**

Out[14]//MatrixForm=

$$\begin{pmatrix} 1649 & 315 & 65 \\ 315 & 65 & 15 \\ 65 & 15 & 4 \end{pmatrix}$$

In[15]:= **B = {{X[2, 1]}, {X[1, 1]}, {X[0, 1]}}**

Out[15]= {{214}, {44}, {10}}

In[16]:= **B // MatrixForm**

Out[16]//MatrixForm=

$$\begin{pmatrix} 214 \\ 44 \\ 10 \end{pmatrix}$$

In[17]:= **Inverse[A].B**

Out[17]= $\left\{\left\{-\frac{5}{44}\right\}, \left\{\frac{367}{220}\right\}, \left\{-\frac{21}{11}\right\}\right\}$

In[18]:= **N[%]**

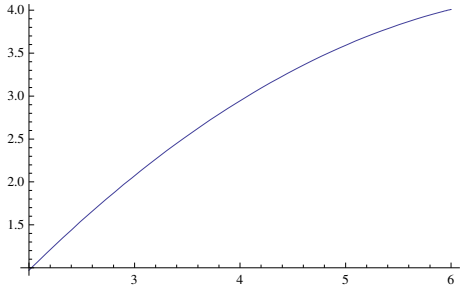
Out[18]= {{-0.113636}, {1.66818}, {-1.90909}}

In[19]:= **pp = Fit[T, {x^2, x, 1}, x]**

Out[19]= $-1.90909 + 1.66818\,x - 0.113636\,x^2$

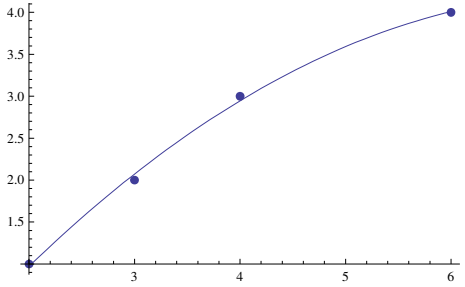
In[20]:= **q3 = Plot[pp, {x, 2, 6}]**

Out[20]=



In[21]:= **Show[q1, q3]**

Out[21]=



In[22]:= **Fit[T, {x^3, x^2, x, 1}, x]**

Out[22]= $-3.06422 \times 10^{-14} - 0.0833333\,x + 0.375\,x^2 - 0.0416667\,x^3$

In[23]:= **InterpolatingPolynomial[T, x]**

Out[23]= $1 + \left(1 + \frac{1}{24}(4 - x)(-3 + x)\right)(-2 + x)$

In[24]:= **Expand[%]**

Out[24]= $-\frac{x}{12} + \frac{3\,x^2}{8} - \frac{x^3}{24}$

In[25]:= **N[%]**

Out[25]= $-0.0833333\,x + 0.375\,x^2 - 0.0416667\,x^3$

In[26]:= **Fit[T, {x^4, x^3, x^2, x, 1}, x]**

Out[26]= $-0.456196 + 0.486912\,x + 0.121558\,x^2 + 0.00585375\,x^3 - 0.00316803\,x^4$

In[1]:=
```
p[k_, x_] := a[k] x^3 + b[k] x^2 + c[k] x + d[k]
```

In[2]:=
```
D[p[k, x], x]
```

In[3]:=
```
p'[k_, x_] := 3 x^2 a[k] + 2 x b[k] + c[k]
```

In[4]:=
```
D[p[1, x], {x, 2}]
```

In[5]:=
```
p''[k_, x_] := 6 x a[k] + 2 b[k]
```

In[6]:=
```
A = {{2, 1}, {3, 2}, {4, 3}, {6, 4}}
```

Out[6]=
```
{{2, 1}, {3, 2}, {4, 3}, {6, 4}}
```

In[7]:=
```
B = {p[1, 2] == 1, p[1, 3] == 2, p[2, 3] == 2, p[2, 4] == 3, p[3, 4] == 3, p[3, 6] == 4,
  p'[1, 3] == p'[2, 3], p'[2, 4] == p'[3, 4],
  p''[1, 3] == p''[2, 3], p''[2, 4] == p''[3, 4],
  p''[1, 2] == 0, p''[3, 6] == 0}
```

Out[7]=
```
{8 a[1] + 4 b[1] + 2 c[1] + d[1] == 1, 27 a[1] + 9 b[1] + 3 c[1] + d[1] == 2, 27 a[2] + 9 b[2] + 3 c[2] + d[2] == 2,
 64 a[2] + 16 b[2] + 4 c[2] + d[2] == 3, 64 a[3] + 16 b[3] + 4 c[3] + d[3] == 3,
 216 a[3] + 36 b[3] + 6 c[3] + d[3] == 4, 27 a[1] + 6 b[1] + c[1] == 27 a[2] + 6 b[2] + c[2],
 48 a[2] + 8 b[2] + c[2] == 48 a[3] + 8 b[3] + c[3], 18 a[1] + 2 b[1] == 18 a[2] + 2 b[2],
 24 a[2] + 2 b[2] == 24 a[3] + 2 b[3], 12 a[1] + 2 b[1] == 0, 36 a[3] + 2 b[3] == 0}
```

In[8]:=
```
B // TableForm
```

Out[8]//TableForm=
```
8 a[1] + 4 b[1] + 2 c[1] + d[1] == 1
27 a[1] + 9 b[1] + 3 c[1] + d[1] == 2
27 a[2] + 9 b[2] + 3 c[2] + d[2] == 2
64 a[2] + 16 b[2] + 4 c[2] + d[2] == 3
64 a[3] + 16 b[3] + 4 c[3] + d[3] == 3
216 a[3] + 36 b[3] + 6 c[3] + d[3] == 4
27 a[1] + 6 b[1] + c[1] == 27 a[2] + 6 b[2] + c[2]
48 a[2] + 8 b[2] + c[2] == 48 a[3] + 8 b[3] + c[3]
18 a[1] + 2 b[1] == 18 a[2] + 2 b[2]
24 a[2] + 2 b[2] == 24 a[3] + 2 b[3]
12 a[1] + 2 b[1] == 0
36 a[3] + 2 b[3] == 0
```

