MA1023-Methods of Mathematics-15S2

Tutorial 1

There will be a spot test on something similar to this during the Tutorial hour on the week starting from 19/09/2016.

Q1. Consider the function $f(x) = x^{\frac{1}{10}}$

1. Write down the nth degree Taylor Polynomial near $c > 0$.
2. Show that the remainder satisfies

$$|R_n(x,c)| < \begin{cases} \dfrac{x^{\frac{1}{10}}}{10(n+1)}\left(\dfrac{x-c}{c}\right)^{n+1} & \text{if } x > c > 0 \\ \dfrac{c^{\frac{1}{10}}}{10(n+1)}\left(\dfrac{c-x}{x}\right)^{n+1} & \text{if } c > x > 0 \end{cases}$$

3. Show that the value of $1000^{\frac{1}{10}}$ accurate to 3 decimal places is 1.995.
4. Find the value of $1025^{\frac{1}{10}}$ accurate to 10 decimal places.

Q2. Consider the biquadratic equation $x^4 - x - 1 = 0$.

1. Find real intervals that contains real roots.
2. Propose a method to narrow down the intervals that contains roots.
3. Repeat the process in 2 and find the roots.
4. Arrange the equation as $x = g(x)$ in 4 different ways.
5. For each , perform iterations according to $x_{k+1} = g(x_k)$.
6. Which g gives convergence?
7. Explain the above behavior by a diagram.
8. Can we use the same method to find complex roots, give it a try!

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Tutorial 3

There will be a spot test on something similar to this during the Tutorial hour on the week starting from 17/10/2016.

Q1. Consider finding roots of $f(x) = x^5 - x - 1 = 0$.

1. Write down the method to be solved by the Newton's method.
2. Analyze the convergence treating it as an iterative method i.e. taking $g(x) = x - \frac{f(x)}{f'(x)}$
3. Find the no of iterations needed to find the real root in $[1,2]$ to an accuracy of 0.001 and find the root as an iterative method described in (2).
4. Analyze the convergence and find the no of iterations needed to find a real root in $[1,2]$ to an accuracy of 0.001 treating the process as the original Newton's method.
5. Suggest and implement, possibly a faster method by taking more terms in the Taylor series expansion of f.
6. What modifications/generalizations are needed in the Banach Fixed point theorem to find the complex roots by the iterative method?
7. Predict convergence and try to find complex roots using ideas in (6).
8. Can we use the Newton's method or the method described in (5) to find complex roots?

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Tutorial 4

There will be a spot test on something similar to this during the Tutorial hour on the week starting from 07/11/2016.

Numerical solutions to non-linear equations of one variable.

Q1. Find the height of the circular sector with arc length $2x$ and chord length x.

Q2. Your CASIO calculator can integrate. $\int_a^b f(x)dx$ is found by $\int(f(x), a, b)$.

Let $X \sim N(0,1)$ which has a PDF $p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. Find the value of z such that $P(X < z) = 0.8$ using the Newton's method.

Q3. Consider finding roots of $f(x) = \tan^{-1}x$ starting from $x_0 > 0$. Find the value of $z = x_0$ such that the Newton's method enters into a cycle. (If $x_0 < z$ the method will converge and if $x_0 > z$ the method will diverge).

Numerical Integration.

Q4. Derive Numerical Integration Formulas that uses the left end point/right end point/middle point of each partition interval to approximate$\int_a^b f(x)dx$.

Q5. Derive error formulas for the same.

Q6. Find the number of intervals required to get the values of $\int_0^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ and $\int_0^1 \sin(x^2)dx$ accurate to 0.001 and find those integrals to that accuracy.

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Tutorial 5

There will be a spot test on something similar to this during the Tutorial hour on the week starting from 14/11/2016.

Numerical Integration.

Q1. Use the error formula for the Trapezoidal rule $-\frac{nh^3}{12}f''(\zeta)$ to find the number of divisions needed to find the value of $\int_0^1 \sin(u^2)du$ accurate to 0.001 and find the integral to that accuracy.

Q2. Use Taylor series to derive an error formula for the Trapezoidal rule.

Q3. Use the error formula for the Simpson's rule $-\frac{nh^5}{180}f^{(4)}(\zeta)$ to find the number of divisions needed to find the value of $\int_0^1 \sin(u^2)du$ accurate to 0.001 and find the integral to that accuracy.

Q4. Use Taylor series to derive an error formula for the Simpson's rule.

Q5. Directly show that the Simpson's formula is exact(zero error) for all cubic polynomials.

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Tutorial 6

There will be a spot test on something similar to this during the Tutorial hour on the week starting from 28/11/2016.

Numerical Integration

Q1. Use the error formula in the Lagrange Interpolation to prove the error formula for the Trapezoidal rule.

