Theorem 1. *Error formula for the Simpson's Rule*
If $f \in \mathcal{C}^4$, error in the Simpson's rule is $-\frac{(b-a)^5}{180n^4} f^{(4)}(\eta)$ where $\eta \in (a,b)$

Proof. With usual notation, let $p(x)$ be the degree two Lagrange polynomial of $f(x)$ agreeing with it on three points x_0, x_1, x_2 which are distance h apart.
Consider error on the first two intervals, $\int_{x_0}^{x_2} (f(x) - p(x))\, dx$

We define a degree three polynomial as follows
$q(x) = p(x) + \frac{1}{h^2} (p'(x_1) - f'(x_1))\, w(x)$ where $w(x) = (x - x_0)(x - x_1)(x - x_2)$
We notice that $q(x_k) = p(x_k) = f(x_k)$ for $k = 0, 1, 2$ and $q'(x_1) = f'(x_1)$

First we claim that $f(x) = q(x) + \frac{f^{(4)}(\zeta_1)}{4!} u(x)$ where $u(x) = (x - x_0)(x - x_1)^2(x - x_2)$ and $\zeta_1 \in (x_0, x_2)$. To prove this define $g(y) = f(y) - q(y) - u(y)\frac{f(x)-p(x)}{u(x)}$
Now $g(y) = 0$ for $y = x_0, x_1, x_2, x$ and $g'(x_1) = 0$
This implies, by the Mean Value Theorem that there is $\zeta_1 \in (x_0, x_2)$ such that $g^{(4)}(\zeta_1) = f^{(4)}(\zeta_1) - 0 - 4!\frac{f(x)-p(x)}{u(x)} = 0$ or $f(x) = q(x) + \frac{f^{(4)}(\zeta_1)}{4!} u(x)$ for some $\zeta_1 \in (x_0, x_2)$ as desired.

Now we notice that
$\int_{x_0}^{x_2} w(x)dx$
$= \int_{x_0}^{x_2} (x - x_0)(x - x_1)(x - x_2)dx$
$= \int_{-h}^{h} (t + h)(t)(t - h)dt = \int_{-h}^{h} (t^3 - h^2 t)dt = 0$
This implies that
$\int_{x_0}^{x_2} (f(x) - p(x))\, dx = \int_{x_0}^{x_2} (f(x) - q(x))\, dx = \int_{x_0}^{x_2} \frac{f^{(4)}(\zeta_1)}{4!} u(x)dx = \frac{f^{(4)}(\eta_1)}{4!} \int_{x_0}^{x_2} u(x)dx$
since $u(x)$ does not change sign on (x_0, x_2) and because $f^{(4)}(\eta_1(x))$ is a continuous function.

Now
$\int_{x_0}^{x_2} u(x)dx$
$= \int_{x_0}^{x_2} (x - x_0)(x - x_1)^2(x - x_2)dx$
$= \int_{-h}^{h} (t + h)(t)^2(t - h)dt = \int_{-h}^{h} (t^4 - h^2 t^2)dt = \left[\frac{t^5}{5} - h^2\frac{t^3}{3}\right]_{-h}^{h} = 2\left[\frac{h^5}{5} - \frac{h^5}{3}\right] = -\frac{4}{15}h^5$
So the error on $[x_0, x_2]$ is
$\frac{f^{(4)}(\eta_1)}{4!} \int_{x_0}^{x_2} u(x)dx = \frac{f^{(4)}(\eta_1)}{4!} \left(-\frac{4}{15}h^5\right) = -\frac{h^5}{90} f^{(4)}(\eta_1)$

We notice here that n needs to be even in order to cover $[a, b]$ by non-overlapping intervals similar to $[x_0, x_2]$ and we need $\frac{n}{2}$ of such intervals. Now adding error terms in each interval we have the total error,
$\sum_{j=1}^{\frac{n}{2}} -\frac{h^5}{90} f^{(4)}(\eta_j) = -\frac{h^5}{90} \sum_{j=1}^{\frac{n}{2}} f^{(4)}(\eta_j) = -\frac{h^5}{90}\frac{n}{2} f^{(4)}(\eta) = -\frac{nh^5}{180} f^{(4)}(\eta)$ for $\eta \in (a, b)$ using Extremum and Intermediate Value Theorems for $f \in \mathcal{C}^4$ ie. for $f^{(4)} \in \mathcal{C}$.
Using $h = \frac{b-a}{n}$, the total error is also equal to $-\frac{(b-a)h^4}{180} f^{(4)}(\eta) = -\frac{(b-a)^5}{180n^4} f^{(4)}(\eta)$