In[9]:=
```
Flatten[Table[{a[k], b[k], c[k], d[k]}, {k, 1, 3}]]
```

In[10]:=
```
T = {a[1], b[1], c[1], d[1], a[2], b[2], c[2], d[2], a[3], b[3], c[3], d[3]}
```

Out[10]=
```
{a[1], b[1], c[1], d[1], a[2], b[2], c[2], d[2], a[3], b[3], c[3], d[3]}
```

In[11]:=
```
Solve[B, T]
```

In[12]:=
$$S = \left\{a[1] \to \frac{1}{46},\ b[1] \to -\frac{3}{23},\ c[1] \to \frac{57}{46},\ d[1] \to -\frac{26}{23},\ a[2] \to -\frac{5}{46},\ b[2] \to \frac{24}{23},\ c[2] \to -\frac{105}{46},\ d[2] \to \frac{55}{23},\ a[3] \to \frac{1}{23},\ b[3] \to -\frac{18}{23},\ c[3] \to \frac{231}{46},\ d[3] \to -\frac{169}{23}\right\};$$

In[13]:=
```
p[1, x] /. S
```

In[14]:=
$$p1[x\_] := -\frac{26}{23} + \frac{57\,x}{46} - \frac{3\,x^2}{23} + \frac{x^3}{46}$$

In[15]:=
```
p[2, x] /. S
```

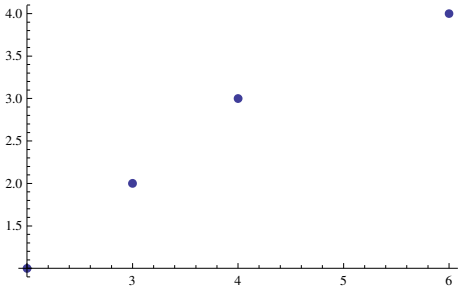
In[16]:=
$$p2[x\_] := \frac{55}{23} - \frac{105\,x}{46} + \frac{24\,x^2}{23} - \frac{5\,x^3}{46}$$

In[17]:=
```
p[3, x] /. S
```

In[18]:=
$$p3[x\_] := -\frac{169}{23} + \frac{231\,x}{46} - \frac{18\,x^2}{23} + \frac{x^3}{23}$$

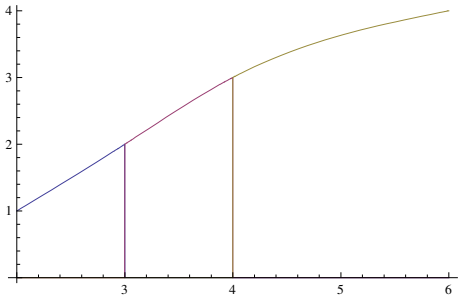
In[19]:=
```
q1 = ListPlot[A, PlotStyle → PointSize[0.02]]
```

Out[19]=



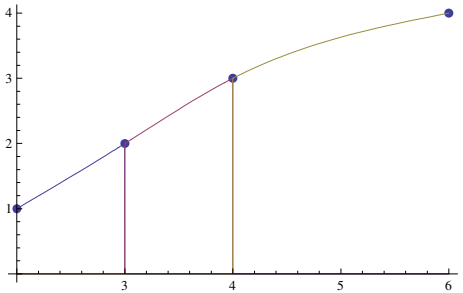
In[20]:=
```
q2 = Plot[{p1[x] UnitBox[(x - 2.5)], p2[x] UnitBox[(x - 3.5)], p3[x] UnitBox[(x - 5) / 2]}, {x, 2, 6}]
```

Out[20]=



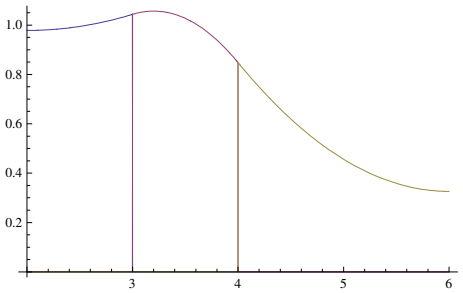
In[21]:=
```
Show[q1, q2]
```

Out[21]=



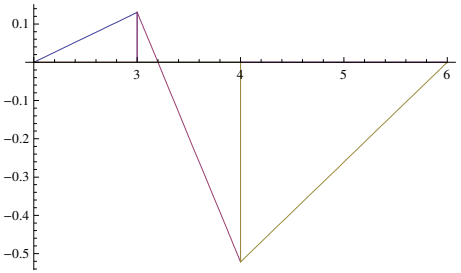
In[22]:=
```
Plot[{p1'[x] UnitBox[(x - 2.5)], p2'[x] UnitBox[(x - 3.5)], p3'[x] UnitBox[(x - 5) / 2]}, {x, 2, 6}]
```

Out[22]=



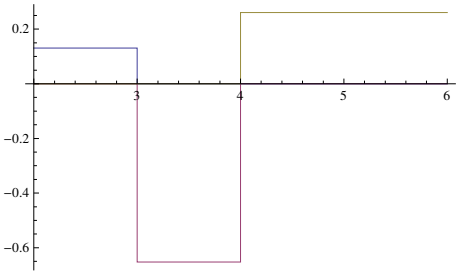
In[23]:=
```
Plot[{p1''[x] UnitBox[(x - 2.5)], p2''[x] UnitBox[(x - 3.5)], p3''[x] UnitBox[(x - 5) / 2]},
 {x, 2, 6}, PlotRange → All]
```

Out[23]=



In[24]:=
```
Plot[{p1'''[x] UnitBox[(x - 2.5)], p2'''[x] UnitBox[(x - 3.5)], p3'''[x] UnitBox[(x - 5) / 2]},
 {x, 2, 6}, PlotRange → All]
```

