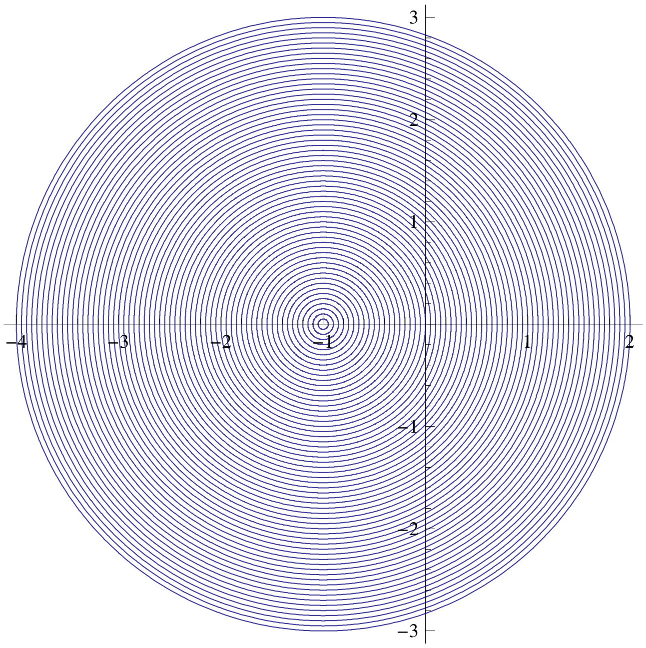
% We want to find the complex roots of z ^ 5 - z - 1 = 0,
using the iterative method z = (1 + z) ^ (1 / 5) = g (z).
To use the Banach Fixed Point Theorem we want to see
1) g : A → A where A is a closed subset of C
2) g is a contraction (Lipschitz continuous with L < 1)

Lets try a circular region A, around z = -1. To do this we let 1 + z = R E ^ (i t),
so that z = R E ^ (I t) - 1, and let R goes from 0 to 3, gives us a disk aroud z = -1.

In[1]:=
```
A =
 ParametricPlot[Table[{Re[R E^(I t) - 1], Im[R E^(I t) - 1]}, {R, 0, 3, 0.05}], {t, 0, 2 Pi}]
```

Out[1]=



% Is this region a contraction for g?

In[2]:=
```
g[z_] := (1 + z) ^ (1 / 5)
```

In[3]:=
```
g'[z]
```

Out[3]= $\frac{1}{5\,(1+z)^{4/5}}$

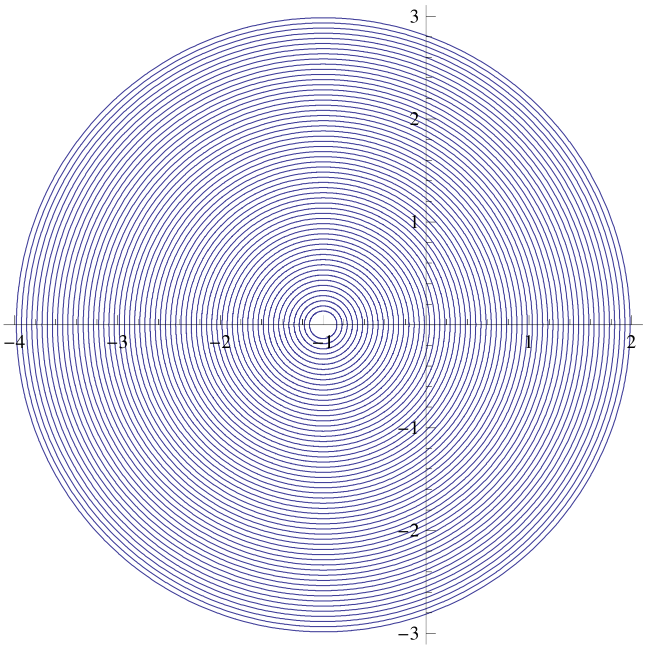
% Now g' (z) =
1 / 5 (1 + z) ^ (-4 / 5) = 1 / 5 (R E^ (I t)) ^ (-4 / 5) = 1 / (5 R ^ (4 / 5)) E ^ (-4 I t / 5),
So that the complex Mod (this is the Matric now), | g' (z) | = 1/(5 R ^(4/5)),
this is less than 1 means,
R ^ (4 / 5) > 1 / 5 or R > (1 / 5) ^ (5 / 4),
so we get rid of radii less than this ie less than

In[4]:=
```
N[(1 / 5) ^ (5 / 4)]
```

Out[4]= 0.133748

In[5]:=
```
A = ParametricPlot[
  Table[{Re[R E^(I t) - 1], Im[R E^(I t) - 1]}, {R, (1 / 5) ^ (5 / 4), 3, 0.05}], {t, 0, 2 Pi}]
```

Out[5]=



% But how actually g ' decides the convergence in the complex case ? . What does the MVT looks like in complex ?
Complex line integral over a curve c (with end point z1 and z2) of g ' is

$g(z2) - g(z1) = \int_{z1}^{z2} g' \, dz$ provided that g is analytic and the curve c is smooth,

(the integral will remain the same for any smooth curve with the same end points, that can be collapsed to c, ie homeomorphic to c) . Now for a straight line c

$$\left| g(z2) - g(z1) \right| <= \int_{z1}^{z2} \left| g' \right| \left| dz \right| \le \text{Max}\{ | g' | \} \int_{z1}^{z2} \left| dz \right|$$

= Max{|g'|}Lenth of c=Max{|g'|}|z2−z1|.
There will be a maximum (since g is analytic) and,
L = Max { | g ' |} is a Lipchitz constant with the requirement for a contraction is ,
L < 1 (ie .R > (1 / 5) ^ (5 / 4)) .
A suffficient requirement here is to able to join any two points by a straight line which lies entirely within the set . Such sets are called Convex, here the set A is clearly not convex due to its hole at the center .

But now we can directly work with the given function for Lipschitz continuity as follows
| g (z2) - g (z1) | = |(1+z1)^(1/5)−(1+z2)^(1/5)|
Notice that
(x - y) = (x^ (1 / 5) - y^ (1 / 5))
(x^ (4 / 5) + x^ (3 / 5) y^ (1 / 5) + x^ (2 / 5) y^ (2 / 5) + x^ (1 / 5) y^ (3 / 5) + y^ (4 / 5))
and that,
| x - y | >= | x^ (1 / 5) - y^ (1 / 5) | (| x^ (4 / 5) | + | x^ (3 / 5) || y^ (1 / 5) |
+ | x^ (2 / 5) || y^ (2 / 5) | + | x^ (1 / 5) || y^ (3 / 5) | + | y^ (4 / 5) |)

