**Note 1.**
*$\mathcal{B}$=set of Bounded functions*
*$\mathcal{C}$=set of Continuous functions*
*$\mathcal{D}$=set of Differentiable functions*
*$\mathcal{D}^n$=set of $n$ times Differentiable functions*
*$\mathcal{C}^n$=set of $n$ times Continuously Differentiable functions*
*$\mathcal{R}$=set of Riemann Integrable functions*
*$\mathcal{P}$=set of all possible Partitions of an interval*
*$\mathcal{LC}$=set of Lipchitz Continuous functions*
*$\mathcal{UC}$=set of Uniformly Continuous functions*

Color Index

For additional knowledge, these topics will NOT be tested at the exam.

## 1 Preliminaries

**Axiom 1.** *Completeness Axiom*
*If $A$ is a bounded non-empty subset of real numbers $\mathbb{R}$ then $\sup A$ and $\inf A$ exists.*

**Theorem 1.** *Intermediate Value Theorem*
*Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Let $c$ be between $f(a)$ and $f(b)$. Then there exists $x \in (a, b)$ such that $f(x) = c$.*

**Theorem 2.** *Extreme Value Theorem*
*Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then $f$ attains its maximum and minimum on $[a, b]$ i.e. there exists $c, d \in [a, b]$ such that $f(c) = \max\{f(x)|x \in [a, b]\}$ and $f(d) = \min\{f(x)|x \in [a, b]\}$.*

**Theorem 3.** *Mean Value Theorem*
*Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$. Then there exists $\zeta \in (a, b)$ such that $\frac{f(b)-f(a)}{b-a} = f'(\zeta)$*

**Theorem 4.** *Generalized Mean Value Theorem*
*Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$ and $g' \neq 0$ on $(a, b)$. Then there exists $\zeta \in (a, b)$ such that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\zeta)}{g'(\zeta)}$*

**Theorem 5.** *L'Hopital Rule*
*Let $f, g \in \mathcal{D}$, $f(a) = g(a) = 0$ and $g' \neq 0$ except possibly at $a$.*
*If $\lim_{x\to a} \frac{f(x)}{g(x)} = L$ then $\frac{f'(x)}{g'(x)} = L$.*

**Theorem 6.** *Ratio Test for absolute convergence*
*Let $u_k \neq 0$ and $\lim_{k\to\infty} \left|\frac{u_{k+1}}{u_k}\right| = L$*
*1. If $L < 1$ then $\sum_{k=1}^{\infty} u_k$ converges absolutely*
*2. If $L > 1$ then $\sum_{k=1}^{\infty} u_k$ diverges*

**Theorem 7.** *Root Test for absolute convergence*
*Let $u_k \neq 0$ and $\lim_{k\to\infty} |u_k|^{1/k} = L$*

*1. If $L < 1$ then $\sum_{k=1}^{\infty} u_k$ converges absolutely*
*2. If $L > 1$ then $\sum_{k=1}^{\infty} u_k$ diverges*

**Theorem 8.** *Bounded Convergence theorems*
*If $f$ is bounded above and increasing then $\lim_{x\to\infty} f(x)$ exist finitely and equal to $\sup\{f(x)\}$.*
*If $f$ is increasing and $\lim_{x\to\infty} f(x)$ exists finitely then $\sup\{f(x)\}$ exists and equal to it*

**Note 2.**
*1. It is convenient to assume $f \in \mathcal{D}$ in a larger open interval containing $(a, b)$ for Mean Value Theorems.*
*2. L'Hopital rule applies in all other cases of limits of $x$ and even when $f(x), g(x) \to 0$ or $\infty$ at the limit with conditions suitably modified.*
*3. Similar Bounded Convergence theorems exists for functions which are bounded below and decreasing.*

## 2 Riemann Integral

Proof of theorems on this section will NOT be tested at the exam.

**Definition 1.** *Riemann Integral*
*Let $f$ be bounded on $[a, b]$. i.e. $f \in \mathcal{B}[a, b]$*
*Let $P = \{x_0, x_1, \cdots, x_n\}$ with $x_0 = a, x_n = b$ and $\Delta x_k = x_k - x_{k-1} > 0$ for $1 \leq k \leq n$ be a partition of $[a, b]$, i.e. $P \in \mathcal{P}[a, b]$.*
*Let $M_k = M_k(P, f) = \sup\{f(x)|x \in [x_{k-1}, x_k]\}$ and*
*$m_k = m_k(P, f) = \inf\{f(x)|x \in [x_{k-1}, x_k]\}$*
*Define Upper Riemann sum $U(P, f) = \sum_{k=1}^{n} M_k \Delta x_k$ and*
*Lower Riemann Sum $L(P, f) = \sum_{k=1}^{n} m_k \Delta x_k$*
*Define Upper Riemann Integral $U(f) = \inf\{U(P, f)|P \in \mathcal{P}[a, b]\}$ and*
*Lower Riemann Integral $L(f) = \sup\{L(P, f)|P \in \mathcal{P}[a, b]\}$.*
*Iff $U(f) = L(f)$ we say that $f$ is Riemann Integrable on $[a, b]$. i.e. $f \in \mathcal{R}[a, b]$ and write $\int_a^b f(x)dx$ for the common value and call it the Riemann integral of $f$ on $[a, b]$.*

**Note 3.** *It is clear that $m_k \leq M_k$ therefore $L(P, f) \leq U(P, f)$*
*We will later show that even $L(f) \leq U(f)$ is true.*

**Example 1.**
*1. Consider the function $f(x) = 1$ if $x \in \mathbb{Q}$ and $2$ if $x \in \mathbb{R} - \mathbb{Q}$. Is $f \in \mathcal{R}[a, b]$?*
*2. Consider equispaced partitions of $[a, b]$ for the function $f(x) = e^x$. Show that $U(f) = L(f) = e^b - e^a$ without direct integration.*

**Theorem 9.** *Let $P, P^* \in \mathcal{P}[a, b]$.*
*We say that $P^*$ is a refinement of $P$ iff $P \subset P^*$*
*We have $L(P^*, f) \geq L(P, f)$ and $U(P^*, f) \leq U(P, f)$*
*Also $L(f) \leq U(f)$ for any $P \in \mathcal{P}[a, b]$*

**Theorem 10.** *Riemann condition for Riemann Integrability*
$f \in \mathcal{R}[a,b]$ *iff* $\forall \epsilon > 0, \exists P \in \mathcal{P}[a,b]; U(P,f) - L(P,f) < \epsilon$

**Theorem 11.** *If* $f, g \in \mathcal{R}[a,b]$ *then*
1. $f + g \in \mathcal{R}[a,b]$ *and* $\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
2. $fg \in \mathcal{R}[a,b]$
3. *If* $f \le g$ *on* $[a,b]$ *then* $\int_a^b f(x)dx \le \int_a^b g(x)dx$
4. $|f| \in \mathcal{R}[a,b]$ *and* $|\int_a^b f(x)dx| \le \int_a^b |f(x)|dx$
5. *If* $c \in (a,b)$ *then* $f \in \mathcal{R}[a,c], f \in \mathcal{R}[c,b]$ *and* $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

**Definition 2.** *Motivated by the above, we define*
$\int_a^a f(x)dx = 0$ *and* $\int_b^a f(x)dx = -\int_a^b f(x)dx$ *for* $b > a$.