Out[24]=



In[1]:= `f[x_] := Sin[x^2]`

In[2]:= `D[f[x], {x, 2}]`

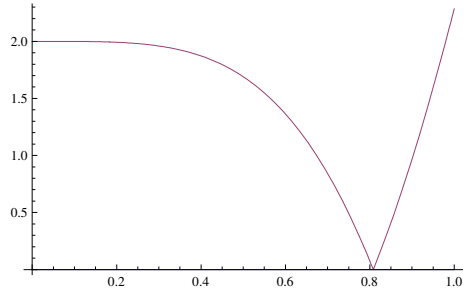
In[3]:= $g[x\_] := 2\,\mathrm{Cos}[x^2] - 4\,x^2\,\mathrm{Sin}[x^2]$

In[4]:= `M = 2 + 4`

Out[4]= 6

In[5]:= `Plot[{0, Abs[g[x]]}, {x, 0, 1}]`

Out[5]=



In[6]:= `Abs[g[1]]`

Out[6]= `-2 Cos[1] + 4 Sin[1]`

In[7]:= `N[%]`

Out[7]= 2.28528

In[8]:= `N[(1000 M / 12) ^ (1 / 2)]`

Out[8]= 22.3607

In[9]:= `n = 23`

Out[9]= 23

In[10]:= `h = (1 - 0) / n`

Out[10]= $\frac{1}{23}$

In[11]:= $\frac{h}{2}\,(f[0] + 2\,\mathrm{Sum}[f[k / 23], \{k, 1, 22\}] + f[1])$

Out[11]=

$$\frac{1}{46}\left(2\left(\mathrm{Sin}\left[\frac{1}{529}\right]+\mathrm{Sin}\left[\frac{4}{529}\right]+\mathrm{Sin}\left[\frac{9}{529}\right]+\mathrm{Sin}\left[\frac{16}{529}\right]+\mathrm{Sin}\left[\frac{25}{529}\right]+\mathrm{Sin}\left[\frac{36}{529}\right]+\mathrm{Sin}\left[\frac{49}{529}\right]+\mathrm{Sin}\left[\frac{64}{529}\right]+\mathrm{Sin}\left[\frac{81}{529}\right]+\mathrm{Sin}\left[\frac{100}{529}\right]+\mathrm{Sin}\left[\frac{121}{529}\right]+\mathrm{Sin}\left[\frac{144}{529}\right]+\mathrm{Sin}\left[\frac{169}{529}\right]+\mathrm{Sin}\left[\frac{196}{529}\right]+\mathrm{Sin}\left[\frac{225}{529}\right]+\mathrm{Sin}\left[\frac{256}{529}\right]+\mathrm{Sin}\left[\frac{289}{529}\right]+\mathrm{Sin}\left[\frac{324}{529}\right]+\mathrm{Sin}\left[\frac{361}{529}\right]+\mathrm{Sin}\left[\frac{400}{529}\right]+\mathrm{Sin}\left[\frac{441}{529}\right]+\mathrm{Sin}\left[\frac{484}{529}\right]\right)+\mathrm{Sin}[1]\right)$$

In[12]:= `a = N[%, 20]`

Out[12]= 0.31043860088116767628

In[13]:= $\int_0^1 f[x]\,dx$

Out[13]= $\sqrt{\frac{\pi}{2}}\ \mathrm{FresnelS}\left[\sqrt{\frac{2}{\pi}}\right]$

In[14]:= `b = N[%, 20]`

Out[14]= 0.31026830172338110181

In[15]:= `a - b`

Out[15]= 0.00017029915778657447

In[16]:= `% < 10^(-3)`

Out[16]= True

In[1]:= `f[x_] := Sin[x^2]`

In[2]:= `D[f[x], {x, 4}]`

Out[2]= $-48\,x^2\,\mathrm{Cos}[x^2] - 12\,\mathrm{Sin}[x^2] + 16\,x^4\,\mathrm{Sin}[x^2]$

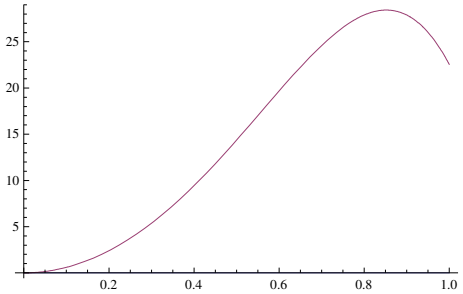
In[3]:= $g[x\_] := -48\,x^2\,\mathrm{Cos}[x^2] - 12\,\mathrm{Sin}[x^2] + 16\,x^4\,\mathrm{Sin}[x^2]$

In[4]:= `M = 12 + 48`

Out[4]= 60

In[5]:= `Plot[{0, Abs[g[x]]}, {x, 0, 1}]`

Out[5]=



In[6]:= `FindMaximum[Abs[g[x]], {x, 1}]`

Out[6]= {28.4285, {x → 0.852077}}

In[7]:= `N[(1000 M / 180)^(1 / 4)]`

Out[7]= 4.27287

In[8]:= `n = 6`

Out[8]= 6

In[9]:= `h = (1 - 0) / n`

Out[9]= $\frac{1}{6}$

In[10]:= $\frac{h}{3}\,(f[0] + 4\,(f[1/6] + f[3/6] + f[5/6]) + 2\,(f[2/6] + f[4/6]) + f[1])$

Out[10]= $\frac{1}{18}\left(2\left(\mathrm{Sin}\left[\frac{1}{9}\right] + \mathrm{Sin}\left[\frac{4}{9}\right]\right) + 4\left(\mathrm{Sin}\left[\frac{1}{36}\right] + \mathrm{Sin}\left[\frac{1}{4}\right] + \mathrm{Sin}\left[\frac{25}{36}\right]\right) + \mathrm{Sin}[1]\right)$

In[11]:= `a = N[%, 20]`

Out[11]= 0.31020534474963330655

In[12]:= $\int_0^1 f[x]\,dx$

Out[12]= $\sqrt{\frac{\pi}{2}}\ \mathrm{FresnelS}\left[\sqrt{\frac{2}{\pi}}\right]$

In[13]:= `b = N[%, 20]`

Out[13]= 0.31026830172338110181

In[14]:= `b - a`

Out[14]= 0.00006295697374779526

In[15]:= `% < 10^(-3)`

Out[15]= True

In[1]:=
```
Table[∫_{-1}^{1} x^n dx == Sum[w[k] x[k]^n, {k, 0, 2}], {n, 0, 5}]
```