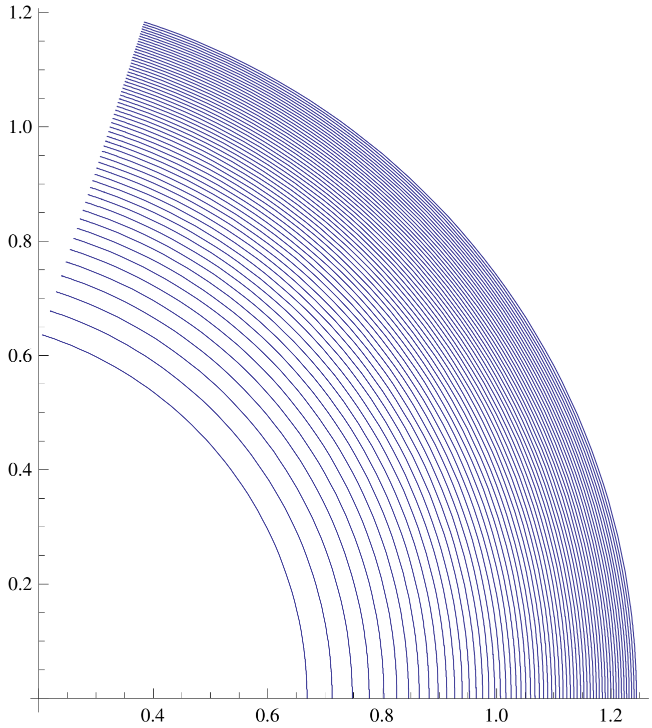
>= | x^ (1 / 5) - y^ (1 / 5) | 5 R^ (4 / 5) for | x |, | y | > R

So with x = 1 + z2 and y = 1 + z1,
we get | g (z2) - g (z1) | ≤ 1 / (5 R^ (4 / 5)) | z2 - z1 | giving us the same region A with,
R > (1 / 5) ^ (5 / 4) for g to be a contraction (without assuming the convexity of A) .

% Now the image of g (z) = (1 + z) ^ (1 / 5) = (R E^ (I t)) ^ (1 / 5) = R^ (1 / 5) E^ (I t / 5)

In[6]:=
```
B = ParametricPlot[Table[{Re[R^(1/5) E^(I t/5)], Im[R^(1/5) E^(I t/5)]},
    {R, (1/5)^(5/4), 3, 0.05}], {t, 0, 2 Pi}]
```

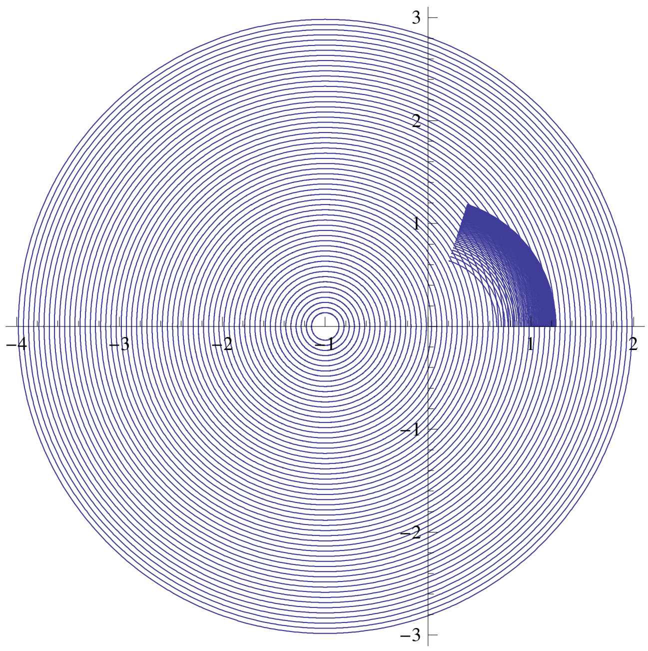
Out[6]=



% To see if g : A → A, we plot them together

In[7]:=
```
Show[{A, B}]
```

Out[7]=



% Lets see what the roots actually are

In[8]:=
```
Roots[z^5 - z - 1 == 0, z]
```

Out[8]=
```
z == Root[-1 - #1 + #1^5 &, 1] || z == Root[-1 - #1 + #1^5 &, 2] ||
  z == Root[-1 - #1 + #1^5 &, 3] || z == Root[-1 - #1 + #1^5 &, 4] || z == Root[-1 - #1 + #1^5 &, 5]
```

In[9]:=
```
N[%]
```

Out[9]=
```
z == 1.1673 || z == -0.764884 - 0.352472 ⅈ ||
 z == -0.764884 + 0.352472 ⅈ || z == 0.181232 - 1.08395 ⅈ || z == 0.181232 + 1.08395 ⅈ
```

In[10]:=
```
T = Table[{Re[%[[n]][[2]]], Im[%[[n]][[2]]]}, {n, 1, 5}]
```

Out[10]=
```
{{1.1673, 0}, {-0.764884, -0.352472},
 {-0.764884, 0.352472}, {0.181232, -1.08395}, {0.181232, 1.08395}}
```

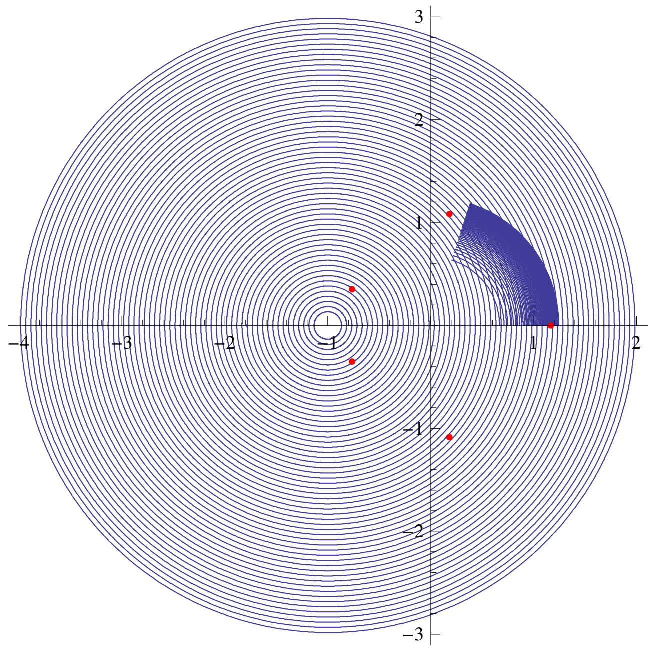
% Lets see where the roots actually located

In[11]:=
```
P = Graphics[{Red, Point[T]}];
```

% Now everyting together

In[12]:=
```
Show[{A, B, P}]
```

Out[12]=



% We can see that we can get the 1 st root by this g. So lets do the iterations.

