Theorem 1. *Error formula for the Simpson's Rule*
If $f \in \mathcal{C}^4$, error in the Simpson's rule is $-\frac{(b-a)^5}{180n^4} f^{(4)}(\eta)$ where $\eta \in (a, b)$

Proof. With usual notation, let $p(x)$ be the degree two Lagrange polynomial of $f(x)$ agreeing with it on three points x_0, x_1, x_2 which are distance h apart.
Consider error on the first two intervals, $\int_{x_0}^{x_2} (f(x) - p(x))\, dx$

We define a degree three polynomial as follows
$q(x) = p(x) + \frac{1}{h^2} (p'(x_1) - f'(x_1))\, w(x)$ where $w(x) = (x - x_0)(x - x_1)(x - x_2)$
We notice that $q(x_k) = p(x_k) = f(x_k)$ for $k = 0, 1, 2$ and $q'(x_1) = f'(x_1)$

First we claim that $f(x) = q(x) + \frac{f^{(4)}(\zeta_1)}{4!} u(x)$ where $u(x) = (x - x_0)(x - x_1)^2(x - x_2)$ and $\zeta_1 \in (x_0, x_2)$. To prove this define $g(y) = f(y) - q(y) - u(y)\frac{f(x)-p(x)}{u(x)}$
Now $g(y) = 0$ for $y = x_0, x_1, x_2, x$ and $g'(x_1) = 0$
This implies, by the Mean Value Theorem that there is $\zeta_1 \in (x_0, x_2)$ such that $g^{(4)}(\zeta_1) = f^{(4)}(\zeta_1) - 0 - 4!\frac{f(x)-p(x)}{u(x)} = 0$ or $f(x) = q(x) + \frac{f^{(4)}(\zeta_1)}{4!} u(x)$ for some $\zeta_1 \in (x_0, x_2)$ as desired.

Now we notice that
$\int_{x_0}^{x_2} w(x)dx$
$= \int_{x_0}^{x_2} (x - x_0)(x - x_1)(x - x_2)dx$
$= \int_{-h}^{h} (t + h)(t)(t - h)dt = \int_{-h}^{h} (t^3 - h^2 t)dt = 0$
This implies that
$\int_{x_0}^{x_2} (f(x) - p(x))\, dx = \int_{x_0}^{x_2} (f(x) - q(x))\, dx = \int_{x_0}^{x_2} \frac{f^{(4)}(\zeta_1)}{4!} u(x)dx = \frac{f^{(4)}(\eta_1)}{4!} \int_{x_0}^{x_2} u(x)dx$
since $u(x)$ does not change sign on (x_0, x_2) and because $f^{(4)}(\eta_1(x))$ is a continuous function.

Now
$\int_{x_0}^{x_2} u(x)dx$
$= \int_{x_0}^{x_2} (x - x_0)(x - x_1)^2(x - x_2)dx$
$= \int_{-h}^{h} (t + h)(t)^2(t - h)dt = \int_{-h}^{h} (t^4 - h^2 t^2)dt = \left[\frac{t^5}{5} - h^2\frac{t^3}{3}\right]_{-h}^{h} = 2\left[\frac{h^5}{5} - \frac{h^5}{3}\right] = -\frac{4}{15} h^5$
So the error on $[x_0, x_2]$ is
$\frac{f^{(4)}(\eta_1)}{4!} \int_{x_0}^{x_2} u(x)dx = \frac{f^{(4)}(\eta_1)}{4!} \left(-\frac{4}{15} h^5\right) = -\frac{h^5}{90} f^{(4)}(\eta_1)$

We notice here that n needs to be even in order to cover $[a, b]$ by non-overlapping intervals similar to $[x_0, x_2]$ and we need $\frac{n}{2}$ of such intervals. Now adding error terms in each interval we have the total error,
$\sum_{j=1}^{\frac{n}{2}} -\frac{h^5}{90} f^{(4)}(\eta_j) = -\frac{h^5}{90} \sum_{j=1}^{\frac{n}{2}} f^{(4)}(\eta_j) = -\frac{h^5}{90}\frac{n}{2} f^{(4)}(\eta) = -\frac{nh^5}{180} f^{(4)}(\eta)$ for $\eta \in (a, b)$ using Extremum and Intermediate Value Theorems for $f \in \mathcal{C}^4$ ie. for $f^{(4)} \in \mathcal{C}$.
Using $h = \frac{b-a}{n}$, the total error is also equal to $-\frac{(b-a)h^4}{180} f^{(4)}(\eta) = -\frac{(b-a)^5}{180n^4} f^{(4)}(\eta)$