Out[1]=
$$\left\{2 == w[0] + w[1] + w[2],\ 0 == w[0]\,x[0] + w[1]\,x[1] + w[2]\,x[2],\ \frac{2}{3} == w[0]\,x[0]^2 + w[1]\,x[1]^2 + w[2]\,x[2]^2,\ 0 == w[0]\,x[0]^3 + w[1]\,x[1]^3 + w[2]\,x[2]^3,\ \frac{2}{5} == w[0]\,x[0]^4 + w[1]\,x[1]^4 + w[2]\,x[2]^4,\ 0 == w[0]\,x[0]^5 + w[1]\,x[1]^5 + w[2]\,x[2]^5\right\}$$

In[2]:=
```
Solve[%, {w[0], w[1], w[2], x[0], x[1], x[2]}]
```

Out[2]=
$$\left\{\left\{w[0] \to \frac{5}{9},\ w[1] \to \frac{5}{9},\ w[2] \to \frac{8}{9},\ x[2] \to 0,\ x[1] \to -\sqrt{\frac{3}{5}},\ x[0] \to \sqrt{\frac{3}{5}}\right\},\right.$$
$$\left\{w[0] \to \frac{5}{9},\ w[1] \to \frac{5}{9},\ w[2] \to \frac{8}{9},\ x[2] \to 0,\ x[1] \to \sqrt{\frac{3}{5}},\ x[0] \to -\sqrt{\frac{3}{5}}\right\},$$
$$\left\{w[0] \to \frac{5}{9},\ w[1] \to \frac{8}{9},\ w[2] \to \frac{5}{9},\ x[2] \to -\sqrt{\frac{3}{5}},\ x[1] \to 0,\ x[0] \to \sqrt{\frac{3}{5}}\right\},$$
$$\left\{w[0] \to \frac{5}{9},\ w[1] \to \frac{8}{9},\ w[2] \to \frac{5}{9},\ x[2] \to \sqrt{\frac{3}{5}},\ x[1] \to 0,\ x[0] \to -\sqrt{\frac{3}{5}}\right\},$$
$$\left\{w[0] \to \frac{8}{9},\ w[1] \to \frac{5}{9},\ w[2] \to \frac{5}{9},\ x[2] \to -\sqrt{\frac{3}{5}},\ x[1] \to \sqrt{\frac{3}{5}},\ x[0] \to 0\right\},$$
$$\left.\left\{w[0] \to \frac{8}{9},\ w[1] \to \frac{5}{9},\ w[2] \to \frac{5}{9},\ x[2] \to \sqrt{\frac{3}{5}},\ x[1] \to -\sqrt{\frac{3}{5}},\ x[0] \to 0\right\}\right\}$$

In[3]:=
```
%[[4]]
```

Out[3]=
$$\left\{w[0] \to \frac{5}{9},\ w[1] \to \frac{8}{9},\ w[2] \to \frac{5}{9},\ x[2] \to \sqrt{\frac{3}{5}},\ x[1] \to 0,\ x[0] \to -\sqrt{\frac{3}{5}}\right\}$$

In[4]:=
```
f[x_] := Sin[x^2]
```

In[5]:=
```
g[t_] := Sin[((t + 1) / 2)^2] / 2
```

In[6]:=
$$\frac{5}{9}\,g\left[-\sqrt{\frac{3}{5}}\right] + \frac{8}{9}\,g[0] + \frac{5}{9}\,g\left[\sqrt{\frac{3}{5}}\right]$$

Out[6]=
$$\frac{4}{9}\,\mathrm{Sin}\left[\frac{1}{4}\right] + \frac{5}{18}\,\mathrm{Sin}\left[\frac{1}{4}\left(1 - \sqrt{\frac{3}{5}}\right)^2\right] + \frac{5}{18}\,\mathrm{Sin}\left[\frac{1}{4}\left(1 + \sqrt{\frac{3}{5}}\right)^2\right]$$

In[7]:=
```
N[%, 20]
```

Out[7]= 0.31027688512104177889

In[8]:=
```
NIntegrate[f[x], {x, 0, 1}, WorkingPrecision → 20]
```

Out[8]= 0.31026830172338110179

In[9]:=
```
% - %%
```

Out[9]= $-8.58339766067710 \times 10^{-6}$

In[1]:=
```
A = Table[∫_{-1}^{1} x^n dx == Sum_{k=0}^{2} w[k] x[k]^n, {n, 0, 5}]
```

Out[1]=
$$\left\{2 == w[0] + w[1] + w[2],\ 0 == w[0]\,x[0] + w[1]\,x[1] + w[2]\,x[2],\ \frac{2}{3} == w[0]\,x[0]^2 + w[1]\,x[1]^2 + w[2]\,x[2]^2,\ 0 == w[0]\,x[0]^3 + w[1]\,x[1]^3 + w[2]\,x[2]^3,\ \frac{2}{5} == w[0]\,x[0]^4 + w[1]\,x[1]^4 + w[2]\,x[2]^4,\ 0 == w[0]\,x[0]^5 + w[1]\,x[1]^5 + w[2]\,x[2]^5\right\}$$

In[2]:=
```
A // MatrixForm
```

Out[2]//MatrixForm=
$$\begin{pmatrix} 2 == w[0] + w[1] + w[2] \\ 0 == w[0]\,x[0] + w[1]\,x[1] + w[2]\,x[2] \\ \frac{2}{3} == w[0]\,x[0]^2 + w[1]\,x[1]^2 + w[2]\,x[2]^2 \\ 0 == w[0]\,x[0]^3 + w[1]\,x[1]^3 + w[2]\,x[2]^3 \\ \frac{2}{5} == w[0]\,x[0]^4 + w[1]\,x[1]^4 + w[2]\,x[2]^4 \\ 0 == w[0]\,x[0]^5 + w[1]\,x[1]^5 + w[2]\,x[2]^5 \end{pmatrix}$$