Q2. One method of doing numerical integration is Gaussian quadrature. Note that both the Trapezoidal and the Simpson's rules looks like $\int_a^b f(x)dx \approx \sum_k w_k f(x_k)$ and we knew x_k and found w_k . In this method we find both x_k and w_k so that the integral and the sum are equal for a given n degree polynomial (x) , by forcing both sides equal for each power of x^j for $j = 0,1,2, \dots, n$. For $n = 3$ how many x_k and w_k can we find?. For $[a, b] = [-1,1]$ find them and use it to approximate $\int_{-1}^1 e^{-\frac{x^2}{2}} dx$ and thereby $\int_0^1 e^{-\frac{x^2}{2}} dx$.

Interpolation

Q1. Consider the data set $\{(1,1), (2,1), (3,2), (5,5)\}$
1. Find the Lagrange polynomial directly by matrix inversion.
2. Find the Lagrange polynomial by the above formula.
3. Use it to find the values at 0, 4 and 6.
4. Use it to find the derivative at 0, 4 and 6.
5. Use it to find the integral from 0 to 5.
6. Assume that the above data are obtained from the Fibonacci sequence with $F(n) = \frac{1}{\sqrt{5}}(\phi^n - \varphi^n)$ where $\phi > 0$ and $\varphi < 0$ are the roots of $y^2 - y - 1 = 0$. If the continuous version is $f(x) = \text{Re } F(x) = \frac{1}{\sqrt{5}}(\phi^x - (-\varphi)^x \cos \pi x)$ find an upper bound for the error.

Q2. One method of finding the Lagrange polynomial is to use the Newton's divided differences. For $\{x_0, x_1, x_2\}$ we define $f[x_0, x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0}$ and $f[x_0, x_1, x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$ and so on and the Lagrange polynomial is given by $p(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$. See why the formula is working and use it to find the Lagrange polynomial for $\{(1,1), (2,1), (3,2), (5,5)\}$.

Q3. Yet another way of finding the Lagrange polynomial is to define it as the iterative process $p(x) = A_1 + P_1(x)(x - x_0)$ and $P_1(x) = A_2 + P_2(x)(x - x_1)$ and so on. Use this to find the find the Lagrange polynomial for $\{(1,1), (2,1), (3,2), (5,5)\}$.

Q4. Consider the data set $\{(1,1), (2,1), (3,2)\}$. Find a polynomial (Hermite polynomial) that goes though the above points and satisfying $p'(1) = 0, p'(2) = 1, p'(3) = 0$.

Q5. Consider the data set $\{(1,1), (2,1), (3,2)\}$. Find a piecewise (defined on each sub interval),twice differentiable cubic polynomial (Cubic Spline) that goes though the above points and satisfying$p''(1) = 0, p''(3) = 0$.

Tutorial 7-Numerical Questions

Note: You are supposed to know the answers to the bolded questions only which are also given with regular Tutorial 7. The rest is given to introduce you to the other related topic in Numerical Analysis that can be done with your S2 knowledge. However those topics will not be tested at the final exam and you are encouraged to solve them as a challenge after the exam. You are also highly encouraged to take the other two specialized courses MA3023-Numerical Methods, MA4053-Numerical Methods for Scientific Computing in your future semesters, where a detailed discussion of these topics will be done.

Q1. Use a suitable numerical method to solve $f'(x) = 0$ and find the global maximum of

$f(x) = |(x-2)(x-3)(x-4)(x-6)|$ on $[2,6]$.

Q2. One numerical method of finding a local maximum is called the **Golden Section Search**. Here we start with an interval $[a, b]$ containing the unique maximum. Then we find $[c, d] \subset (a, b)$. If $f(c) < f(d)$, maximum is on $[c, b]$ and otherwise the maximum is in $[a, d]$. So we rename $[a, b]$ accordingly and repeat the process. We can select $[c, d] = \left[\frac{a+b}{2} - \epsilon, \frac{a+b}{2} + \epsilon\right]$ for small ϵ. But it is advantageous to select $[c, d]$ such that $\frac{d-a}{b-a} = \frac{c-a}{d-a} = \frac{1}{\phi}$ where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio (hence the name of the method) which satisfies $\phi^2 - \phi - 1 = 0$, why? Note that f need not be differentiable to employ this method.

Q3. Use the above method to find the maximum in Q1.

Q4. Suppose we are going to fit $f(x) = 4\sin^2\left(\frac{\pi x}{12}\right)$ by a n th degree interpolating polynomial $p(x)$ on $[0,6]$ with equispaced points such that the error is smaller than 0.001 in magnitude. Find n.

Q5. Find the **Natural Cubic Spline**s for the data $\{(1,1), (2,1), (3,2)\}$ and $\{(1,1), (2,1), (3,2), (5,5)\}$.

Q6. The least square parabola $ax^2 + bx + c$ that minimizes the error
$E(a, b, c) = \sum_{k=0}^{n}(ax_k^2 + bx_k + c - y_k)^2$ for the data set $\{(x_k, y_k)\}, k = 1, \dots, n$ is given by the formula

$$\begin{pmatrix} \sum x_k^4 & \sum x_k^3 & \sum x_k^2 \\ \sum x_k^3 & \sum x_k^2 & \sum x_k \\ \sum x_k^2 & \sum x_k & \sum 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \sum x_k^2 y_k \\ \sum x_k y_k \\ \sum y_k \end{pmatrix}.$$

Find a condition for x_k such that a, b, c can be found (ie. matrix is invertible).