In[13]:=
```
z = 1 + 2 I; S = {};
For[k = 0, k ≤ 5, k++, {Print[N[z]], S = Append[S, {Re[z], Im[z]}], z = N[g[z], 10]}]
```

```
1. + 2. ⅈ

1.21599 + 0.192593 ⅈ

1.17321 + 0.0203438 ⅈ

1.16795 + 0.00218662 ⅈ

1.16737 + 0.000235485 ⅈ

1.16731 + 0.0000253657 ⅈ
```

% S is the point set

In[14]:= **S**

Out[14]= {{1, 2}, {1.2159869826, 0.1925934177},
{1.1732078739, 0.0203438340}, {1.1679474384, 0.0021866166},
{1.1673733782, 0.0002354852}, {1.1673114550, 0.0000253657}}

% These are the points in 2 D

In[15]:= **Q = Graphics[Point[S]];**

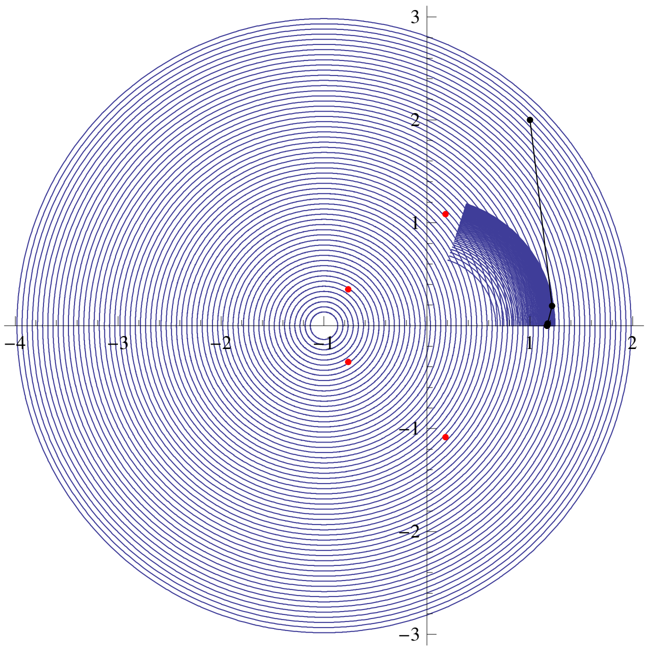
% This is the line connected version

In[16]:= **U = Graphics[Line[S]];**

% See all of them in the same picture

In[17]:= **Show[{A, B, P, Q, U}]**

Out[17]=



% It is clear that we cannot get the other roots by using this g. This is because our root taking is not general (ie there are other branches). We know E^(I t) = Cos[t] + I Sin[t], so that E^(I 2 n Pi) = 1 where n is an integer. Now

g[z_] := (1 + z)^(1 / 5) = ((1 + z) .1)^(1 / 5) = (1 + z)^(1 / 5) E^(I 2 n Pi / 5)

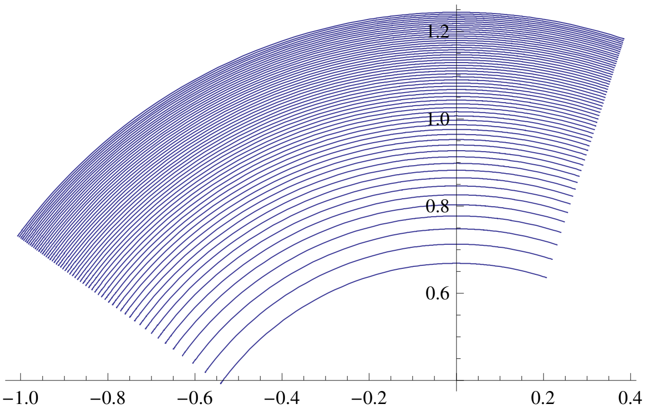
Lets define a new g with n = 1

In[18]:= **g[z_] := (1 + z)^(1 / 5) E^(I 2 Pi / 5)**

% Now the image of g (z) with 1 + z = R E^(I t) is

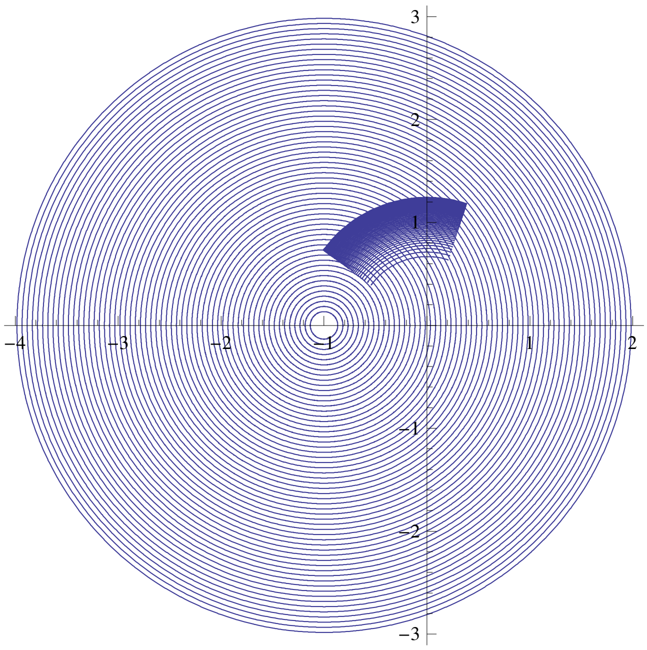
In[19]:=
```
B = ParametricPlot[
  Table[{Re[R^(1 / 5) E^(I t / 5) E^(I 2 Pi / 5)], Im[R^(1 / 5) E^(I t / 5) E^(I 2 Pi / 5)]},
   {R, (1 / 5)^(5 / 4), 3, 0.05}], {t, 0, 2 Pi}]
```

Out[19]=



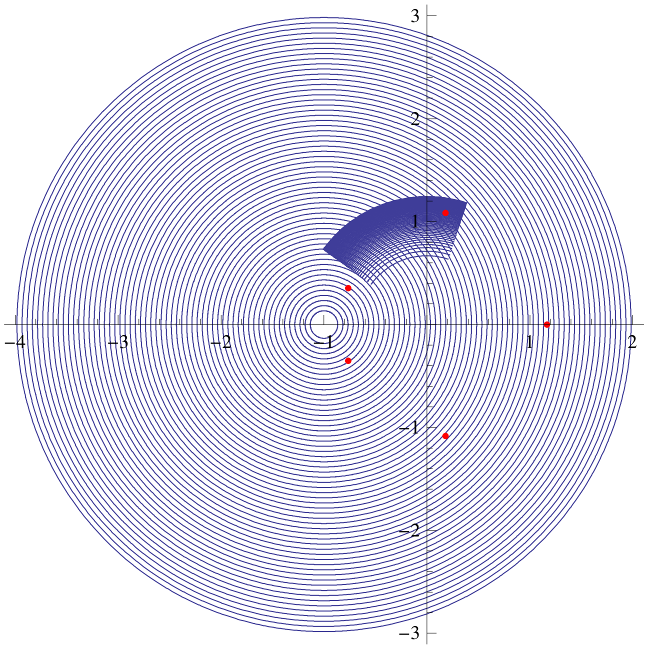
In[20]:=
```
Show[{A, B}]
```

Out[20]=



In[21]:= **Show[{A, B, P}]**

Out[21]=



% We can see that now we can get the 2 nd root by this g. So lets do the iterations

In[22]:=
```
z = 1 + 2 I;  S = {};
For[k = 0, k ≤ 5, k++, {Print[N[z]], S = Append[S, {Re[z], Im[z]}], z = N[g[z], 10]}]
```