□

Theorem 2. *Newton-Kantorovich Theorem*
1.Let $f : [a, b] \to \mathbb{R}$
2.f' *is Lipschitz continuous with constant* γ
3.$f'(x) \neq 0$ *and* $\frac{1}{|f'(x)|} \leq \beta$
4.$x_0 \in [a, b]$ *and* $\left|\frac{f(x_0)}{f'(x_0)}\right| = \alpha$
5.$q = \alpha\beta\gamma < \frac{1}{2}$
6.$[x_0 - 2\alpha, x_0 + 2\alpha] \subset [a, b]$
7. $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, k \geq 0$
Then
1. $\lim_{k\to\infty} x_k = z \in [x_0 - 2\alpha, x_0 + 2\alpha]$*.*
2. $f(z) = 0$ *and* z *is a unique root of* f
3. $|x_k - z| \leq 2\alpha q^{2^k-1}$

Proof. By definition we have, $|x_{k+1} - x_k| = \left|\frac{f(x_k)}{f'(x_k)}\right| \leq \beta|f(x_k)| = \beta|f(x_k) - f(x_{k-1}) - (x_k - x_{k-1})f'(x_{k-1})| = \beta\left|\int_{x_{k-1}}^{x_k} f'(t)dt - f'(x_{k-1})\int_{x_{k-1}}^{x_k} dt\right| \leq \beta\left|\int_{x_{k-1}}^{x_k} |f'(t) - f'(x_{k-1})|dt\right| \leq \beta\gamma\left|\int_{x_{k-1}}^{x_k} |t - x_{k-1}|dt\right| = \frac{1}{2}\beta\gamma|x_k - x_{k-1}|^2 = p|x_k - x_{k-1}|^2$

Also by using the formula k times, $|x_{k+1} - x_k| \leq p|x_k - x_{k-1}|^2 \leq p(p|x_{k-1} - x_{k-2}|^2)^2 = p^{1+2^1}|x_{k-1} - x_{k-2}|^{2^2} \leq p^{1+2^1+2^2+\cdots+2^{k-1}}|x_1 - x_0|^{2^k} = p^{\frac{2^k-1}{2-1}}|x_1 - x_0|^{2^k} = \frac{1}{p}(p\alpha)^{2^k} = \frac{\alpha}{\alpha\beta\gamma}(\alpha\beta\gamma)^{2^k} = \alpha q^{2^k-1}$

So we have $|x_{k+1} - x_k| \leq \alpha q^{2^k-1}$

Now a general difference,
$|x_{k+j} - x_k| = |x_{k+j} - x_{k+j-1} + x_{k+j-1} - x_{k+j-2} + \cdots + x_{k+1} - x_k| \leq |x_{k+j} - x_{k+j-1}| + |x_{k+j-1} - x_{k+j-2}| + \cdots + |x_{k+1} - x_k| \leq \alpha q^{2^{k+j-1}-1} + \alpha q^{2^{k+j-2}-1} + \cdots + \alpha q^{2^k-1} = \alpha q^{-1}\left(q^{2^k} + q^{2^{k+1}} + \cdots + q^{2^{k+j-1}}\right) \leq \alpha q^{-1}\left(q^{2^k} + (q^{2^k})^2 + (q^{2^k})^4 + \dots\right) = \alpha q^{2^k-1}\left(1 + (q^{2^k}) + (q^{2^k})^3 + \dots\right) \leq \alpha q^{2^k-1}\left(1 + (q^{2^k}) + (q^{2^k})^2 + \dots\right)$
$\leq \alpha q^{2^k-1}\left(1 + q + q^2 + \dots\right) = \alpha q^{2^k-1}\frac{1}{1-q} \leq \alpha q^{2^k-1}\frac{1}{1-\frac{1}{2}} = 2\alpha q^{2^k-1}$ since $0 \leq q < \frac{1}{2} < 1$

Finally we have $|x_{k+j} - x_k| \leq 2\alpha q^{2^k-1}(1)$

Note that by $k = 0$ in (1)we have $|x_j - x_0| \leq 2\alpha q^{2^0-1} = 2\alpha$ so $x_k \in [x_0 - 2\alpha, x_0 + 2\alpha]$

Also by (1) and $0 \leq q < 1$, x_k is a Cauchy sequence and converges to $z \in [x_0 - 2\alpha, x_0 + 2\alpha]$.
By taking $j \to \infty$ in (1), we have $|z - x_k| \leq 2\alpha q^{2^k-1}$.

Note that f' is continuous with $f'(x) \neq 0$. Taking $k \to \infty$ in $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$, we get $z = z - \frac{f(z)}{f'(z)}$ or $f(z) = 0$, so z is a root of f.