In[3]:=
```
B = Flatten[Table[{w[k], x[k]}, {k, 0, 2}]]
```

Out[3]= {w[0], x[0], w[1], x[1], w[2], x[2]}

In[4]:=
```
Solve[A, B]
```

Out[4]=
$$\left\{\left\{w[0] \to \frac{5}{9},\ w[1] \to \frac{5}{9},\ w[2] \to \frac{8}{9},\ x[2] \to 0,\ x[1] \to -\sqrt{\frac{3}{5}},\ x[0] \to \sqrt{\frac{3}{5}}\right\},\right.$$
$$\left\{w[0] \to \frac{5}{9},\ w[1] \to \frac{5}{9},\ w[2] \to \frac{8}{9},\ x[2] \to 0,\ x[1] \to \sqrt{\frac{3}{5}},\ x[0] \to -\sqrt{\frac{3}{5}}\right\},$$
$$\left\{w[0] \to \frac{5}{9},\ w[1] \to \frac{8}{9},\ w[2] \to \frac{5}{9},\ x[2] \to -\sqrt{\frac{3}{5}},\ x[1] \to 0,\ x[0] \to \sqrt{\frac{3}{5}}\right\},$$
$$\left\{w[0] \to \frac{5}{9},\ w[1] \to \frac{8}{9},\ w[2] \to \frac{5}{9},\ x[2] \to \sqrt{\frac{3}{5}},\ x[1] \to 0,\ x[0] \to -\sqrt{\frac{3}{5}}\right\},$$
$$\left\{w[0] \to \frac{8}{9},\ w[1] \to \frac{5}{9},\ w[2] \to \frac{5}{9},\ x[2] \to -\sqrt{\frac{3}{5}},\ x[1] \to \sqrt{\frac{3}{5}},\ x[0] \to 0\right\},$$
$$\left.\left\{w[0] \to \frac{8}{9},\ w[1] \to \frac{5}{9},\ w[2] \to \frac{5}{9},\ x[2] \to \sqrt{\frac{3}{5}},\ x[1] \to -\sqrt{\frac{3}{5}},\ x[0] \to 0\right\}\right\}$$

In[5]:=
```
N[%]
```

Out[5]=
```
{{w[0.] → 0.555556, w[1.] → 0.555556, w[2.] → 0.888889, x[2.] → 0.,
  x[1.] → -0.774597, x[0.] → 0.774597}, {w[0.] → 0.555556, w[1.] → 0.555556,
  w[2.] → 0.888889, x[2.] → 0., x[1.] → 0.774597, x[0.] → -0.774597},
 {w[0.] → 0.555556, w[1.] → 0.888889, w[2.] → 0.555556, x[2.] → -0.774597,
  x[1.] → 0., x[0.] → 0.774597}, {w[0.] → 0.555556, w[1.] → 0.888889,
  w[2.] → 0.555556, x[2.] → 0.774597, x[1.] → 0., x[0.] → -0.774597},
 {w[0.] → 0.888889, w[1.] → 0.555556, w[2.] → 0.555556, x[2.] → -0.774597,
  x[1.] → 0.774597, x[0.] → 0.}, {w[0.] → 0.888889, w[1.] → 0.555556,
  w[2.] → 0.555556, x[2.] → 0.774597, x[1.] → -0.774597, x[0.] → 0.}}
```

In[6]:=
```
M = Table[Table[D[Sum_{k=0}^{2} w[k] x[k]^n, B[[j]]], {j, 1, 6}], {n, 0, 5}]
```

Out[6]=
$$\left\{\{1, 0, 1, 0, 1, 0\}, \{x[0], w[0], x[1], w[1], x[2], w[2]\},\ \{x[0]^2, 2\,w[0]\,x[0], x[1]^2, 2\,w[1]\,x[1], x[2]^2, 2\,w[2]\,x[2]\},\ \{x[0]^3, 3\,w[0]\,x[0]^2, x[1]^3, 3\,w[1]\,x[1]^2, x[2]^3, 3\,w[2]\,x[2]^2\},\ \{x[0]^4, 4\,w[0]\,x[0]^3, x[1]^4, 4\,w[1]\,x[1]^3, x[2]^4, 4\,w[2]\,x[2]^3\},\ \{x[0]^5, 5\,w[0]\,x[0]^4, x[1]^5, 5\,w[1]\,x[1]^4, x[2]^5, 5\,w[2]\,x[2]^4\}\right\}$$

In[7]:= **M // MatrixForm**

Out[7]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ x[0] & w[0] & x[1] & w[1] & x[2] & w[2] \\ x[0]^2 & 2\,w[0]\,x[0] & x[1]^2 & 2\,w[1]\,x[1] & x[2]^2 & 2\,w[2]\,x[2] \\ x[0]^3 & 3\,w[0]\,x[0]^2 & x[1]^3 & 3\,w[1]\,x[1]^2 & x[2]^3 & 3\,w[2]\,x[2]^2 \\ x[0]^4 & 4\,w[0]\,x[0]^3 & x[1]^4 & 4\,w[1]\,x[1]^3 & x[2]^4 & 4\,w[2]\,x[2]^3 \\ x[0]^5 & 5\,w[0]\,x[0]^4 & x[1]^5 & 5\,w[1]\,x[1]^4 & x[2]^5 & 5\,w[2]\,x[2]^4 \end{pmatrix}$$

In[8]:= $\mathbf{F = Table\left[\sum_{k=0}^{2} w[k]\,x[k]\,\hat{}\,n - \int_{-1}^{1} x\,\hat{}\,n\,dx,\ \{n,\ 0,\ 5\}\right]}$

Out[8]=

$$\left\{-2 + w[0] + w[1] + w[2],\ w[0]\,x[0] + w[1]\,x[1] + w[2]\,x[2],\ -\frac{2}{3} + w[0]\,x[0]^2 + w[1]\,x[1]^2 + w[2]\,x[2]^2,\ w[0]\,x[0]^3 + w[1]\,x[1]^3 + w[2]\,x[2]^3,\ -\frac{2}{5} + w[0]\,x[0]^4 + w[1]\,x[1]^4 + w[2]\,x[2]^4,\ w[0]\,x[0]^5 + w[1]\,x[1]^5 + w[2]\,x[2]^5\right\}$$