1. + 2. i

0.192593 + 1.21599 i

0.17188 + 1.09902 i

0.178956 + 1.0848 i

0.180915 + 1.08385 i

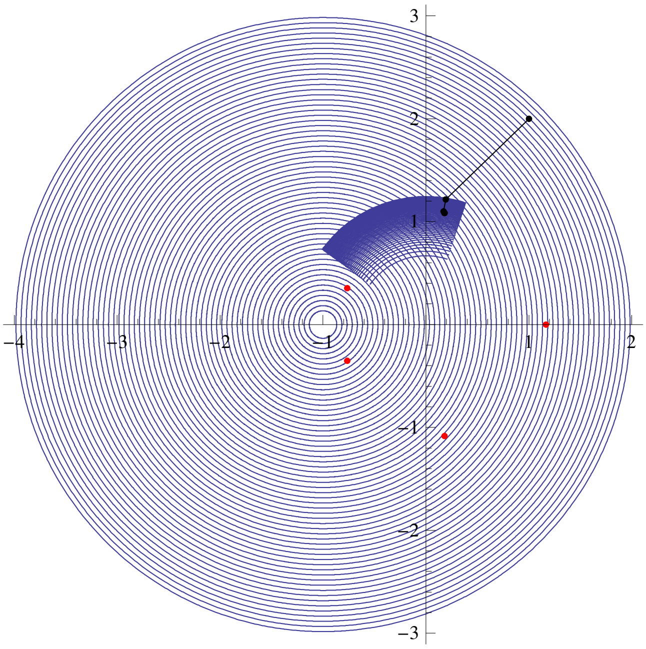
0.181207 + 1.08392 i

In[23]:= **Q = Graphics[Point[S]];**

In[24]:= **U = Graphics[Line[S]];**

In[25]:=
```
Show[{A, B, P, Q, U}]
```

Out[25]=

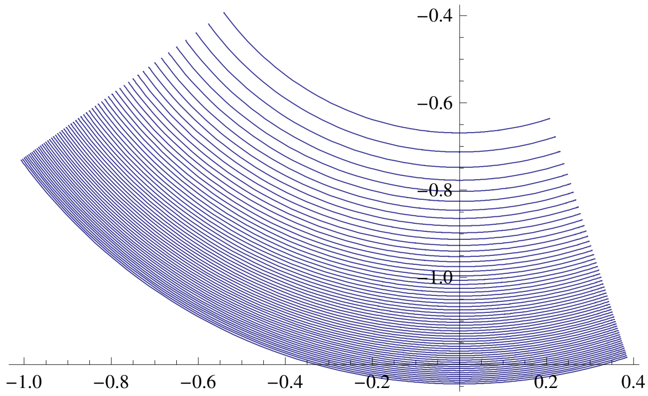


% Now to find the 5 th root, lets define a new g with n = 3

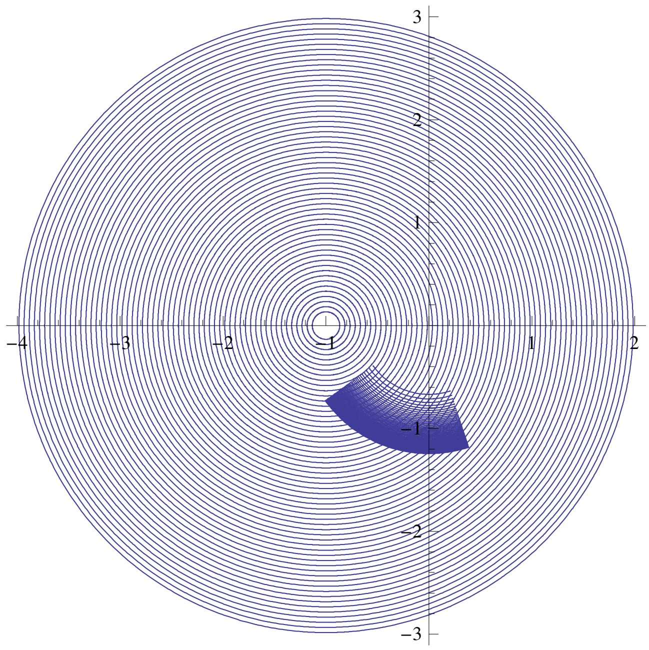
In[26]:=
```
g[z_] := (1 + z) ^ (1 / 5) E^ (I 6 Pi / 5)
```

In[27]:=
```
B = ParametricPlot[
  Table[{Re[R^ (1 / 5) E^ (I t / 5) E^ (I 6 Pi / 5)], Im[R^ (1 / 5) E^ (I t / 5) E^ (I 6 Pi / 5)]},
   {R, (1 / 5) ^ (5 / 4), 3, 0.05}], {t, 0, 2 Pi}]
```

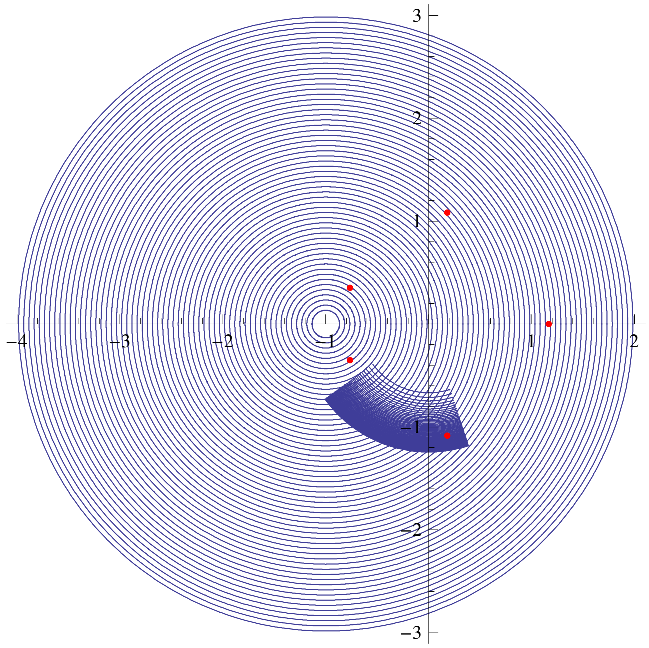
Out[27]=



In[28]:= `Show[{A, B}]`

Out[28]=


In[29]:= `Show[{A, B, P}]`

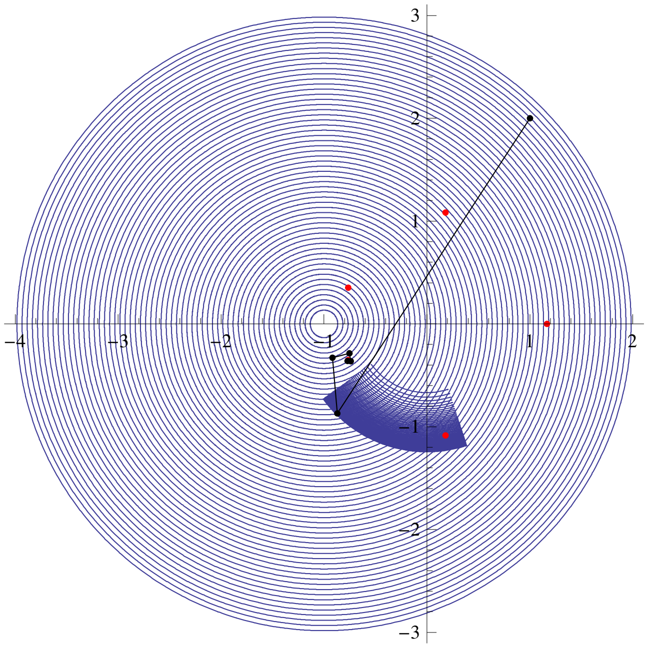
Out[29]=


In[30]:=
```
z = 1 + 2 I;  S = {};
For[k = 0, k ≤ 5, k++, {Print[N[z]], S = Append[S, {Re[z], Im[z]}], z = N[g[z], 10]}]
```

```
1. + 2. i
-0.870551 - 0.870551 i
-0.917779 - 0.32846 i
-0.752724 - 0.286094 i
-0.738875 - 0.363051 i
-0.770659 - 0.361744 i
```

```
In[31]:= Q = Graphics[Point[S]];
In[32]:= U = Graphics[Line[S]];
In[33]:= Show[{A, B, P, Q, U}]
```

Out[33]=



% We have a serious problem,
iterations goes outside the region and we don ' t get the root we want . How did that happen ?