**Theorem 12.** $f \in \mathcal{R}[a,b] \Rightarrow \forall \epsilon > 0, \exists P \in \mathcal{P}[a,b], \forall t_k \in [x_{k-1}, x_k]; |\sum_{k=1}^n f(t_k)\Delta x_k - \int_a^b f(x)dx| < \epsilon$

**Note 4.**
1. *The converse of the above theorem is also true, so*
$f \in \mathcal{R}[a,b]$ *and* $I = \int_a^b f(x)dx$ *iff*
$\forall \epsilon > 0, \exists P \in \mathcal{P}[a,b], \forall t_k \in [x_{k-1}, x_k]; |\sum_{k=1}^n f(t_k)\Delta x_k - I| < \epsilon$
*So one can use the above as the definition of the Riemann integral.*
2. *Also one can show that it is sufficient to use the equispaced partitins, i.e.*
*Let* $\overline{\mathcal{P}}[a,b]$ *be equispaced partitions(i.e.* $\Delta x_k = (b-a)/n$*). Then* $f \in \mathcal{R}[a,b]$ *iff*
$\forall \epsilon > 0, \exists \overline{P} \in \overline{\mathcal{P}}[a,b]; U(\overline{P}, f) - L(\overline{P}, f) < \epsilon$
3. *These lead to the following simple definition*
$\overline{P} \in \overline{\mathcal{P}}[a,b]$ *and* $\Delta x_k = (b-a)/n$. *Then*
$f \in \mathcal{R}[a,b]$ *and* $I = \int_a^b f(x)dx$ *iff* $\forall t_k \in [x_{k-1}, x_k]; \lim_{n\to\infty} \sum_{k=1}^n f(t_k)\Delta x_k = I$

**Theorem 13.** *Fundamental Theorem of Calculus*
*If* $f \in \mathcal{R}[a,b]$ *and there exists* $F \in \mathcal{D}[a,b]$ *such that* $f = F'$
*then* $\int_a^b f(x)dx = F(b) - F(a)$.

**Definition 3.** *Strong forms of Continuity*
1. $f$ *is uniformly continuous on* $A$ *i.e.* $f \in \mathcal{UC}(A)$
*iff* $\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in A; |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$
2. $f$ *is Lipchitz continuous on* $A$ *i.e.* $f \in \mathcal{LC}(A)$
*iff* $\exists L > 0, \forall x, y \in A; |f(x) - f(y)| \le L|x - y|$

**Theorem 14.** $f \in \mathcal{LC}(A) \Rightarrow f \in \mathcal{UC}(A) \Rightarrow f \in \mathcal{C}(A)$

**Theorem 15.** $f \in \mathcal{C}[a,b] \Rightarrow f \in \mathcal{UC}[a,b] \Rightarrow f \in \mathcal{R}[a,b]$

**Example 2.** *Show that* $\frac{1}{x}$ *is not uniformly continuous on* $(0,1]$ *but* $x^2$ *is.*

**Theorem 16.** *Second Fundamental Theorem of Calculus*
*Let* $f \in \mathcal{R}[a,b]$, $x \in [a,b]$ *and* $F(x) = \int_a^x f(t)dt$.
*If* $f \in \mathcal{R}[a,b]$ *then* $F \in \mathcal{C}[a,b]$
*If* $s \in (a,b)$ *and* $f \in \mathcal{C}(s)$ *then* $F \in \mathcal{D}(s)$ *and* $F'(s) = f(s)$

**Theorem 17.** *Integration by Parts*
*If $F, G$ differentiable on $[a, b]$, $F' = f \in \mathcal{R}[a, b]$ and $G' = g \in \mathcal{R}[a, b]$. Then*
$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx$

**Theorem 18.** *Change of Variable*
*$g$ has continuous derivative $g'$ on $[c, d]$. $f$ is continuous on $g([c, d])$ and let $F(x) = \int_{g(c)}^x f(t)dt, x \in g([c, d])$. Then for each $x \in [c, d], \int_c^x f(g(t))g'(t)dt$ exists and has value $F(g(x))$.*

**Theorem 19.** *Leibniz Rule*
*Let $f$ be continuous and the functions $a(x)$ and $b(x)$ be differentiable. Then*
$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)dt = f(b(x))b'(x) - f(a(x))a'(x)$

**Theorem 20.** *Mean Value Theorem for Integrals*
*$f \in \mathcal{C}[a, b]$. Then there exists $\zeta \in (a, b)$ such that $\int_a^b f(x)dx = f(\zeta)(b - a)$.*

**Theorem 21.** *Generalized Mean Value Theorem for Integrals*
*$f \in \mathcal{C}[a, b]$, $g \in \mathcal{R}[a, b]$ and $g$ does not change sign on $[a, b]$. Then there exists $\zeta \in (a, b)$ such that $\int_a^b f(x)g(x)dx = f(\zeta)\int_a^b g(x)dx$.*

## 3 Improper Riemann Integral

**Definition 4.** *Improper Integrals of the First Kind*
*Suppose $\int_a^b f(x)dx$ exists for each $b \geq a$*
*Iff $\lim_{b\to\infty}\int_a^b f(x)dx$ exists and equal to $I \in \mathbb{R}$ we say that $\int_a^\infty f(x)dx$ converges to the value $I$ and diverges otherwise.*

**Theorem 22.** *Direct Comparison Test for Integrals*
*Let $f, g \in \mathcal{R}[a, b]$ for all $b > a$ and $0 \leq f(x) \leq g(x)$ for all $x > a$.*
*If $\int_a^\infty g(x)dx$ converges, then $\int_a^\infty f(x)dx$ converges.*

**Theorem 23.** *Limit Comparison Test for Integrals*
*Let $f, g \in \mathcal{R}[a, b]$ for all $b > a$ and $f(x), g(x) > 0$ for all $x > a$.*
*If $\lim_{x\to\infty}\frac{f(x)}{g(x)} = c$, then if*
*1. $c \in (0, \infty)$ then $\int_a^\infty f(x)dx$ conv.$\Leftrightarrow \int_a^\infty g(x)dx$ conv.*
*2. $c = 0$ and $\int_a^\infty g(x)dx$ conv.$\Rightarrow \int_a^\infty f(x)dx$ conv.*
*3. $c = \infty$ and $\int_a^\infty g(x)dx$ div.$\Rightarrow \int_a^\infty f(x)dx$ div.*

**Definition 5.** *Other types of Improper Integrals*
*1. $f : [a, \infty) \to \mathbb{R} : \int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$*
*2. $f : (-\infty, b] \to \mathbb{R} : \int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx$*
*3. $f : \mathbb{R} \to \mathbb{R} : \int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx, c \in \mathbb{R}$*
*4. $f : (a, b] \to \mathbb{R} : \int_{a^+}^b f(x)dx = \lim_{t\to a^+}\int_t^b f(x)dx$*
*5. $f : [a, b) \to \mathbb{R} : \int_a^{b^-} f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx$*
*6. $f : [a, c) \cup (c, b] \to \mathbb{R} : \int_a^b f(x)dx = \int_a^{c^-} f(x)dx + \int_{c^+}^b f(x)dx$*

**Note 5.**
*1. If types 3 and 6 lead to $\infty - \infty$ we try the Cauchy Principal Value given by $\lim_{b\to\infty} \int_{-b}^{b} f(x)dx$ for case 3 and $\lim_{\delta\to 0}\left(\int_a^{c-\delta} f(x)dx + \int_{c+\delta}^{b} f(x)dx\right)$ for case 6.*
*2. Similar types of convergence tests exists for above types of improper integrals.*