In[9]:= **F // MatrixForm**

Out[9]//MatrixForm=

$$\begin{pmatrix} -2 + w[0] + w[1] + w[2] \\ w[0]\,x[0] + w[1]\,x[1] + w[2]\,x[2] \\ -\frac{2}{3} + w[0]\,x[0]^2 + w[1]\,x[1]^2 + w[2]\,x[2]^2 \\ w[0]\,x[0]^3 + w[1]\,x[1]^3 + w[2]\,x[2]^3 \\ -\frac{2}{5} + w[0]\,x[0]^4 + w[1]\,x[1]^4 + w[2]\,x[2]^4 \\ w[0]\,x[0]^5 + w[1]\,x[1]^5 + w[2]\,x[2]^5 \end{pmatrix}$$

In[10]:=
```
X = {-2, 3, 4, -5, 6, -7}; For[i = 1, i ≤ 20, i++,
 {X = N[X - Inverse[M /. Table[B[[i]] → X[[i]], {i, 1, 6}]].F /. Table[B[[i]] → X[[i]], {i, 1, 6}], 50],
  Print[X]}]
```



```
{0.91209166666666666666666666666666666666666666666667,
 3.5632500000000000000000000000000000000000000000000,
 -8.9739583333333333333333333333333333333333333333333,
 -0.88229166666666666666666666666666666666666666666667,
 10.061866666666666666666666666666666666666666666667,
 -5.7528888888888888888888888888888888888888888888889}

{0.15342884912611663837855811886182446117383753902,
 3.3948128786358004599700097682586566588216766 1343, 1.7974383651522149350736982468499330036795543896 9,
 -1.0397474794878728128068433479529658492945806 9336, 0.04913278572166842654774363428824253514660807128,
 -5.7466829636312722018066682250268840393343160958 86}

{0.19582971772752679618838691382218654460418863 7, 2.1433525948606206890105725346262642112964567 2,
 1.7392731378214610423405781099454408185516180 79, -0.13394160301227572447010184576972153673451588 5,
 0.06489714445101216147103497623237263684419328 43, -4.1098369358191239730087008094956405612595862 3}

{0.20339811486566694431395834735009267752398 53, 1.5653467151033898358621952924251941170160 35,
 1.7773450476754521635262150638262985606005103, -0.14734497338658668705506915309420646755256 21,
 0.01925683745888089215982658882360876187550443, -3.8362982487742996187445749489086256823359165}

{0.38136072172369181542089484592295404421130, 0.80837199273911502456257110770120177019033,
 1.6027969411644580368442137190042801246899 8, -0.23689909161068920082424021144184212062423,
 0.01584233711185014773489143507276583109872 5, -3.1552907977101472342523840317815145879004 2}

{0.89468500836576277600545776615131150862 1, 0.48161025111162507077751935836329909122 3,
 1.0926004899293279231095561393525682914 1, -0.42882539908690345711148474882530069897 58,
 0.01271450170430443168358661991343166223780, -2.6535527769301682658952990366604547562 07}

{0.66209037529201425061703809146566838237, 0.74520385305192155855866056774649371414,
 1.3166760265004555008819415787349267890 4, -0.37822800597150492994210836488615273705,
 0.02123359820753024850102032979940482859 3, -1.7897278384227827065936778239638293226}

{0.75421823155230620381137129970262923 81, 0.68979536365198610617494473731741574 42,
 1.1671480146841432539785789565304256268, -0.35480915380111967765556559597978913913,
 0.07863375376355054221004974376694513 51, -0.56595322003732732023586939100628622 0}

{0.60639260009062568644051952008927991, 0.74914843086027125652113945823951613,
 -1.3464042983152829859426878834848 44, 0.25112229268000936417086378493471 92,
 2.7400116982246572995021683633955 64, -1.8028492835175775378947850721263 82}

{4.2484282826006596215479593916 07, -0.32526129948635864683439174435 9,
 -2.3918688537672376202782439198 61, 1.5321619085225063365056115203 85,
 0.14344057116657799873028452825 43, -1.7835569225270828328035434675 6826}

{1.4747000244548167276402198797, -0.22575351636287384163190429 55, 0.30638429774552746270011163 92,
 1.5768422482204860555440124834 7, 0.21891567779965580965966848 10, -1.1923789833297912953848249 72}

{1.0986764387078847530877547 3, 0.17604725973936484409351393, 0.22944246442163134519790476 3,
 1.3285164628344868935310703 33, 0.67188109687048390171434050, -0.41162218283922406090776797}

{2.5754183878117634156783 38, -0.24263358465579069613073 0, 0.30274592708478768183009 31,
 1.0029252958844495993302 86, -0.87816431489655109750843 1, -1.4272349074690118958751 45}

{1.2373936425673754683593, -0.20927954514452262447 46, 0.55169973854631746856 53,
 0.69205900755739779025 91, 0.21090661888630706307 54, -1.4911703411204516990595}

{1.1193283447606369458 1, -0.28777333453576766100, 0.73112616015941945374,
 0.70809216984301858654, 0.14954549507994360044 6, -1.3036460532975713557 8}

{1.0868514519985102241, -0.23292920252750801 83, 0.68748893424477382 52,
 0.72065946852968610 10, 0.22565961375671595 07, -0.97221027318725818 95}

{0.88723266633637615, -0.08420676796811803, 0.61679381279193050,
 0.74881697068619215 8, 0.49597352087169335, -0.69144455628109174}

{0.918311521329908, -0.019927676729264, 0.568680802505770,
 0.7687049034292183, 0.513007676164322, -0.8186503169237499}

{0.89421006566687, -0.00588720356927, 0.55865487006556,
 0.77332301991883, 0.54713506426758, -0.77816101541547}

{0.888839915397, -0.000069822715, 0.555617688321, 0.7745702581476, 0.555542396282, -0.7745722638102}
```

w[0.`] → 0.8888888888888888`, w[1.`] → 0.5555555555555556`, w[2.`] → 0.5555555555555556`,
x[2.`] → -0.7745966692414834`, x[1.`] → 0.7745966692414834`, x[0.`] → 0.`