This happens because of the way Mathematica defines Arg,
it goes from - Pi to Pi, not from 0 to 2 Pi . We were good
as long as we were on the upper half plane . Now redefine Arg as arg

```
In[34]:= arg[z_] := 2 Pi (1 - Sign[Arg[z]]) / 2 + Arg[z]
```

% Redefine g

```
In[35]:= g[z_] := Abs[(1 + z)]^(1 / 5) E^(I arg[1 + z] / 5) E^(I 6 Pi / 5)
```

% Lets do the iterations now

```
In[36]:= z = 1 + 2 I; S = {};
For[k = 0, k ≤ 5, k++, {Print[N[z]], S = Append[S, {Re[z], Im[z]}], z = N[g[z], 10]}]
```

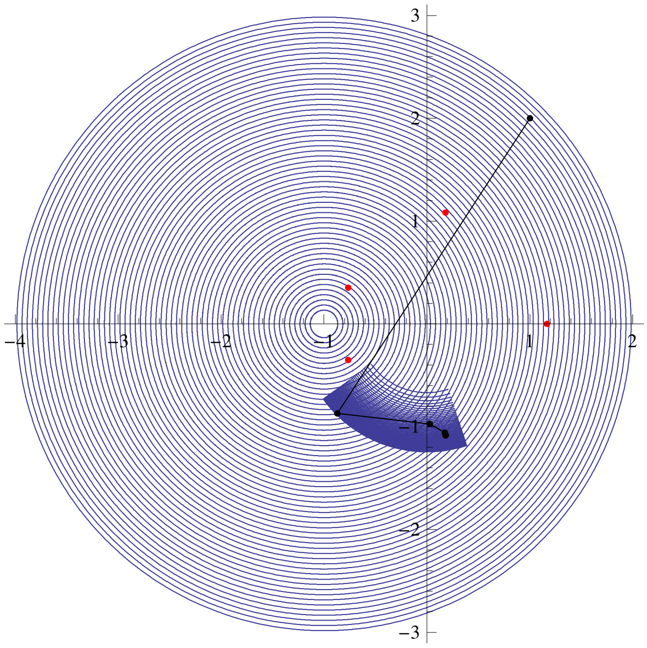
```
1. + 2. ⅈ

-0.870551 - 0.870551 ⅈ

0.0287747 - 0.97436 ⅈ

0.173477 - 1.05806 ⅈ

0.182598 - 1.08049 ⅈ

0.181673 - 1.08369 ⅈ
```

In[37]:= **Q = Graphics[Point[S]];**

In[38]:= **U = Graphics[Line[S]];**

In[39]:= **Show[{A, B, P, Q, U}]**

Out[39]=



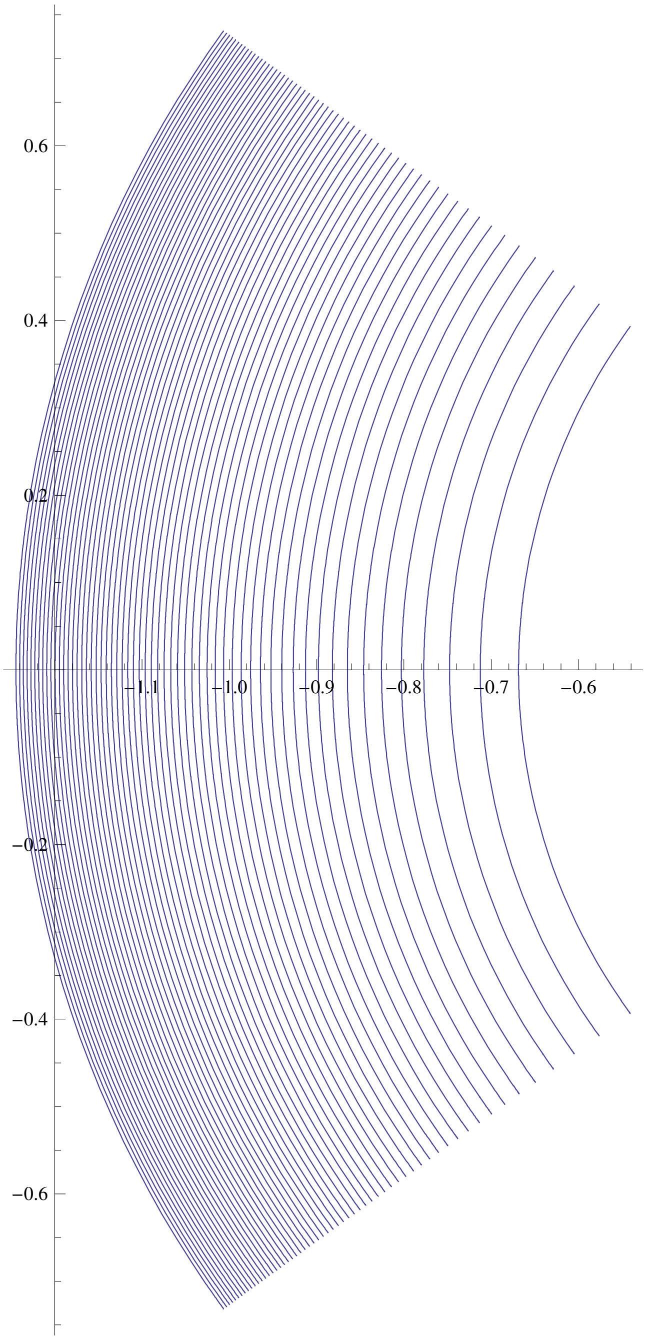
% To get the 3 rd root Lets define new g with n = 2

In[40]:= **g[z_] := Abs[(1 + z)]^(1/5) E^(I arg[1 + z]/5) E^(I 4 Pi/5)**

% Now the image of g (z) with 1 + z = R E^(I t)