**Example 3.**
*1. Find $\int_{-1}^{1} \frac{1}{x^2}dx$ if it exists.*
*2. Prove that $\int_a^{\infty} |f(x)|dx$ conv. $\Rightarrow \int_a^{\infty} f(x)dx$ conv.*
*3. Prove that if $|f(x)| \le Me^{ax}$, then the Laplace Transform of $f(x), F(s) = \int_0^{\infty} f(x)e^{-sx}dx$ exists for all $s > a$.*

**Example 4.** *Gamma function is defined by $\Gamma(x) = \int_0^{\infty} e^{-t}t^{x-1}dt$. Show that*
*1. $F(x)$ exists iff $x > 0$.*
*2. $\Gamma(x) = (x-1)\Gamma(x-1), x > 1$.*
*3. $\Gamma(n) = (n-1)!$ for all integer $n \ge 1$.*
*4. $\lim_{x\to\infty} \Gamma(x) = \infty = \lim_{x\to 0^+} \Gamma(x)$*
*5. we can use (2) to define $\Gamma(x)$ for $x < 0$.*
*6. $\Gamma(x)$ does not exist for $x = 0, -1, -2, -3, \cdots$*
*7. Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\int_0^{\infty} e^{-t^2}dt = \sqrt{\pi}$ using $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$*
*8. Use the formula for the $n$ dimensional ball $V_n(r) = \frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})}$ to find the volumes of $2, 3, 4, 5$ dimensional balls.*
*9. Show that $\Gamma(x)$ is continuous on $(0, \infty)$.*
*10. Prove that the Beta Function $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$ exists iff $x, y > 0$. It can be shown that $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.*

**Example 5.**
*1. Test the convergence of $\int_0^{\infty} \frac{\sin x}{x}dx$*
*2. Fresnel Integrals are defined by $S(x) = \int_0^x \sin(t^2)dt$ and $C(x) = \int_0^x \cos(t^2)dt$. Test the convergence of $S(\infty)$ and $C(\infty)$.*
*3. The Logarithmic Integral is defined by $li(x) = \int_0^x \frac{dt}{\log t}$. Check the convergence of each of the following: $\int_0^{1/2} \frac{dt}{\log t}, \int_{1/2}^{1} \frac{dt}{\log t}, \int_1^2 \frac{dt}{\log t}, \int_2^{\infty} \frac{dt}{\log t}$*

**Example 6.** *The exponential integral is defined by $Ei(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t}dt$. We are interested in the related integral $F(x) = \int_1^x \frac{e^{-t}}{t}dt$ which is equal to $Ei(-x) - Ei(-1)$.*
*1. Show that $F(\infty)$ is a converging improper Riemann integral.*
*2. Show that $F(0)$ is a diverging improper Riemann integral.*
*3. Write the 2nd degree ($n = 2$) Taylor polynomial and the integral form of the remainder of the Taylor series of $F(x)$ at $a = 1$.*
*4. What can be the radius of convergence of the Taylor series of $F(x)$ at $a = 1$? Direct proof is not needed, use the previous.*

## 4 Taylor Series with Remainder

**Theorem 24.** *Taylor series of $f \in \mathcal{D}^{n+1}$ at $a$.*
*$f(x) = T_n(x, a) + R_n(x, a)$*

*Taylor Polynomial* $T_n(x,a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$
*Lagrange Remainder* $R_n(x,a) = \frac{f^{(n+1)}(\zeta)}{(n+1)!}(x-a)^{n+1}$
*where* $\zeta$ *between* $x$ *and* $a$

*Proof.* Use Generalized Mean Value Theorem on $F(t) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(t)(x-t)^k$ and $G(t) = (x-t)^{n+1}$. □

**Example 7.** *Let* $f(x) = \ln(1+x)$. *Show that*
*1.* $T_n(x,0) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} x^k$
*2. Find the range of convergence of* $x$
*3. Show that* $R_n(x,0) \to 0$ *as* $n \to \infty$ *for* $-\frac{1}{2} < x < 1$
*4. Find the value of* $\ln(1.5)$ *accurate to* $0.000001$

**Theorem 25.** *Integral form of the Remainder.* $f \in \mathcal{C}^{n+1}$
$R_n(x,a) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt$

**Example 8.**
*1. Show that* $R_n(x,0) \to 0$ *as* $n \to \infty$ *for* $x < 0$
*2.Find the value of* $\ln(0.2)$ *accurate to* $0.000001$

**Theorem 26.** *Other forms of Remainders.* $f \in \mathcal{C}^{n+1}$
$R_n(x,a) = \frac{f^{(n+1)}(\zeta)}{n!(p+1)}(x-\zeta)^{n-p}(x-a)^{p+1}, 0 \le p \le n$
$p = n$, *Lagrange Remainder* $R_n(x,a) = \frac{f^{(n+1)}(\zeta)}{(n+1)!}(x-a)^{n+1}$
$p = 0$, *Cauchy Remainder* $R_n(x,a) = \frac{f^{(n+1)}(\zeta)}{n!}(x-\zeta)^n(x-a)$
*where* $\zeta$ *between* $x$ *and* $a$

**Theorem 27.** *When* $a$ *is fixed,* $\zeta$ *in the error term is a function of* $x$, *and can be written as* $\zeta(x)$. *If* $f \in \mathcal{C}^{n+1}$ *then* $f^{(n+1)}(\zeta(x)) \in \mathcal{C}$

*Proof.* We have $f^{(n+1)}(\zeta(x)) = (n+1)!\frac{f(x)-T_n(x,a)}{(x-a)^{n+1}}$, so clearly $f^{(n+1)}(\zeta(x))$ is continuous for $x \neq a$. For $x = a$, this leads to the $0/0$ situation. But either $a < \zeta(x) < x$ or $x < \zeta(x) < a$ is true and since $\lim_{x\to a} x = a$, we have by the Sandwitch theorem,$\lim_{x\to a}\zeta(x) = a$. Now If $f \in \mathcal{C}^{n+1}$ and if we define $\zeta(a) = a$, we have $\lim_{x\to a} f^{(n+1)}(\zeta(x)) = f^{(n+1)}(\lim_{x\to a}\zeta(x)) = f^{(n+1)}(a) = f^{(n+1)}(\zeta(a))$, so $\zeta(x)$ is continuous at $a$ too. Note that this means $\zeta(x)$ is continuous at □

**Example 9.**
*1. Find the value of* $\ln(0.02)$ *accurate to* $10^{-6}$ *using the Taylor series expansion at* $0$ *for* $x < 0$.
*2. Find the values of* $\ln 2, \ln 5, \ln 500, \ln(0.2), \ln(0.02)$ *accurate to* $10^{-6}$ *using the Taylor series expansion at* $0$ *for* $x > 0$.
*3. Find the Taylor series expansions for* $e^x, \sin x, \tan^{-1} x, \tan x$ *at* $0$ *with remainder.*
*4. Find the range of convergence.*
*5. Show that the remainder* $R_n(x,a) \to 0$ *as* $n \to \infty$ *within the range of convergence.*
*6. Find the values of* $e, \sin 1, \tan^{-1} 1$ *accurate to the 10th decimal place.*

*7. Find the values of $e^4, \sin 4, \tan^{-1} 4$ accurate to the 6th decimal place using a suitable Taylor series expansion.*
*8. Deduce the values of 7. from 6. whenever it is possible.*
*9. Show that e is irrational.*