Now assume that the root z is not unique, i.e. there exists $w \neq z$ with $f(w) = 0$. By MVT, for some ζ between z, w we have $f'(\zeta) = \frac{f(z)-f(w)}{z-w} = \frac{0}{z-w} = 0$ or $f'(\zeta) = 0$ which is a contradiction. Therefore $z = w$ and the root is unique.

□

Theorem 1.
If $\lim_{(x,y)\to(a,b)} f(x,y) = L$, $\lim_{x\to a} f(x,y) = g(y)$ in a dnbd of b and $\lim_{y\to b} f(x,y) = h(x)$ in a dnbd of a, Then $\lim_{x\to a} h(x) = \lim_{y\to b} g(y) = L$.

Proof.
Let $\epsilon > 0$, then $\lim_{(x,y)\to(a,b)} f(x,y) = L$ implies $\exists\delta > 0, \forall(x,y)$;
$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x,y) - L| < \epsilon/2$.
$\lim_{x\to a} f(x,y) = g(y)$ in a dnbd of b implies $\exists\delta_1$, whenever $0 < |y-b| < \delta_1$, $\exists\delta_2(y) > 0 \forall x$ such that $0 < |x-a| < \delta_2(y) \Rightarrow |f(x,y) - g(y)| < \epsilon/2$
Let $\delta_3(y) = \min\{\delta_2(y), \delta/\sqrt{2}\} > 0$ and $\delta_4 = \min\{\delta_1, \delta/\sqrt{2}\} > 0$

Now when $0 < |y-b| < \delta_4$ select x such that $0 < |x-a| < \delta_3(y)$. Then
$0 < |x-a| < \delta_2(y)$ so $|f(x,y) - g(y)| < \epsilon/2$.
We also have $0 < |x-a| < \delta/\sqrt{2}$ and $0 < |y-b| < \delta/\sqrt{2}$ so $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ which implies $|f(x,y) - L| < \epsilon/2$.
Now $|g(y) - L| \leq |f(x,y) - g(y)| + |f(x,y) - L| < \epsilon$ which means
$L = \lim_{y\to b} g(y) = \lim_{y\to b}\lim_{x\to a} f(x,y)$
In the same way we can prove $L = \lim_{x\to a} h(x) = \lim_{x\to a}\lim_{y\to b} f(x,y)$ □

Note 1.
When the one variable limits exist, but the iterated limits are different, this theorem can be used to show that the double limit does not exist.
For example let $f(x,y) = \frac{x-y}{x+y}$ for $(x,y) \neq (0,0)$. Then $\lim_{x\to 0} f(x,y) = -1 = g(y)$ exists in a dnbd of 0 and $\lim_{y\to 0} g(y) = -1$. Also $\lim_{y\to 0} f(x,y) = 1 = h(x)$ exists in a dnbd of 0 and $\lim_{y\to 0} h(x) = 1$. Therefore $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Theorem 2.
If $\lim_{(x,y)\to(a,b)} f(x,y) = L$ and $\lim_{x\to a} g(x) = b$ then $\lim_{x\to a} f(x, g(x)) = L$

Proof. Let $\epsilon > 0$, then $\lim_{(x,y)\to(a,b)} f(x,y) = L$ implies $\exists\delta > 0, \forall(x,y)$;
$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x,y) - L| < \epsilon$.
$\lim_{x\to a} g(x) = b$ implies $\exists\delta_1 > 0 \forall x, 0 < |x-a| < \delta_1 \Rightarrow |g(x) - b| < \delta/\sqrt{2}$
Let $\delta_3 = \min\{\delta_1, \delta/\sqrt{2}\} > 0$.
Now if $0 < |x-a| < \delta_3$ we have $0 < |x-a| < \delta/\sqrt{2}$ and $|y-b| < \delta/\sqrt{2}$ where $y = g(x)$.
Therefore we have $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ which implies $|f(x,g(x)) - L| < \epsilon$.
This means $\lim_{x\to a} f(x, g(x)) = L$. □

Note 2.
If g is continuous such that $g(a) = b$ we get $\lim_{x\to a} g(x) = g(a) = b$ which is a required condition.
If we can find two different functions g, h such that $\lim_{x\to a} g(x) = b = \lim_{x\to a} h(x)$ but if $\lim_{x\to a} f(x, g(x)) \neq \lim_{x\to a} f(x, h(x))$ this means the double limit does not exist.
For example let $f(x,y) = \frac{xy}{x^2+y^2}$ for $(x,y) \neq (0,0)$. Let $g(x) = mx$ which gives $\lim_{x\to a} g(x) = 0$. But $\lim_{x\to 0} f(x, g(x)) = \frac{m}{1+m^2}$, so the answer depends on m. Therefore $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.