Theorem 1. *Newton-Kantorovich Theorem*
1.Let $f:[a,b]\to\mathbb{R}$
2.$f'\neq 0$ *and there exists* $\beta>0$ *such that* $\frac{1}{|f'(x)|}\leq\beta$
3.f' *is Lipschitz continuous with constant* γ
4.$x_0\in[a,b]$ *and* $x_{k+1}=x_k-\frac{f(x_k)}{f'(x_k)}, k\geq 0$
5.$\left|\frac{f(x_0)}{f'(x_0)}\right|=\alpha$
6.$q=\alpha\beta\gamma<\frac{1}{2}$
7.$[x_0-2\alpha,x_0+2\alpha]\subset[a,b]$
Then
1. $\lim_{k\to\infty}x_k=z\in[x_0-2\alpha,x_0+2\alpha]$ *is a unique root of* f
2. $|x_k-z|\leq 2\alpha q^{2^k-1}$

Proof. By definition we have, $|x_{k+1}-x_k|=\left|\frac{f(x_k)}{f'(x_k)}\right|\leq\beta|f(x_k)|=\beta|f(x_k)-f(x_{k-1})-(x_k-x_{k-1})f'(x_{k-1})|=\beta\left|\int_{x_{k-1}}^{x_k}f'(t)dt-f'(x_{k-1})\int_{x_{k-1}}^{x_k}dt\right|\leq\beta\left|\int_{x_{k-1}}^{x_k}|f'(t)-f'(x_{k-1})|dt\right|\leq\beta\gamma\left|\int_{x_{k-1}}^{x_k}|t-x_{k-1}|dt\right|=\frac{1}{2}\beta\gamma|x_k-x_{k-1}|^2=p|x_k-x_{k-1}|^2$

Also by using the formula k times, $|x_{k+1}-x_k|\leq p|x_k-x_{k-1}|^2\leq p(p|x_{k-1}-x_{k-2}|^2)^2=p^{1+2^1}|x_{k-1}-x_{k-2}|^{2^2}\leq p^{1+2^1+2^2+\cdots+2^{k-1}}|x_1-x_0|^{2^k}=p^{\frac{2^k-1}{2-1}}|x_1-x_0|^{2^k}=\frac{1}{p}(p\alpha)^{2^k}=\frac{\alpha}{\alpha\beta\gamma}(\alpha\beta\gamma)^{2^k}=\alpha q^{2^k-1}$

So we have $|x_{k+1}-x_k|\leq\alpha q^{2^k-1}$

Now a general difference,
$|x_{k+j}-x_k|=|x_{k+j}-x_{k+j-1}+x_{k+j-1}-x_{k+j-2}+\cdots+x_{k+1}-x_k|\leq|x_{k+j}-x_{k+j-1}|+|x_{k+j-1}-x_{k+j-2}|+\cdots+|x_{k+1}-x_k|\leq\alpha q^{2^{k+j-1}-1}+\alpha q^{2^{k+j-2}-1}+\cdots+\alpha q^{2^k-1}=\alpha q^{-1}\left(q^{2^k}+q^{2^{k+1}}+\cdots+q^{2^{k+j-1}}\right)\leq\alpha q^{-1}\left(q^{2^k}+(q^{2^k})^2+(q^{2^k})^4+\dots\right)=\alpha q^{2^k-1}\left(1+(q^{2^k})+(q^{2^k})^3+\dots\right)\leq\alpha q^{2^k-1}\left(1+(q^{2^k})+(q^{2^k})^2+\dots\right)$
$\leq\alpha q^{2^k-1}\left(1+q+q^2+\dots\right)=\alpha q^{2^k-1}\frac{1}{1-q}\leq\alpha q^{2^k-1}\frac{1}{1-\frac{1}{2}}=2\alpha q^{2^k-1}$ since $0\leq q<\frac{1}{2}<1$

Finally we have $|x_{k+j}-x_k|\leq 2\alpha q^{2^k-1}(1)$

Note that by $k=0$ in (1)we have $|x_j-x_0|\leq 2\alpha q^{2^0-1}=2\alpha$ so $x_k\in[x_0-2\alpha,x_0+2\alpha]$

Also by (1) and $0\leq q<1$, x_k is a Cauchy sequence and converges to $z\in[x_0-2\alpha,x_0+2\alpha]$ (take $j\to\infty$ in (1))

Let $g:[x_0-2\alpha,x_0+2\alpha]\to[x_0-2\alpha,x_0+2\alpha]$ be defined by $g(x)=x-\frac{f(x)}{f'(x)}\in\mathcal{C}$. Then $z=g(z)$ and $f(z)=0$.
Also $|g(y)-g(x)|=\left|y-x-\frac{f(y)-f(x)}{f'(x)}\right|=\frac{1}{|f'(x)|}\left|\int_x^y(f'(x)-f'(t))dt\right|$
$\leq\beta\left|\int_x^y|f'(x)-f'(t)|dt\right|\leq\beta\gamma\left|\int_x^y|x-t|dt\right|\leq\frac{1}{2}\beta\gamma|x-y|^2\leq 2\alpha\beta\gamma|x-y|=2q|x-y|$
Therefore g is a contraction since $(2q<1)$.
Now assume that $z_1\neq z$ with $f(z_1)=0$. Then $|z_1-z|=|g(z_1)-g(z)|\leq 2q|z_1-z|<|z_1-z|$ which is a contradiction. Therefore z is unique.