**Definition 6.** *Power Series*
*An infinite series of the form $\sum_{k=0}^{\infty} u_k(x)$ is*
*1. Converges point-wise to $S(x)$ iff*
$\forall\epsilon > 0 \forall x \exists N > 0 \forall n; n > N \Rightarrow |\sum_{k=0}^{n} u_k(x) - S(x)| < \epsilon$
*2. Converges Uniformly to $S(x)$ iff*
$\forall\epsilon > 0 \exists N > 0 \forall x \forall n; n > N \Rightarrow |\sum_{k=0}^{n} u_k(x) - S(x)| < \epsilon$

**Definition 7.** *A Power Series at a*
*is an infinite series of the form $S(x) = \sum_{k=0}^{\infty} a_k(x-a)^k$.*

**Theorem 28.**
*1. There exists $R \in [0, \infty]$ called the radius of convergence such that a power series converges absolutely and uniformly for $|x-a| < R$ and diverges for $|x-a| > R$.*
*2. The power series may converge conditionally or diverge for $|x-a| = R$*
*3. Radius of convergence $R = \frac{1}{\lim_{k\to\infty} |a_k|^{1/k}}$*
*4. Since the power series is uniformly converges for $x \in (a-R, a+R)$ the series may be differentiated term-by-term giving $S'(x) = \sum_{k=1}^{\infty} k a_k (x-a)^{k-1}$ with the same radius of convergence.*
*5. Since the power series is uniformly converges for $x \in (a-R, a+R)$ the series may be integrated term-by-term giving $\int_a^x S(t)dt = \sum_{k=0}^{\infty} \frac{a_k}{k+1}(x-a)^{k+1}$ with the same radius of convergence.*

**Example 10.** *Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$*
*1. Write down the length of the perimeter $C(a, b)$ as a definite integral.*
*2. Convert the above integral into the Incomplete Elliptic Integral of the Second Kind $E(\phi, k) = \int_0^{\phi} \sqrt{1 - k^2 \sin^2 x}\, dx$.*
*3. What is $C(a, a)$?*
*4. Use Taylor Series to calculate $C(a, 2a)/C(a, a)$ accurate to 0.001.*

**Example 11.** *Consider a pendulum of mass $m$ and length $\ell$ oscillating at an angle $2\alpha$ in a gravitational field of strength $g$.*
*1. Write the time period $T(\alpha)$ as a definite integral.*
*2.Convert the above integral into the Incomplete Elliptic Integral of the First Kind $F(\phi, k) = \int_0^{\phi} \frac{1}{\sqrt{1-k^2\sin^2 x}} dx$.*
*3. What is $T(0)$ or the limit?*
*4. Use Taylor Series to calculate $T\left(\frac{\pi}{2}\right)/T(0)$ accurate to 0.001.*

**Note 6.** *We may write the Taylor expansion as*
$f(a+h) = \sum_{k=0}^{n} \frac{1}{k!} D^k f(a) + \frac{1}{(n+1)!} D^{n+1} f(a + \theta h)$
*where $D = h\frac{d}{dx}, D^k = D(D^{k-1}), D^0 = 1$ and $\theta \in (0, 1)$.*

**Theorem 29.** *Second Derivative Test*
$f \in \mathcal{C}^1, f'(a) = 0.$
*1. If $f''(a) > 0$ then $a$ is a local minimum of $f$.*
*2. If $f''(a) < 0$ then $a$ is a local maximum of $f$.*

*Proof.* Use the second order Taylor series $f(a+h) = f(a) + f'(a)h + \frac{1}{2}f''(a+\theta h)$. Note that the first two terms are the Tangent Line at $a$. Since $f'(a) = 0$ we have $f(a+h) - f(a) = \frac{1}{2}f''(a+\theta h)$. Since $f'' \in \mathcal{C}$ and $f''(a) > 0$, we can select $h$ such that $f''(a+\theta h) > 0$ for all $\theta \in (0,1)$. ☐

**Casio 1.** *CASIO fx-991ES*
*Formula:*$(ALPHAX + 1)2x^{\square}ALPHAX - 10x^{\square}6CALCX?10 =$
*Sum: SHIFT* $\sum_{\square}^{\square}\square$*:* $\sum((-1)x^{\square}(ALPHAX-1)(1\div 2)x^{\square}ALPHAX \div ALPHAX, 1, 10)$

**Mathematica 1.**
*Formula:* $f[n\_] := (1+n)2^n - 10^6; Table[\{n, f[n]\}, \{n, 1, 50\}]$
*Sum:* $Sum[(-1)\Lambda(k-1)(1/2)\Lambda k/k, \{k, 1, 10\}]$
*Taylor Series:* $Series[Log[1+x], \{x, 0, 20\}]$

## 5 Numerical Integration

**Theorem 30.** *Trapezoidal Rule*
$f \in \mathcal{C}^2[a,b], h = \frac{b-a}{n}, x_0 = a, x_n = b, x_k = x_0 + kh, 0 \le k \le n, \zeta \in (a,b)$
$\int_a^b f(x)dx = \frac{h}{2}\left[f(x_0) + 2\sum_{k=1}^{n-1} f(x_k) + f(x_n)\right] - \frac{(b-a)^3}{12n^2}f''(\zeta)$

**Theorem 31.** *Simpson's Rule(see Proofs.pdf for the proof for the error)*
$f \in \mathcal{C}^4[a,b], n$ *is even,* $h = \frac{b-a}{n}, x_0 = a, x_n = b, x_k = x_0 + kh, 0 \le k \le n, \zeta \in (a,b)$
$\int_a^b f(x)dx = \frac{h}{3}\left[f(x_0) + 4\sum_{\substack{k=1 \\ kodd}}^{n-1} f(x_k) + 2\sum_{\substack{k=2 \\ keven}}^{n-2} f(x_k) + f(x_n)\right] - \frac{(b-a)^5}{180n^4}f^{(4)}(\zeta)$

**Example 12.**
*1. For each of the following integrals, use the Trapezoidal and Simpson's rules to find the number of divisions needed to find its value accurate to 0.001 and find the integral to that accuracy.*
$\int_0^1 \sin(x^2)dx$
$\int_0^2 \cos(x^2)dx$
$\int_0^1 e^{-x^2}dx$
$\int_0^{\frac{\pi}{2}} \sqrt{2-\cos^2 x}dx$
$\int_2^{10} \frac{x}{\log x}dx$
*2. Derive a numerical integration rule and its error that uses the function value at the mid point (Mid Point Rule), left end point, right end point of each interval.*
*3. Use Mid Point Rule rule to do the integrals in Q1.*
*4. Show directly that cubic polynomials are integrated exactly(error is 0) by the Simpson's rule.*
*5. Use Taylor series to derive an approximate formula for the remainder in Trapezoidal rule.*
*6. Use integration by parts to prove the error formula for the Simpson's rule.*

**Example 13.**
*One method of doing numerical integration is Gaussian Quadrature. Note that both the Trapezoidal and the Simpsons rules looks like $\int_a^b f(x)dx \approx \sum_k w_k f(x_k)$ and we knew $x_k$ and found $w_k$. In this method we find both $x_k$ and $w_k$ so that the integral and the sum are equal for a given $n$ degree polynomial $p(x)$. It is achieved by forcing both sides equal for each power of $x^j$ for $j = 0, 1, 2, , n$. What is the degree of the polynomial we need to use if we want 3 points and the corresponding 3 weights? Find them for $[a, b] = [-1, 1]$ and use it to approximate integrals given above.*
*Casio fx-991ES uses a variant of this method.*