In[1]:=
```
f[x_, y_] := x^3 + 3 y^2 - 75 x - 9 y^2
```

In[2]:=
```
G[X_] := {D[f[x, y], x], D[f[x, y], y]} /. {x → X[[1]], y → X[[2]]}
```

In[3]:=
```
g[X_, t_, u_] := G[X + t u].u
```

In[4]:=
```
X = {-2, -4};
For[i = 1, i ≤ 20, i++,
 {
  u = G[X] / Norm[G[X]],
  S = NSolve[g[X, t, u] == 0],
  T = Table[X + S[[k]][[1]][[2]] u, {k, 1, 2}],
  V = Table[f[T[[k]][[1]], T[[k]][[2]]], {k, 1, 2}],
  If[V[[1]] > V[[2]], X = T[[1]], X = T[[2]]],
  Print[X]
 }
]
```

{-5.41033, -1.40165}

{-4.72733, -0.50521}

{-5.07314, -0.241738}

{-4.95458, -0.086127}

{-5.01273, -0.0418206}

{-4.99219, -0.0148636}

{-5.0022, -0.00723414}

{-4.99865, -0.00257}

{-5.00038, -0.00125132}

{-4.99977, -0.000444508}

{-5.00007, -0.000216443}

{-4.99996, -0.0000768865}

{-5.00001, -0.0000374386}

{-4.99999, -0.0000132992}

$\{-5., -6.47582 \times 10^{-6}\}$

$\{-5., -2.30038 \times 10^{-6}\}$

$\{-5., -1.12014 \times 10^{-6}\}$

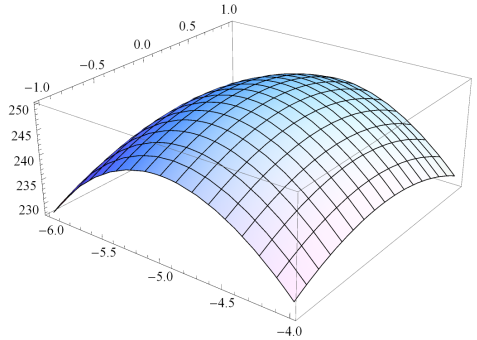
$\{-5., -3.97902 \times 10^{-7}\}$

$\{-5., -1.93752 \times 10^{-7}\}$

$\{-5., -6.88259 \times 10^{-8}\}$

In[6]:=
```
Plot3D[f[x, y], {x, -6, -4}, {y, -1, 1}]
```

Out[6]=



In[1]:=
```
DSolve[{y'[x] + y[x] == x, y[0] == 0}, y[x], x]
```

Out[1]= $\{\{y[x] \to e^{-x} (1 - e^{x} + e^{x} x)\}\}$

In[2]:=
```
z[x_] := e^-x (1 - e^x + e^x x)
```

In[3]:=
```
z[1]
```

Out[3]= $\frac{1}{e}$

In[4]:=
```
N[%, 10]
```

Out[4]= 0.3678794412

In[5]:=
```
f[x_, y_] := x - y
```

In[6]:=
```
h = 1 / 100; Y = 0; For[k = 1, k ≤ 100, k++, {Y = Y + h (k h - Y), Print[N[{k h, Y}, 10]]}]
```