□

Theorem 2. *Error formula for the Simpson's Rule*
If $f \in \mathcal{C}^4$, error in the Simpson's rule is $-\frac{(b-a)^5}{180n^4} f^{(4)}(\eta)$ where $\eta \in (a,b)$

Proof. With usual notation, let $p(x)$ be the degree two Lagrange polynomial of $f(x)$ agreeing with it on three points x_0, x_1, x_2 which are distance h apart.
Consider error on the first two intervals, $\int_{x_0}^{x_2} (f(x) - p(x))\, dx$

We define a degree three polynomial as follows
$q(x) = p(x) + \frac{1}{h^2}(p'(x_1) - f'(x_1))\, w(x)$ where $w(x) = (x - x_0)(x - x_1)(x - x_2)$
We notice that $q(x_k) = p(x_k) = f(x_k)$ for $k = 0, 1, 2$ and $q'(x_1) = f'(x_1)$

First we claim that $f(x) = q(x) + \frac{f^{(4)}(\zeta_1)}{4!} u(x)$ where $u(x) = (x - x_0)(x - x_1)^2(x - x_2)$ and $\zeta_1 \in (x_0, x_2)$. To prove this define $g(y) = f(y) - q(y) - u(y)\frac{f(x)-p(x)}{u(x)}$
Now $g(y) = 0$ for $y = x_0, x_1, x_2, x$ and $g'(x_1) = 0$
This implies, by the Mean Value Theorem that there is $\zeta_1 \in (x_0, x_2)$ such that $g^{(4)}(\zeta_1) = f^{(4)}(\zeta_1) - 0 - 4!\frac{f(x)-p(x)}{u(x)} = 0$ or $f(x) = q(x) + \frac{f^{(4)}(\zeta_1)}{4!} u(x)$ for some $\zeta_1 \in (x_0, x_2)$ as desired.

Now we notice that
$\int_{x_0}^{x_2} w(x)dx$
$= \int_{x_0}^{x_2} (x - x_0)(x - x_1)(x - x_2)dx$
$= \int_{-h}^{h} (t + h)(t)(t - h)dt = \int_{-h}^{h} (t^3 - h^2 t)dt = 0$
This implies that
$\int_{x_0}^{x_2} (f(x) - p(x))\, dx = \int_{x_0}^{x_2} (f(x) - q(x))\, dx = \int_{x_0}^{x_2} \frac{f^{(4)}(\zeta_1)}{4!} u(x)dx = \frac{f^{(4)}(\eta_1)}{4!} \int_{x_0}^{x_2} u(x)dx$
since $u(x)$ does not change sign on (x_0, x_2) and because $f^{(4)}(\eta_1(x))$ is a continuous function.

Now
$\int_{x_0}^{x_2} u(x)dx$
$= \int_{x_0}^{x_2} (x - x_0)(x - x_1)^2(x - x_2)dx$
$= \int_{-h}^{h} (t + h)(t)^2(t - h)dt = \int_{-h}^{h} (t^4 - h^2 t^2)dt = \left[\frac{t^5}{5} - h^2\frac{t^3}{3}\right]_{-h}^{h} = 2\left[\frac{h^5}{5} - \frac{h^5}{3}\right] = -\frac{4}{15}h^5$
So the error on $[x_0, x_2]$ is
$\frac{f^{(4)}(\eta_1)}{4!} \int_{x_0}^{x_2} u(x)dx = \frac{f^{(4)}(\eta_1)}{4!}\left(-\frac{4}{15}h^5\right) = -\frac{h^5}{90} f^{(4)}(\eta_1)$

We notice here that n needs to be even in order to cover $[a, b]$ by non-overlapping intervals similar to $[x_0, x_2]$ and we need $\frac{n}{2}$ of such intervals. Now adding error terms in each interval we have the total error,
$\sum_{j=1}^{\frac{n}{2}} -\frac{h^5}{90} f^{(4)}(\eta_j) = -\frac{h^5}{90}\sum_{j=1}^{\frac{n}{2}} f^{(4)}(\eta_j) = -\frac{h^5}{90}\frac{n}{2} f^{(4)}(\eta) = -\frac{nh^5}{180} f^{(4)}(\eta)$ for $\eta \in (a, b)$ using Extremum and Intermediate Value Theorems for $f \in \mathcal{C}^4$ ie. for $f^{(4)} \in \mathcal{C}$.
Using $h = \frac{b-a}{n}$, the total error is also equal to $-\frac{(b-a)h^4}{180} f^{(4)}(\eta) = -\frac{(b-a)^5}{180n^4} f^{(4)}(\eta)$

$\square$