**Example 14.** *We want to approximate the value of $F(2) = \int_1^2 \frac{e^{-t}}{t} dt$ numerically accurate to $0.001$.*
*1. Consider the method of term-by-term integration after using the Taylor series of $e^{-t}$ at $a = 0$. What is the degree of the Taylor polynomial $(n)$ that we have to use? Note that $f^{(n+1)}(\zeta)$ in the error term is a continuous function of $t$.*
*2. Find the value of $F(2)$ accurate to $0.001$ by this Taylor series method. Use the sum function in the calculator and confirm your answer by integrating function in the calculator.*
*3. Consider method of the Trapezoidal rule. How many intervals $(n)$ needed?*
*4. Find the value of $F(2)$ accurate to $0.001$ by this Trapezoidal method. Use the sum function in the calculator and confirm your answer by integrating function in the calculator.*

**Casio 2.** *$\int(f(X), a, b, m)$ and the default variable is $X$ and $n = 2^m$ for the Simpson's method in model fx-991MS*

**Mathematica 2.** $NIntegrate[f(x), \{x, a, b\}, Method- > TrapezoidalRule]$

## 6 Interpolation

**Theorem 32.** *Lagrange Method of finding the Interpolating Polynomial $p(x)$ of $f(x)$ for the points $x_k, 0 \le k \le n$*
$w_j(x) = \prod_{\substack{i=0 \\ i\neq j}}^n (x - x_i)$
$w(x) = \prod_{i=0}^n (x - x_i)$
$\ell_j(x) = \frac{w_j(x)}{w_j(x_j)} = \prod_{\substack{i=0 \\ i\neq j}}^n \left(\frac{x-x_i}{x_j-x_i}\right) = \frac{w(x)}{(x-x_j)w'(x_j)}$ *and* $\ell_j(x_k) = 0$ *if* $j \neq k$ *and 1 if* $j = k$.
$p(x) = \sum_{k=0}^n f(x_k)\ell_k(x)$ *and* $p(x_j) = f(x_j), 0 \le j \le n$
$f(x) = p(x) + \frac{f^{(n+1)}(\zeta)}{(n+1)!} w(x)$ *with* $\zeta \in (x_0, x_n)$ *when* $f \in \mathcal{C}^{(n+1)}$

**Example 15.** *Consider the data set $A = \{(2, 1), (3, 2), (4, 3), (6, 4)\}$*
*1. Show that that the data set may be generated by the function $f(x) = 4\sin^2(\frac{\pi x}{12})$. Find an upper bound for the error.*
*2. For the same function on $[0, 6]$, find the number of points required to make the error $\le 0.001$ and find the Interpolating Polynomial.*
*3. Use the error formula for the interpolating polynomial to derive the error formula for the Trapezoidal Method. Can you do the same with the Simpsons Method?*

**Example 16.** *Consider the data set* $A = \{(2,1),(3,2),(4,3),(6,4)\}$.
*1. Find the Interpolating polynomial by direct matrix inversion.*
*2. One way of finding the Lagrange polynomial is to define it as the iterative process* $p(x) = p_0(x)(x - x_0) + q_0$ *and* $p_0(x) = p_1(x)(x - x_1) + q_1$ *and so on. See why this method is working and find the Interpolating Polynomial for* $A$. *Mathematica seems to use this method.*
*3. Another method of finding the Interpolating Polynomial is to use the Newtons divided differences. For* $x_0, x_1, x_2$ *we define* $f[x_0, x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0}$ *and* $f[x_0, x_1, x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$ *and so on and the interpolating polynomial is given by* $p(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$. *Use the method in (2) to see why the formula is working and use it to find the interpolating polynomial for* $A$.
*5. Find a polynomial (Hermite polynomial) that goes though the above points of* $A$ *and satisfying* $p'(1) = 0, p'(2) = 1, p'(3) = 0$.
*6. Propose a method to fix a function consisting of a linear combination of* $x, e^x, \sin x, \cos x$ *to the above data.*

**Example 17.**
*1. Get the value of USD/LKR for the 1st of every month for this year 2019. (Write the values).*
*2. Use the Interpolating Polynomial to predict this value on 1st November 2019,1st December 2019 and 1st January 2020. (Write the polynomial and the predicted values. You can use a software, write the code).*
*3. For the data set* $\{(x_k, y_k)\}, k = 0, \ldots, n$ *a natural cubic spline is a twice differentiable piece-wise cubic polynomial* $p(x)$ *which satisfies* $p(x_k) = y_k$ *with* $p''(x_0) = p''(x_n) = 0$. *Let* $p(x) = \sum_{k=1}^{n} p_k(x)$ *where* $p_k(x)$ *is the part of* $p(x)$ *on* $[x_{k-1}, x_k]$ *which is 0 elsewhere. Assume that* $p_k(x) = a_k(x-x_k)^3 + b_k(x-x_k)^2 + c_k(x-x_k) + d_k$ *and that* $\Delta x_k = x_k - x_{k-1} = h$ *is a constant.*
*Derive the formula* $s_{k+1} + 4s_k + s_{k-1} = \frac{6}{h^2}(y_{k+1} - 2y_k + y_{k-1})\,; k = 1, \ldots, n-1$ *where* $s_k = p''(x_k)$.
*Also write the system of equations in matrix form that must be solved to find* $s_k$.
*4. Use Cubic Spline to predict the same values (Write the calculated* $s_k$ *values, the last cubic polynomial* $p_n(x)$ *and the predicted values. You can use a software, write the code).*

**Mathematica 3.** $InterpolatingPolynomial[\{\{2,1\},\{3,2\},\{4,3\},\{6,4\}\},x]$

## 7 Numerical solutions of non-linear equations of one variable

**Algorithm 1.** *Bisection Method*
*1. Find* $a_0, b_0$ *such that* $f(a_0)$ *and* $f(b_0)$ *are of different sine(say* $f(a_0) < 0$ *and* $f(b_0) > 0$*).*
*2.* $k = 0$.
*3.* $x_k = \frac{a_k+b_k}{2}$

*4. If $f(x_k) = 0$ then stop and return $x_k$*
*If $f(x_k) < 0$ then $a_{k+1} = x_k$ and $b_{k+1} = b_k$*
*If $f(x_k) > 0$ then $b_{k+1} = x_k$ and $a_{k+1} = a_k$*
*5. If Stopping Condition(e.g. $|f(x_k)|$ or $|b_k - a_k|$ or $k$ less than a given number) is True then stop and return $x_k$*
*6. $k \leftarrow k + 1$ and goto 3*

**Theorem 33.** *Convergence of the Bisection Method*
*1. $f : [a, b] \to \mathbb{R}$*
*2. $f$ is continuous (ie.$f \in \mathcal{C}$).*
*3. $a_0, b_0 \in [a, b]$ and we select $a_k, b_k, x_k$ for $k \geq 0$ according to the above algorithm.*
*Then*
*1. $\lim_{k\to\infty} x_k = \lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = z \in [a, b]$ is a root of $f$.*
*2. $|x_k - z| \leq \frac{1}{2}|b_k - a_k| = \left(\frac{1}{2}\right)^{k+1} |b_0 - a_0| \leq \left(\frac{1}{2}\right)^{k+1} |b - a|$*