□

Theorem 2. *Newton-Kantorovich Theorem*
1.Let $f : [a, b] \to \mathbb{R}$
2.f' *is Lipschitz continuous with constant* γ
3.$f'(x) \neq 0$ *and* $\frac{1}{|f'(x)|} \leq \beta$
4.$x_0 \in [a, b]$ *and* $\left|\frac{f(x_0)}{f'(x_0)}\right| = \alpha$
5.$q = \alpha\beta\gamma < \frac{1}{2}$
6.$[x_0 - 2\alpha, x_0 + 2\alpha] \subset [a, b]$
7. $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, k \geq 0$
Then
1. $\lim_{k\to\infty} x_k = z \in [x_0 - 2\alpha, x_0 + 2\alpha]$.
2. $f(z) = 0$ *and* z *is a unique root of* f
3. $|x_k - z| \leq 2\alpha q^{2^k-1}$

Proof. By definition we have, $|x_{k+1} - x_k| = \left|\frac{f(x_k)}{f'(x_k)}\right| \leq \beta|f(x_k)| = \beta|f(x_k) - f(x_{k-1}) - (x_k - x_{k-1})f'(x_{k-1})| = \beta\left|\int_{x_{k-1}}^{x_k} f'(t)dt - f'(x_{k-1})\int_{x_{k-1}}^{x_k} dt\right| \leq \beta\left|\int_{x_{k-1}}^{x_k} |f'(t) - f'(x_{k-1})|dt\right| \leq \beta\gamma\left|\int_{x_{k-1}}^{x_k} |t - x_{k-1}|dt\right| = \frac{1}{2}\beta\gamma|x_k - x_{k-1}|^2 = p|x_k - x_{k-1}|^2$

Also by using the formula k times, $|x_{k+1} - x_k| \leq p|x_k - x_{k-1}|^2 \leq p(p|x_{k-1} - x_{k-2}|^2)^2 = p^{1+2^1}|x_{k-1} - x_{k-2}|^{2^2} \leq p^{1+2^1+2^2+\cdots+2^{k-1}}|x_1 - x_0|^{2^k} = p^{\frac{2^k-1}{2-1}}|x_1 - x_0|^{2^k} = \frac{1}{p}(p\alpha)^{2^k} = \frac{\alpha}{\alpha\beta\gamma}(\alpha\beta\gamma)^{2^k} = \alpha q^{2^k-1}$

So we have $|x_{k+1} - x_k| \leq \alpha q^{2^k-1}$

Now a general difference,
$|x_{k+j} - x_k| = |x_{k+j} - x_{k+j-1} + x_{k+j-1} - x_{k+j-2} + \cdots + x_{k+1} - x_k| \leq |x_{k+j} - x_{k+j-1}| + |x_{k+j-1} - x_{k+j-2}| + \cdots + |x_{k+1} - x_k| \leq \alpha q^{2^{k+j-1}-1} + \alpha q^{2^{k+j-2}-1} + \cdots + \alpha q^{2^k-1} = \alpha q^{-1}\left(q^{2^k} + q^{2^{k+1}} + \cdots + q^{2^{k+j-1}}\right) \leq \alpha q^{-1}\left(q^{2^k} + (q^{2^k})^2 + (q^{2^k})^4 + \ldots\right) = \alpha q^{2^k-1}\left(1 + (q^{2^k}) + (q^{2^k})^3 + \ldots\right) \leq \alpha q^{2^k-1}\left(1 + (q^{2^k}) + (q^{2^k})^2 + \ldots\right)$
$\leq \alpha q^{2^k-1}\left(1 + q + q^2 + \ldots\right) = \alpha q^{2^k-1}\frac{1}{1-q} \leq \alpha q^{2^k-1}\frac{1}{1-\frac{1}{2}} = 2\alpha q^{2^k-1}$ since $0 \leq q < \frac{1}{2} < 1$

Finally we have $|x_{k+j} - x_k| \leq 2\alpha q^{2^k-1}(1)$

Note that by $k = 0$ in (1)we have $|x_j - x_0| \leq 2\alpha q^{2^0-1} = 2\alpha$ so $x_k \in [x_0 - 2\alpha, x_0 + 2\alpha]$

Also by (1) and $0 \leq q < 1$, x_k is a Cauchy sequence and converges to $z \in [x_0 - 2\alpha, x_0 + 2\alpha]$.
By taking $j \to \infty$ in (1), we have $|z - x_k| \leq 2\alpha q^{2^k-1}$.

Note that f' is continuous with $f'(x) \neq 0$. Taking $k \to \infty$ in $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$, we get $z = z - \frac{f(z)}{f'(z)}$ or $f(z) = 0$, so z is a root of f.

Now assume that the root z is not unique, i.e. there exists $w \neq z$ with $f(w) = 0$. By MVT, for some ζ between z, w we have $f'(\zeta) = \frac{f(z)-f(w)}{z-w} = \frac{0}{z-w} = 0$ or $f'(\zeta) = 0$ which is a contradiction. Therefore $z = w$ and the root is unique.

□