Q7. Fit the least square line/parabola/cubic (equations are identical to the above) for each of the data $\{(1,1), (2,1), (3,2)\}$ and $\{(1,1), (2,1), (3,2), (5,5)\}$.

Q8. Fit a least square function of the form $p(x) = ae^x + b\sin x + c\cos x$ for each of the above data. The method is identical to finding the least square parabola which also has 3 coefficients.

Q9. Let the data set $\{(x_k, y_k)\}, k = 1, \dots, n$ has a least square line given by $ax + b$. Show that the least square line passes through $(\bar{x}, \bar{y})$ where $\bar{x} = \frac{\sum x_k}{n}, \bar{y} = \frac{\sum x_k}{n}$ are the means of x_k, y_k respectively.

Q10. For the least square line $ax + b$, show that $a = \frac{s_{xy}}{s_{xx}}$ where $s_{xy} = \frac{1}{n}\sum(x_k - \bar{x})(y_k - \bar{y})$ is the covariance of x_k, y_k and $s_{xx} = \frac{1}{n}\sum(x_k - \bar{x})^2$ is the variance of x_k. Show that r given by $r = \frac{s_{xy}}{\sqrt{s_{xx}s_{yy}}}$ is $r = \pm 1$ when all data are on a straight line. r is the correlation coefficient and is a measure of the degree to which a given set of points falling on a straight line.

Q11. One method to find the maximum of a multivariate function $f(x, y)$ is called the **Steepest Descent Method**. Here we start at a given point (x_0, y_0) and select the direction of the maximum slope at (x_0, y_0). Then we follow that maximum slope direction till we get the maximum along that direction as a one variable function, say at (x_1, y_1) and we repeat the process. Show that the maximum directions at (x_1, y_1) and (x_0, y_0) are perpendicular.

Q12. Explicitly write the error functions E for each of the data sets in Q7 and use the steepest descent method to minimize it and find the coefficients of the least square polynomials.

Q13.One application of Taylor series it to solve the ODE $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$. Show that by starting from $y_0 = y(x_0)$ followed by $y_{k+1} = y_k + hf(x_k, y_k)$ will give you a good approximation to $y(x)$ at $x_{k+1} = x_k + h$ if h is suitably small. The above method is called the **Euler's method**.
Use it to solve $\frac{dy}{dx} = x + y$, $y(0) = 1$ and find the value $y(1)$. Directly solve the ODE and compare the results.

Q14.Use the Taylor series for the Euler's method to show that the local truncation error $y(x_0 + h) - y_1 = \frac{h^2}{2} y''(\zeta)$ and express error in terms of f.

Q15. We saw in both the natural cubic spline interpolation and in Gaussian quadrature that we will be forced to solve large systems of linear equations. There are numerical methods for those too. One of them is called the **Jacobi Method**. Suppose we want to solve

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \quad \text{which can be re written as} \quad \begin{aligned} x_{k+1} &= (d_1 - b_1y_k - c_1z_k)/a_1 \\ y_{k+1} &= (d_2 - a_2x_k - c_2z_k)/b_2 \\ c_{k+1} &= (d_3 - a_3x_k - b_3y_k)/c_3 \end{aligned}$$

and start at (x_0, y_0, z_0) and do iterations. Use this method to solve the systems of linear equations you obtain in Q5. Will the method always converge?

Q16. We see that we already have new values for some variables after the 1st and 2nd steps which can be used in the next step immediately. Use this idea to write a better method (**Gauss-Seidel Method**) to solve the systems in Q5. Will the method always converge?

Q17. Suppose we have an inconsistent (more variables than equations) system of equations. Use the ideas of least square curve fitting to come up with a method to find a least square solution.

Q18. **Richardson Extrapolation** is a technique to make a numerical solution $A(h)$ which depends on a step length $h > 0$ for an actual solution A, more accurate.
Assume $A - A(h) = ah^m + bh^n + \cdots$where $< n$. Show that $A - \frac{2^m A\left(\frac{h}{2}\right) - A(h)}{2^m - 1}$ is having a lesser error.
So if $A_k(h)$ is the k th iteration for A a better one is $A_{k+1}(h) = \frac{2^m A_k\left(\frac{h}{2}\right) - A_k(h)}{2^m - 1}$.

Q19. Assume that we are going to approximate $A = f'(x)$ by $A_0(h) = \frac{f(x+h) - f(x)}{h}$. Find a better approximation $A_1(h)$.

Q20. Use the above method to improve the answer (called **Romberg Integration**) obtained from the Trapezoidal method for $\int_0^1 e^{-\frac{x^2}{2}} dx$.