**Casio 3.** *Bisection Method*
*ALPHA X ALPHA = ( ALPHA A + ALPHA B ) ÷ 2 ALPHA : ALPHA X-e$^{\square}$ - ALPHA X CALC*

**Mathematica 4.** *Bisection Method*
*Algorithm*
*f[x_]:=x-E$\hat{(}$-x);a = 0; b = 1; For[k = 0, k <= 19, k++, {x = (a + b)/2, Print[N[k, a, b, x, f[x], Abs[a - b]/2, 10]], If[f[x] == 0, k = 20, If[f[x] > 0, b = x, a = x]]}]*

*Builtin function*
*FindRoot[f[x] == 0, {x, 0}], by iterations starting $x_0 = 0$*

**Theorem 34.** $f : A \to \mathbb{R}$
*$f \in \mathcal{C}^1(A) \Rightarrow f \in \mathcal{LC}(A)$ when $A$ is closed with $L = \max\{|f'(x)| : x \in A\}$ using Mean Value and Extreme Value theorems on $f'$.*

**Definition 8.** *Cauchy Sequence and Completeness.*
*Let $u_n : \mathbb{N} \to A$ be a sequence.*
*1. $u_n$ is converging on $A$ iff $\exists a \in A, \forall \epsilon > 0, \exists N > 0, \forall n > 0; n > N \Rightarrow |u_n - a| < \epsilon$*
*2. $u_n$ is a Cauchy sequence on $A$ iff $\forall \epsilon > 0, \exists N > 0, \forall n, m > 0; m, n > N \Rightarrow |u_m - u_n| < \epsilon$*
*3. $A$ is Complete iff Every Cauchy sequence on $A$ is converging to a point of $A$.*

**Theorem 35.**
*1. All converging sequences are Cauchy.*
*2. $\mathbb{Q}$ is not complete. Take $u_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \to e$ as $n \to \infty$.*
*3. $\mathbb{R}^n$ is complete.*
*4. A closed subset of a complete space is complete.*
*5. A complete space is closed.*

**Definition 9.**
*1. $g$ is a Contraction iff it is Lipchitz continuous with Lipchitz constant $L < 1$.*
*2. $z$ id a Fixed Point of $g$ iff $z = g(z)$*

**Theorem 36.** *Global Convergence of the Fixed Point method(Banach Fixed Point Theorem)*
*1.*$g : [a, b] \to [a, b]$ *i.e.* $g([a, b]) \subset [a, b]$
*2.*$g$ *is a contraction with Lipschitz constant* $L$
*3.*$x_0 \in [a, b]$ *and* $x_{k+1} = g(x_k), k \geq 0$
*Then*
*1.*$\lim_{k\to\infty} x_k = z \in [a, b]$ *is a unique fixed point of* $g$
*2.*$|x_k - z| \leq \frac{L^k}{1-L}|x_1 - x_0| \leq \frac{L^k}{1-L}|b - a|$

**Theorem 37.** *Local convergence of the Fixed Point method*
*Let* $z = g(z)$ *be a fixed point. If* $g \in \mathcal{C}^1$ *with* $|g'(z)| < 1$*, then there exists a neighbourhood of* $z$ *such that the fixed point method is converging.*
*Note that* $z = g(z)$ *and* $g \in \mathcal{C}$ *implies that there is a neighbourhood of* $z$ *that the condition 1. is met.*

**Algorithm 2.** *Fixed Point Method*
*1. Select* $x_0$
*2.* $k = 0.$
*3.* $x_{k+1} = g(x_k)$
*4. If Stopping Condition(e.g.* $|f(x_k)|$ *or or* $k$ *less than a given number) is True then stop and return* $x_k$
*5.* $k \leftarrow k + 1$ *and goto 3.*

**Example 18.** *Consider the equations* $x = e^{-x}$ *and* $x^5 - x - 1 = 0$*. For each case*
*1.Find intervals that contains real roots.*
*2.Find number of iterations needed to find each root to an accuracy of 0.0001 using each of the methods Bisection/Fixed Point*
*3.Do the iterations and find all real roots.*

**Example 19.** *Let* $T_n(x) = \sum_{k=1}^{n} \frac{(-x)^k}{k!}$ *be the n th degree Taylor polynomial of* $e^{-x}$ *at* $x = 0$ *and* $\lim_{x\to\infty} T_n(x) = e^{-x}$*. Solve* $x = T_2(x)$ *and find an approximate solution to* $x = e^{-x}$*. Also find a* $n$ *for which the difference in the solutions to* $x = T_n(x)$ *and* $x = e^{-x}$ *is less than 0.001. Assume that one real solution to* $x = T_n(x)$ *remain in* $[0.5, 0.61]$ *for all* $n \geq 2$*.*

**Example 20.** *Find the global maximums of*
*1.* $w(x)$ *on* $[2, 6]$*.* $w(x) = (x - 2)(x - 3)(x - 4)(x - 6)$
*2.* $f''(x)$ *on* $[0, 1]$*.* $f(x) = e^{-x^2}, \sin(x^2)$
*3.* $f^{(4)}(x)$ *on* $[0, 1]$*.* $f(x) = e^{-x^2}, \sin(x^2)$

**Casio 4.** *Fixed Point Method*
*ALPHA X ALPHA* $= e^{\square}$ *-ALPHA X CALC =*

**Mathematica 5.** *Fixed Point Method*
*Algorithm*
*g[x_]:=E(-x);x = 0; For[k = 0, k <= 19, k++, {x = g[x], Print[N[{k, x, Abs[x - g[x]]}, 10]]}]*
*Builtin function*
*FindRoot[f[x] == 0, {x, 0}], by iterations starting* $x_0 = 0$

**Note 7.** *See the note AllRoots.pdf on finding the complex roots of* $x^5 - x - 1 = 0$ *using the Fixed Point method.*

**Definition 10.** *Newton's method for finding roots of* $f(x) = 0$
$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$

**Note 8.** *We can analyze the Newtons method as a Fixed Point Method with* $g(x) = x - \frac{f(x)}{f'(x)}$ *when* $f'(x) \neq 0$. *Then* $g'(x) = \frac{f(x)f''(x)}{(f'(x))^2}$ *indicates that local convergence is guaranteed.*

**Theorem 38.** *Local convergence of the Newton's Method*
*Let* $z$ *be a root of* $f$. *If* $f \in \mathcal{C}^2$ *and* $f'(z) \neq 0$ *then there exists a neighbourhood of* $z$ *where the Newton's method is converging.*

**Theorem 39.** *Global convergence of the Newton's method(Newton-Kantorovich Theorem, see Proofs.pdf for the proof)*
*1.* $f : [a, b] \to \mathbb{R}$
*2.* $f' \neq 0$ *and there exists* $\beta > 0$ *such that* $\frac{1}{|f'(x)|} \leq \beta$
*3.* $f'$ *is Lipschitz continuous with constant* $\gamma$
*4.* $x_0 \in [a, b]$ *and* $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, k \geq 0$
*5.* $\left|\frac{f(x_0)}{f'(x_0)}\right| = \alpha$
*6.* $q = \alpha\beta\gamma < \frac{1}{2}$
*7.* $[x_0 - 2\alpha, x_0 + 2\alpha] \subset [a, b]$
*Then*
*1.* $\lim_{k\to\infty} x_k = z \in [x_0 - 2\alpha, x_0 + 2\alpha]$ *is a unique root of* $f$
*2.* $|x_k - z| \leq 2\alpha q^{2^k-1}$