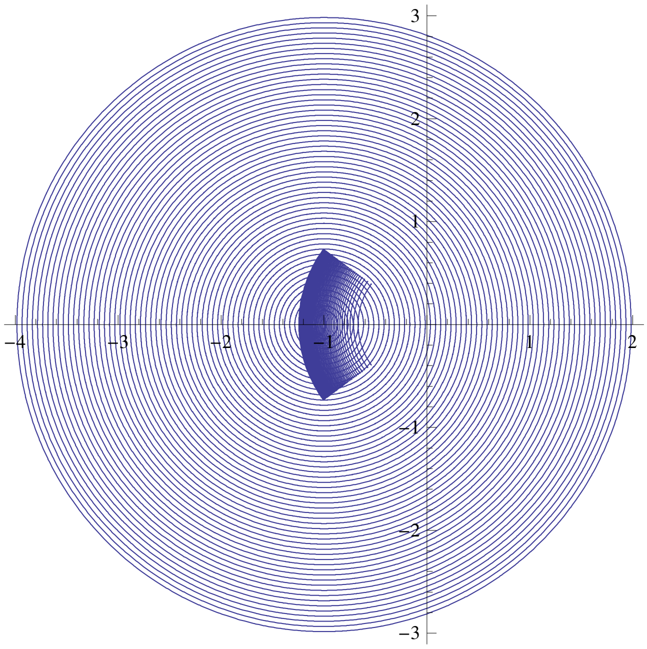
In[41]:=
```
B = ParametricPlot[
  Table[{Re[R^(1 / 5) E^(I t / 5) E^(I 4 Pi / 5)], Im[R^(1 / 5) E^(I t / 5) E^(I 4 Pi / 5)]},
   {R, (1 / 5)^(5 / 4), 3, 0.05}], {t, 0, 2 Pi}]
```

Out[41]=



In[42]:=
```
Show[{A, B}]
```

Out[42]=

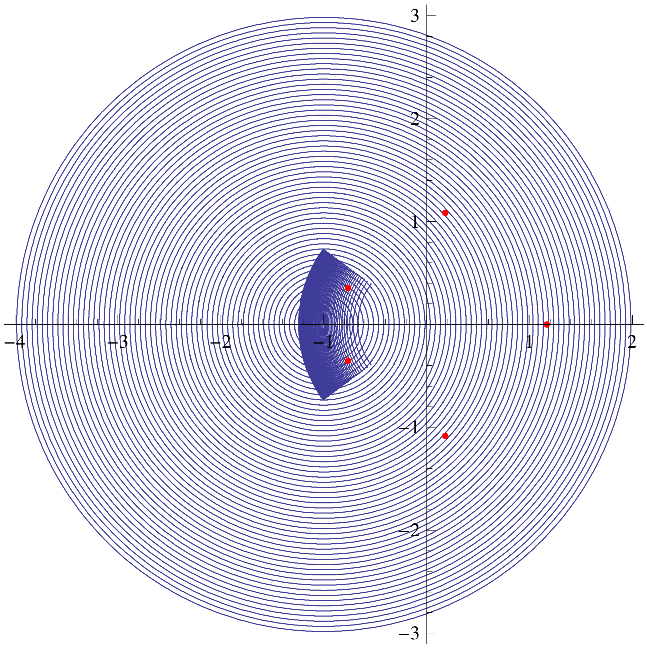


% This time however, the image of g is not within A (there is ahole in A at the center), violating the requirements of the Banach Fixed Point Theorem. So things can go wrong.

% Lets plot with roots

In[43]:=
```
Show[{A, B, P}]
```

Out[43]=



% We can see that there are two roots, violating uniqueness. Lets try iterations anyway

In[44]:=
```
z = 1 + 2 I; S = {};
For[k = 0, k ≤ 5, k++, {Print[N[z]], S = Append[S, {Re[z], Im[z]}], z = N[g[z], 10]}]
```

1. + 2. ⅈ

-1.09696 + 0.558928 ⅈ

-0.858087 + 0.246567 ⅈ

-0.710552 + 0.316129 ⅈ

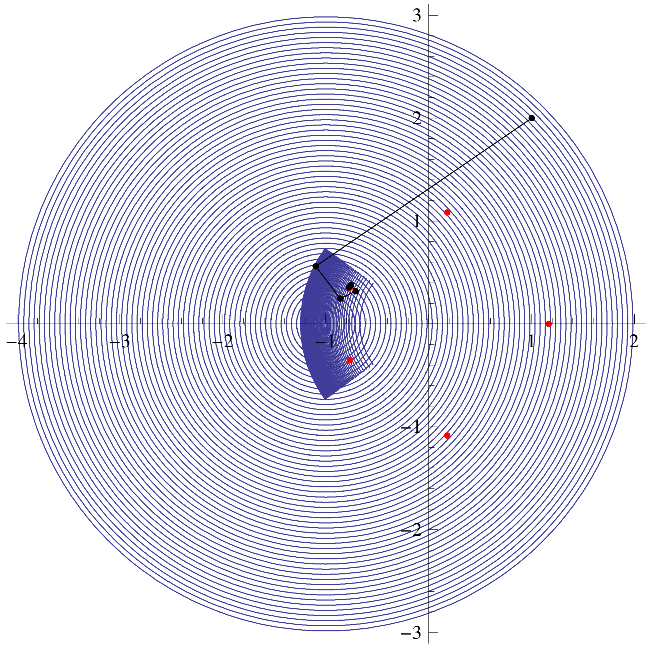
-0.755481 + 0.376594 ⅈ

-0.774697 + 0.354669 ⅈ

In[45]:= **Q = Graphics[Point[S]];**

In[46]:= **U = Graphics[Line[S]];**

In[47]:= **Show[{A, B, P, Q, U}]**

Out[47]=


% We will never get the 4 th root if we start anywhere on the upper half plane. But inspired by symmetry, lets start with z = 1 - 2 I to get the other root.

```
In[48]:= z = 1 - 2 I; S = {};
For[k = 0, k ≤ 5, k++, {Print[N[z]], S = Append[S, {Re[z], Im[z]}], z = N[g[z], 10]}]
```

1. - 2. ⅈ

-1.09696 - 0.558928 ⅈ

-0.858087 - 0.246567 ⅈ

-0.710552 - 0.316129 ⅈ

-0.755481 - 0.376594 ⅈ

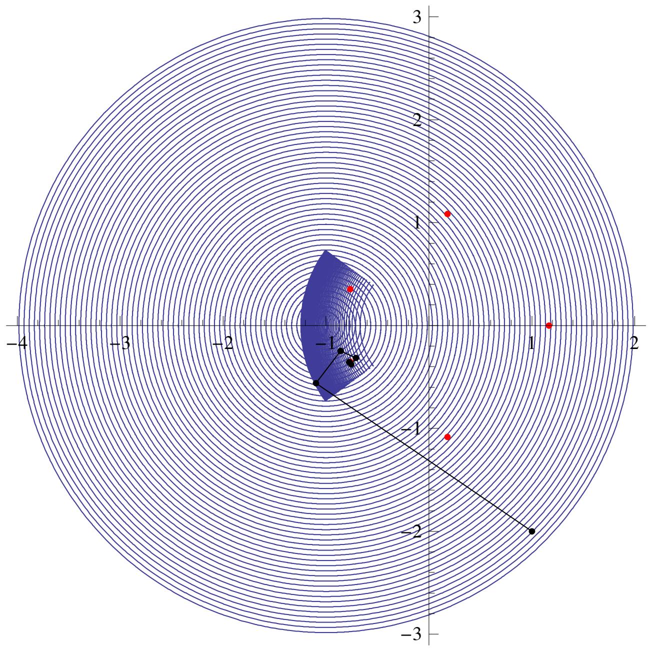
-0.774697 - 0.354669 ⅈ

In[49]:= **Q = Graphics[Point[S]];**

In[50]:= **U = Graphics[Line[S]];**

In[51]:=
```
Show[{A, B, P, Q, U}]
```

Out[51]=



% We had all those trouble and guesswork since we failed to meet requirements of the Banach Fixed Point Theorem. Now we try to change the region A to account for thse requirements. I have figured out that it can be done by reducing the radious to 2 and angle less than (see why?)