{0.01000000000, 0.0001000000000}

{0.02000000000, 0.0002990000000}

{0.03000000000, 0.0005960100000}

{0.04000000000, 0.0009900499000}

{0.05000000000, 0.001480149401}

{0.06000000000, 0.002065347907}

{0.07000000000, 0.002744694428}

{0.08000000000, 0.003517247484}

{0.09000000000, 0.004382075009}

{0.1000000000, 0.005338254259}

{0.1100000000, 0.006384871716}

{0.1200000000, 0.007521022999}

{0.1300000000, 0.008745812769}

{0.1400000000, 0.01005835464}

{0.1500000000, 0.01145777109}

{0.1600000000, 0.01294319338}

{0.1700000000, 0.01451376145}

{0.1800000000, 0.01616862384}

{0.1900000000, 0.01790693760}

{0.2000000000, 0.01972786822}

{0.2100000000, 0.02163058954}

{0.2200000000, 0.02361428364}

{0.2300000000, 0.02567814081}

{0.2400000000, 0.02782135940}

{0.2500000000, 0.03004314581}

{0.2600000000, 0.03234271435}

{0.2700000000, 0.03471928720}

{0.2800000000, 0.03717209433}

{0.2900000000, 0.03970037339}

{0.3000000000, 0.04230336965}

{0.3100000000, 0.04498033596}

{0.3200000000, 0.04773053260}

{0.3300000000, 0.05055322727}

{0.3400000000, 0.05344769500}

{0.3500000000, 0.05641321805}

{0.3600000000, 0.05944908587}

{0.3700000000, 0.06255459501}

{0.3800000000, 0.06572904906}

{0.3900000000, 0.06897175857}

{0.4000000000, 0.07228204098}

{0.4100000000, 0.07565922057}

{0.4200000000, 0.07910262837}

{0.4300000000, 0.08261160208}

{0.4400000000, 0.08618548606}

{0.4500000000, 0.08982363120}

{0.4600000000, 0.09352539489}

{0.4700000000, 0.09729014094}

{0.4800000000, 0.1011172395}

{0.4900000000, 0.1050060671}

{0.5000000000, 0.1089560065}

{0.5100000000, 0.1129664464}

{0.5200000000, 0.1170367819}

{0.5300000000, 0.1211664141}

{0.5400000000, 0.1253547500}

{0.5500000000, 0.1296012025}

{0.5600000000, 0.1339051905}

{0.5700000000, 0.1382661385}

{0.5800000000, 0.1426834772}

{0.5900000000, 0.1471566424}

{0.6000000000, 0.1516850760}

{0.6100000000, 0.1562682252}

{0.6200000000, 0.1609055430}

{0.6300000000, 0.1655964875}

{0.6400000000, 0.1703405227}

{0.6500000000, 0.1751371174}

{0.6600000000, 0.1799857462}

{0.6700000000, 0.1848858888}

{0.6800000000, 0.1898370299}

{0.6900000000, 0.1948386596}

{0.7000000000, 0.1998902730}

{0.7100000000, 0.2049913703}

{0.7200000000, 0.2101414566}

{0.7300000000, 0.2153400420}

{0.7400000000, 0.2205866416}

{0.7500000000, 0.2258807752}

{0.7600000000, 0.2312219674}

{0.7700000000, 0.2366097477}

{0.7800000000, 0.2420436503}

{0.7900000000, 0.2475232138}

{0.8000000000, 0.2530479816}

{0.8100000000, 0.2586175018}

{0.8200000000, 0.2642313268}

{0.8300000000, 0.2698890135}

{0.8400000000, 0.2755901234}

{0.8500000000, 0.2813342222}

{0.8600000000, 0.2871208799}

{0.8700000000, 0.2929496711}

{0.8800000000, 0.2988201744}

{0.8900000000, 0.3047319727}

{0.9000000000, 0.3106846530}

{0.9100000000, 0.3166778064}

{0.9200000000, 0.3227110284}

{0.9300000000, 0.3287839181}

{0.9400000000, 0.3348960789}

{0.9500000000, 0.3410471181}

{0.9600000000, 0.3472366469}

{0.9700000000, 0.3534642805}

{0.9800000000, 0.3597296376}

{0.9900000000, 0.3660323413}

{1.000000000, 0.3723720179}

In[7]:=
```
N[Y - z[1], 10]
```

Out[7]= 0.004492576689

In[8]:=
```
D[f[x, y], x] + D[f[x, y], y] f[x, y]
```

Out[8]= 1 - x + y

In[9]:=
```
h = 1 / 100; Y = 0; For[k = 1, k ≤ 100, k++, {Y = Y + h (k h - Y) + (1 - k h + Y) h^2 / 2, Print[N[{k h, Y}, 10]]}]
```

{0.01000000000, 0.0001495000000}

{0.02000000000, 0.0003970124750}

{0.03000000000, 0.0007415622009}

{0.04000000000, 0.001182183657}

{0.05000000000, 0.001717920930}

{0.06000000000, 0.002347827616}

{0.07000000000, 0.003070966732}

{0.08000000000, 0.003886410613}

{0.09000000000, 0.004793240827}

{0.1000000000, 0.005790548081}

{0.1100000000, 0.006877432127}

{0.1200000000, 0.008053001678}

{0.1300000000, 0.009316374311}

{0.1400000000, 0.01066667639}

{0.1500000000, 0.01210304296}

{0.1600000000, 0.01362461768}

{0.1700000000, 0.01523055273}

{0.1800000000, 0.01692000873}

{0.1900000000, 0.01869215465}

{0.2000000000, 0.02054616771}

{0.2100000000, 0.02248123334}

{0.2200000000, 0.02449654507}

{0.2300000000, 0.02659130444}

{0.2400000000, 0.02876472096}

{0.2500000000, 0.03101601199}

{0.2600000000, 0.03334440267}

{0.2700000000, 0.03574912587}

{0.2800000000, 0.03822942206}

{0.2900000000, 0.04078453931}

{0.3000000000, 0.04341373315}

{0.3100000000, 0.04611626650}

{0.3200000000, 0.04889140965}

{0.3300000000, 0.05173844012}

{0.3400000000, 0.05465664265}

{0.3500000000, 0.05764530905}

{0.3600000000, 0.06070373823}

{0.3700000000, 0.06383123603}

{0.3800000000, 0.06702711523}

{0.3900000000, 0.07029069544}

{0.4000000000, 0.07362130302}

{0.4100000000, 0.07701827105}

{0.4200000000, 0.08048093925}

{0.4300000000, 0.08400865391}

{0.4400000000, 0.08760076780}

{0.4500000000, 0.09125664016}

{0.4600000000, 0.09497563659}

{0.4700000000, 0.09875712901}

{0.4800000000, 0.1026004956}

{0.4900000000, 0.1065051206}

{0.5000000000, 0.1104703947}

{0.5100000000, 0.1144957143}

{0.5200000000, 0.1185804819}

{0.5300000000, 0.1227241061}

{0.5400000000, 0.1269260013}

{0.5500000000, 0.1311855875}

{0.5600000000, 0.1355022910}

{0.5700000000, 0.1398755432}

{0.5800000000, 0.1443047815}

{0.5900000000, 0.1487894489}

{0.6000000000, 0.1533289939}

{0.6100000000, 0.1579228704}

{0.6200000000, 0.1625705379}

{0.6300000000, 0.1672714610}

{0.6400000000, 0.1720251100}

{0.6500000000, 0.1768309601}

{0.6600000000, 0.1816884921}

{0.6700000000, 0.1865971916}

{0.6800000000, 0.1915565495}

{0.6900000000, 0.1965660619}

{0.7000000000, 0.2016252295}

{0.7100000000, 0.2067335585}

{0.7200000000, 0.2118905596}

{0.7300000000, 0.2170957485}

{0.7400000000, 0.2223486458}

{0.7500000000, 0.2276487768}

{0.7600000000, 0.2329956715}

{0.7700000000, 0.2383888645}

{0.7800000000, 0.2438278953}

{0.7900000000, 0.2493123078}

{0.8000000000, 0.2548416503}

{0.8100000000, 0.2604154759}

{0.8200000000, 0.2660333419}

{0.8300000000, 0.2716948102}

{0.8400000000, 0.2773994468}

{0.8500000000, 0.2831468223}

{0.8600000000, 0.2889365114}

{0.8700000000, 0.2947680931}

{0.8800000000, 0.3006411506}

{0.8900000000, 0.3065552712}

{0.9000000000, 0.3125100462}

{0.9100000000, 0.3185050713}

{0.9200000000, 0.3245399458}

{0.9300000000, 0.3306142733}

{0.9400000000, 0.3367276613}

{0.9500000000, 0.3428797211}

{0.9600000000, 0.3490700679}

{0.9700000000, 0.3552983207}

{0.9800000000, 0.3615641024}

{0.9900000000, 0.3678670396}

{1.000000000, 0.3742067625}

In[10]:= `N[Y - z[1], 10]`

Out[10]= 0.006327321358