**Example 21.** *Consider the Newton's method of finding the real roots of* $x - e^{-x}$ *and* $x^5 - x - 1 = 0$
*1. Treat the method as an fixed point method and find the no of iterations needed to calculate the root to an accuracy of* $10^{-4}$ *and find the root.*
*2. Use the error formula above for the Newton's method and find the no of iterations needed to calculate the root to an accuracy of* $10^{-4}$ *and find the root.*
*3. Use more terms in the Taylor series(instead of 2 terms used in the Newton's method) and propose a possibly faster method to find the root.*
*4. If* $f$ *was not differentiable, propose a method which uses the secant(instead of the tangent) joining two successive points.*
*5. Try to find complex roots using the Newton's method(see Note 3).*

**Example 22.**
*1. Do Example 5 for* $\sin x = 2x$ *and* $x = e^{-x}$
*2. Try to solve* $x^m = 0$ *for* $m \in \mathbb{R}$. *What is going wrong/right?.*
*3. Show that the sequence* $x_{k+1} = \frac{x_k}{2} + \frac{a}{2x_k}$ *converges to* $\sqrt{a}$, *provided we select* $x_0$ *on a suitable range. What is such a range?*
*4. Suppose we want to solve* $\tan^{-1} x = 0$ *by the Newton's method. Find the value* $z$

*such that the Newton's method is converging for* $0 < x_0 < z$*, diverging for* $x_0 > z$ *and enters into a cycle for* $x_0 = z$*.*
*5. Your CASIO calculator can integrate,* $\int_a^b f(x)dx$ *is evaluated as* $\int(f(x), a, b)$*. Find the* $z$ *value for which* $P(x < z) = 0.8$ *when* $X \sim N(0,1)$*, ie when* $X$ *is Normally distributed with mean 0 and standard deviation 1 which is having a PDF* $p(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$*.*
*6. Find the height of the circular sector with arc length* $2x$ *and chord length* $x$*.*
*7. What is the height if the shape is a parabola?*

**Example 23.** *An iterative method of finding solutions to a non-linear equation* $f(x) = 0$ *is said to have a convergence of order* $p$ *iff* $|x_{k+1} - z| \leq r|x_k - z|^p$ *where* $x_k$ *is the* $k$*th iteration,* $z$ *is the solution and* $r$ *is a constant. Show that* $p = 1$ *for the fixed point method and* $p = 2$ *for the Newton?s method.*

**Casio 5.** *Solving cubic* $x^3 + 2x^2 + 3x + 4 = 0$ *with roots* $x_1, x_2, x_3$
*MODE 5:EQN 4:*$ax^3 + bx^2 + cx + d$ $1 = 2 = 3 = 4 == x1 = x2 = x3$

**Mathematica 6.**
$NRoots[x^5 - x - 1 == 0, x]$
$NSolve[f[x] == 0, x]$
$FindRoot[f[x] == 0, \{x, x0\}]$

## 8 Bivariable Real Analysis

**Definition 11.** *Functions of two variables* $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$

**Example 24.** *Draw the graphs of the following functions*
*1.* $f(x,y) = x^2 + y^2$ *2.* $f(x,y) = \sqrt{x^2 + y^2}$ *3.* $\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 1$

**Definition 12.** *Limit*
$\lim_{(x,y)\to(a,b)} f(x,y) = L \Leftrightarrow$
$\forall\epsilon > 0 \exists\delta > 0 \forall(x,y), 0 < d((x,y),(a,b)) < \delta \Rightarrow |f(x,y) - L| < \epsilon$

**Note 9.** *Matric*
$d : \mathbb{R}^2 \to [0,\infty)$ *is a distance measuring function, called a Matric in* $\mathbb{R}^2$*. Some options are*
*1.* $\sqrt{(x-a)^2 + (y-b)^2}$ *2.* $|x-a| + |y-b|$ *3.* $\max\{|x-a|, |y-b|\}$
*One can show all these are matrics.We will use the first matric.*
$0 < d((x,y),(a,b)) < \delta$ *is an open set(without its boundary) around and excluding* $(a,b)$*. Such a set is called a deleted neighbourhood which we write in short as dnbd.*

**Example 25.** *Use the definition to show that* $\lim_{(x,y)\to(2,3)} xy = 6$

**Theorem 40.**
*If* $\lim_{(x,y)\to(a,b)} f(x,y) = L$*,* $\lim_{x\to a} f(x,y) = g(y)$ *in a dnbd of* $b$ *and* $\lim_{y\to b} f(x,y) = h(x)$ *in a dnbd of* $a$*, Then* $\lim_{x\to a} h(x) = \lim_{y\to b} g(y) = L$*.*

*Proof.*
Let $\epsilon > 0$, then $\lim_{(x,y)\to(a,b)} f(x,y) = L$ implies $\exists\delta > 0, \forall(x,y);$

$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x, y) - L| < \epsilon/2$.
$\lim_{x \to a} f(x, y) = g(y)$ in a dnbd of $b$ implies $\exists \delta_1$, whenever $0 < |y - b| < \delta_1$, $\exists \delta_2(y) > 0 \forall x$ such that $0 < |x - a| < \delta_2(y) \Rightarrow |f(x, y) - g(y)| < \epsilon/2$
Let $\delta_3(y) = \min\{\delta_2(y), \delta/\sqrt{2}\} > 0$ and $\delta_4 = \min\{\delta_1, \delta/\sqrt{2}\} > 0$

Now when $0 < |y - b| < \delta_4$ select $x$ such that $0 < |x - a| < \delta_3(y)$. Then $0 < |x - a| < \delta_2(y)$ so $|f(x, y) - g(y)| < \epsilon/2$.
We also have $0 < |x-a| < \delta/\sqrt{2}$ and $0 < |y-b| < \delta/\sqrt{2}$ so $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ which implies $|f(x, y) - L| < \epsilon/2$.
Now $|g(y) - L| \leq |f(x, y) - g(y)| + |f(x, y) - L| < \epsilon$ which means $L = \lim_{y \to b} g(y) = \lim_{y \to b} \lim_{x \to a} f(x, y)$
In the same way we can prove $L = \lim_{x \to a} h(x) = \lim_{x \to a} \lim_{y \to b} f(x, y)$ □

**Note 10.**
*When the one variable limits exist, but the iterated limits are different, this theorem can be used to show that the double limit does not exist in* $\mathbb{R}$.
*For example let* $f(x, y) = \frac{x-y}{x+y}$ *for* $(x, y) \neq (0, 0)$. *Then* $\lim_{x \to 0} f(x, y) = -1 = g(y)$ *exists in a dnbd of* $0$ *and* $\lim_{y \to 0} g(y) = -1$. *Also* $\lim_{y \to 0} f(x, y) = 1 = h(x)$ *exists in a dnbd of* $0$ *and* $\lim_{y \to 0} h(x) = 1$. *Therefore* $\lim_{(x,y) \to (a,b)} f(x, y)$ *does not exist in* $\mathbb{R}$.