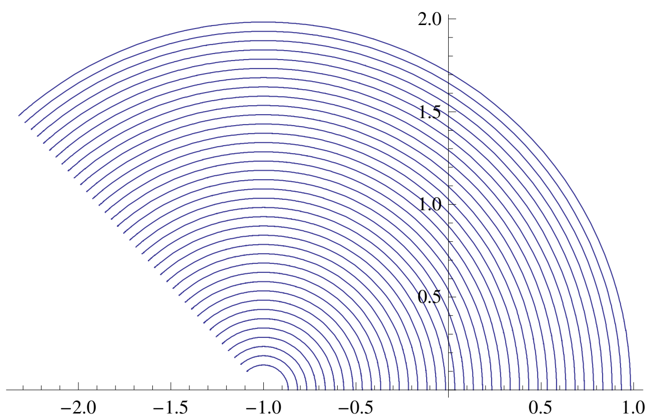
In[52]:=
```
N[5 (Pi - ArcSin[(1 / 5) ^ (5 / 4)] - 4 Pi / 5)]
```

Out[52]=
```
2.47084
```

% Taking angles upto 2.3 the new domain is

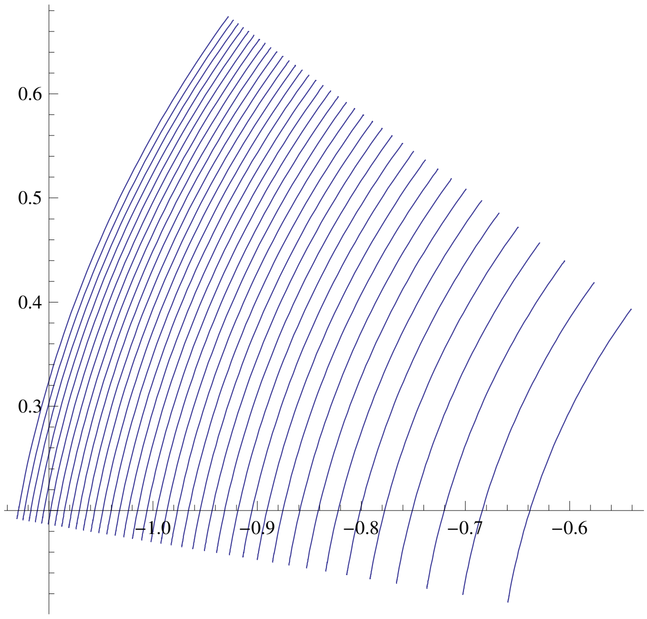
In[53]:=
```
A = ParametricPlot[
  Table[{Re[R E^(I t) - 1], Im[R E^(I t)]}, {R, (1 / 5) ^ (5 / 4), 2, 0.05}], {t, 0, 2.3}]
```

Out[53]=



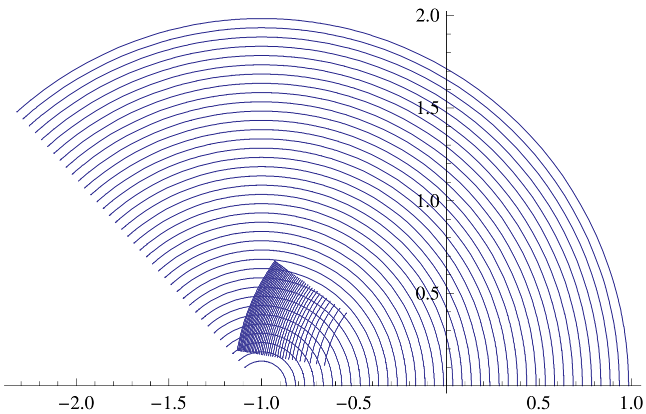
In[54]:=
```
B = ParametricPlot[
  Table[{Re[R^(1/5) E^(I t/5) E^(I 4 Pi/5)], Im[R^(1/5) E^(I t/5) E^(I 4 Pi/5)]},
   {R, (1/5)^(5/4), 2, 0.05}], {t, 0, 2.3}]
```

Out[54]=



In[55]:=
```
Show[{A, B}]
```

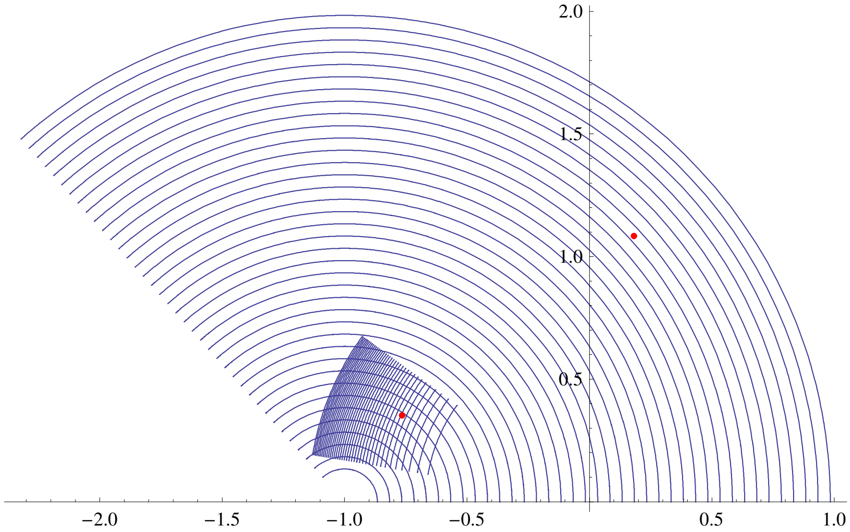
Out[55]=



% Now we meet both requirements of the theorem

In[56]:=
```
Show[{A, B, P}]
```

Out[56]=



% We can see that there is a unique root in the overlapping regions, we must be able to get it. Try iterations

In[57]:=
```
z = -0.5 + 1.5 I; S = {};
For[k = 0, k ≤ 5, k++, {Print[N[z]], S = Append[S, {Re[z], Im[z]}], z = N[g[z], 10]}]
```

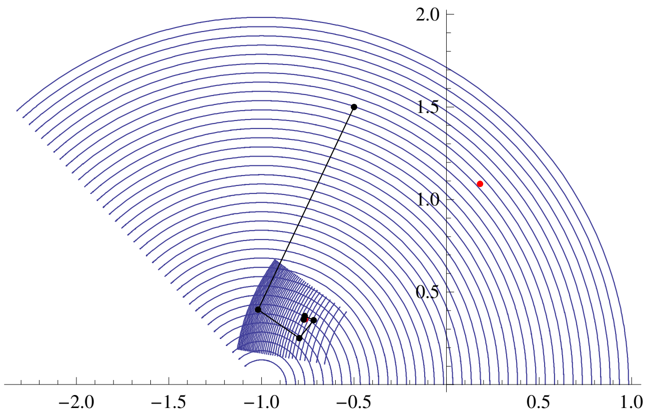
```
-0.5 + 1.5 i
-1.01838 + 0.404996 i
-0.796244 + 0.250753 i
-0.718113 + 0.347448 i
-0.766478 + 0.370704 i
-0.771869 + 0.350813 i
```

In[58]:=
```
Q = Graphics[Point[S]];
```

In[59]:=
```
U = Graphics[Line[S]];
```

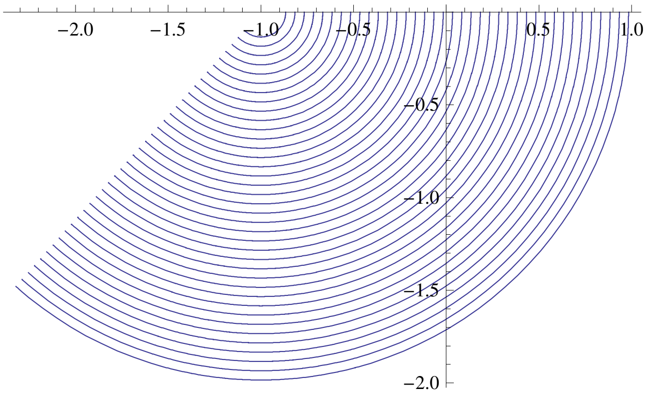
In[60]:=
```
Show[{A, B, P, Q, U}]
```

Out[60]=

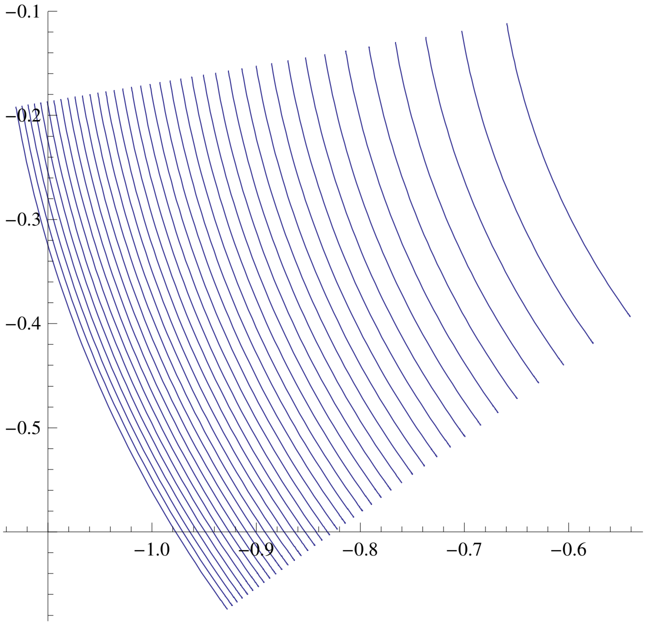


% Now we try to get the 4 th root acoording to the Banach Fixed Point Theorem. We find a domain on the lower half plane which is a mirror image of the previous one

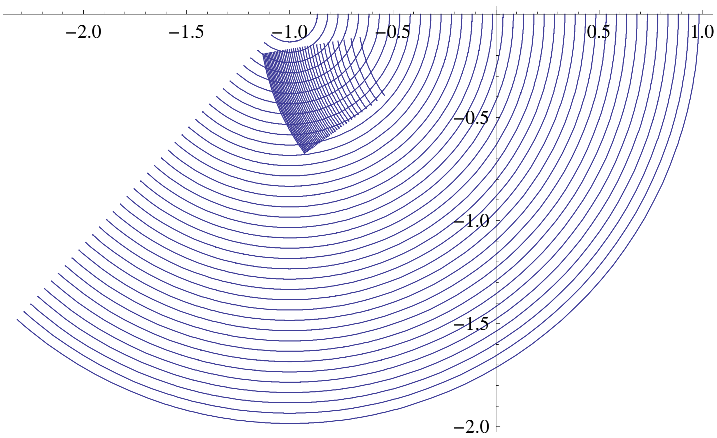
In[61]:=
```
A = ParametricPlot[Table[{Re[R E^(I t) - 1], Im[R E^(I t)]},
    {R, (1 / 5) ^ (5 / 4), 2, 0.05}], {t, 2 Pi - 2.3, 2 Pi}]
```

Out[61]=


In[62]:=
```
B = ParametricPlot[
  Table[{Re[R^(1 / 5) E^(I t / 5) E^(I 4 Pi / 5)], Im[R^(1 / 5) E^(I t / 5) E^(I 4 Pi / 5)]},
   {R, (1 / 5) ^ (5 / 4), 2, 0.05}], {t, 2 Pi - 2.3, 2 Pi}]
```

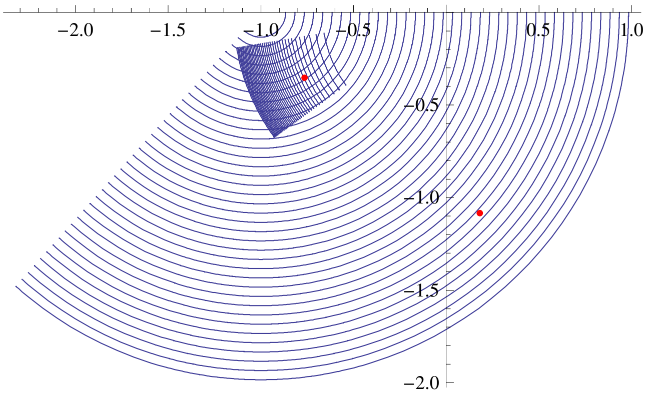
Out[62]=


In[63]:=
```
Show[{A, B}]
```

Out[63]=


In[64]:= **Show[{A, B, P}]**

Out[64]=



In[65]:=
```
z = -0.5 - 1.5 I; S = {};
For[k = 0, k ≤ 5, k++, {Print[N[z]], S = Append[S, {Re[z], Im[z]}], z = N[g[z], 10]}]
```

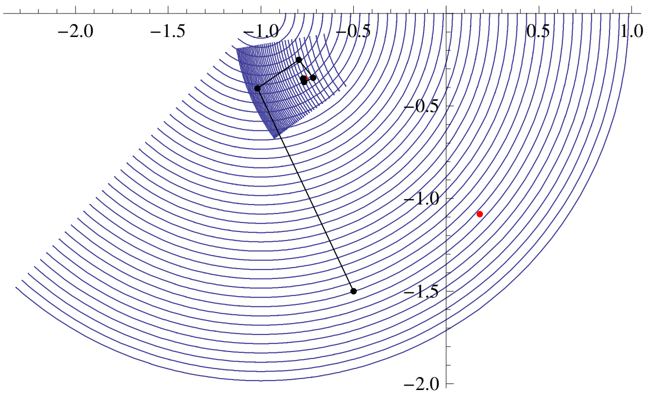
```
-0.5 - 1.5 i
-1.01838 - 0.404996 i
-0.796244 - 0.250753 i
-0.718113 - 0.347448 i
-0.766478 - 0.370704 i
-0.771869 - 0.350813 i
```

In[66]:= **Q = Graphics[Point[S]];**

In[67]:= **U = Graphics[Line[S]];**

In[68]:= **Show[{A, B, P, Q, U}]**

Out[68]=



% Finally we found all the roots and saw Banach Fixed Point Theorem in action ! We are happy now !