**Theorem 41.**
*If* $\lim_{(x,y) \to (a,b)} f(x, y) = L$ *and* $\lim_{x \to a} g(x) = b$ *then* $\lim_{x \to a} f(x, g(x)) = L$

*Proof.* Let $\epsilon > 0$, then $\lim_{(x,y) \to (a,b)} f(x, y) = L$ implies $\exists \delta > 0, \forall (x, y)$;
$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x, y) - L| < \epsilon$.
$\lim_{x \to a} g(x) = b$ implies $\exists \delta_1 > 0 \forall x, 0 < |x - a| < \delta_1 \Rightarrow |g(x) - b| < \delta/\sqrt{2}$
Let $\delta_3 = \min\{\delta_1, \delta/\sqrt{2}\} > 0$.
Now if $0 < |x-a| < \delta_3$ we have $0 < |x-a| < \delta/\sqrt{2}$ and $|y-b| < \delta/\sqrt{2}$ where $y = g(x)$.
Therefore we have $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ which implies $|f(x, g(x)) - L| < \epsilon$.
This means $\lim_{x \to a} f(x, g(x)) = L$. □

**Note 11.**
*If* $g$ *is continuous such that* $g(a) = b$ *we get* $\lim_{x \to a} g(x) = g(a) = b$ *which is a required condition.*
*If we can find two different functions* $g, h$ *such that* $\lim_{x \to a} g(x) = b = \lim_{x \to a} h(x)$ *but if* $\lim_{x \to a} f(x, g(x)) \neq \lim_{x \to a} f(x, h(x))$ *this means the double limit does not exist in* $\mathbb{R}$.
*For example let* $f(x, y) = \frac{xy}{x^2+y^2}$ *for* $(x, y) \neq (0, 0)$. *Let* $g(x) = mx$ *which gives* $\lim_{x \to a} g(x) = 0$. *But* $\lim_{x \to 0} f(x, g(x)) = \frac{m}{1+m^2}$, *so the answer depends on* $m$. *Therefore* $\lim_{(x,y) \to (a,b)} f(x, y)$ *does not exist in* $\mathbb{R}$.

**Example 26.** *Investigate the existence of the limit,* $\lim_{(x,y) \to (0,0)}$ *for the following functions.*

1. $f(x, y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & ,(x, y) = (0, 0) \\ 0 & ,(x, y) = (0, 0) \end{cases}$

2. $f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & ,(x, y) \neq (0, 0) \\ 0 & ,(x, y) = (0, 0) \end{cases}$

*3.* $f(x,y) = \begin{cases} \frac{x^2y^2}{x^2y^2+(x-y)^2} & ,(x,y) \neq (0,0) \\ 0 & ,(x,y) = (0,0) \end{cases}$ *4.* $f(x,y) = \begin{cases} x\sin\frac{1}{y} & ,y \neq 0 \\ 0 & ,y = 0 \end{cases}$

**Definition 13.** *Continuity of $f$ at $(a,b)$ i.e. $f \in \mathcal{C}(a,b)$*
$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

**Definition 14.** *Partial Derivatives*
$f_x(a,b) = f_1(a,b) = \frac{\partial f}{\partial x}(a,b) = \lim_{x\to a} \frac{f(x,b)-f(a,b)}{x-a} = \lim_{\Delta x\to a} \frac{f(a+\Delta x,b)-f(a,b)}{\Delta x}$
$f_y(a,b) = f_2(a,b) = \frac{\partial f}{\partial y}(a,b) = \lim_{y\to b} \frac{f(a,y)-f(a,b)}{y-b} = \lim_{\Delta y\to b} \frac{f(a,b+\Delta y)-f(a,b)}{\Delta y}$

**Definition 15.** $f \in \mathcal{C}^1 \Leftrightarrow f_x \in \mathcal{C}$ *and* $f_y \in \mathcal{C}$

**Theorem 42.** *Mean Value Theorem*
*1.* $f : \mathbb{D} \to \mathbb{R}, \mathbb{D} = \{(x,y)|(x-a)^2 + (y-b)^2 < \delta^2\}$
*2.* $f_x$ *and* $f_y$ *exists on* $\mathbb{D}$
*3.* $\Delta x^2 + \Delta y^2 < \delta^2$
*Then*
*1.* $f(a+\Delta x, b+\Delta y) = f(a,b) + \Delta x f_x(a+\theta\Delta x, b) + \Delta y f_y(a+\Delta x, b+\alpha\Delta y)$
*2.* $0 < \theta, \alpha < 1$

**Definition 16.** *Differentiability of $f$ at $(a,b)$ i.e. $f \in \mathcal{D}(a,b)$*
*1.* $f_x, f_y$ *exists at* $(a,b)$
*2. There exists $\delta > 0$ and a function $\phi$ such that for all $\sqrt{\Delta x^2 + \Delta y^2} < \delta$ we have* $f(a+\Delta x, b+\Delta y) = f(a,b) + \Delta x f_x(a,b) + \Delta y f_y(a,b) + \phi(\Delta x, \Delta y)$ *and*
*3.*$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\phi(\Delta x,\Delta y)}{\sqrt{\Delta x^2+\Delta y^2}} = 0.$

**Note 12.** *Frechet Derivative*
*If we write $\boldsymbol{h} = \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$ and $\boldsymbol{a} = \begin{pmatrix} a \\ b \end{pmatrix}$, we have $f(\boldsymbol{a}+\boldsymbol{h}) = f(\boldsymbol{a}) + f'(\boldsymbol{a})\boldsymbol{h} + \phi(\boldsymbol{h})$ and* $\lim_{||\boldsymbol{h}||\to 0} \frac{f(\boldsymbol{a}+\boldsymbol{h})-f(\boldsymbol{a})-f'(\boldsymbol{a})\boldsymbol{h}}{||\boldsymbol{h}||} = \lim_{||\boldsymbol{h}||\to 0} \frac{\phi(\boldsymbol{h})}{||\boldsymbol{h}||} = 0$ *where* $f'(\boldsymbol{a}) = \nabla f(\boldsymbol{a}) = (f_x(\boldsymbol{a}), f_y(\boldsymbol{a}))$.
*Such $f'(\boldsymbol{a})$ are called Frechet Derivatives. In 2D it is called the Gradient.*

**Theorem 43.** $f \in \mathcal{C}^1 \Rightarrow f \in \mathcal{D} \Rightarrow f \in \mathcal{C}$

**Example 27.** *Let* $f(x,y) = g(\sqrt{x^2+y^2}), g(x) = x\sin\frac{1}{x}, g(0) = 0$. *Show that* $f \in \mathcal{D}$ *but* $f \notin \mathcal{C}^1$

**Theorem 44.** *Chain Rule.* $f = f(x,y) \in \mathcal{C}^1$.
*1. If* $y = y(t), x = x(t) \in \mathcal{C}^1$ *then* $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$
*2.* $y = y(u,v), x = x(u,v) \in \mathcal{C}^1$ *then* $\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$ *and* $\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$

**Note 13.** *The above may be written as*
$\frac{df}{dt} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \frac{\partial f}{\partial(x,y)}\frac{\partial(x,y)}{\partial t}$ *and* $\frac{\partial f}{\partial(u,v)} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{\partial f}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)}$

*With $\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\boldsymbol{u} = \begin{pmatrix} u \\ v \end{pmatrix}$, the above may also be written as*
$(f \circ \boldsymbol{x})'(t) = (f' \circ \boldsymbol{x})(t)\boldsymbol{x}'(t)$ *and* $(f \circ \boldsymbol{x})'(\boldsymbol{u}) = (f' \circ \boldsymbol{x})(\boldsymbol{u})\boldsymbol{x}'(\boldsymbol{u})$

*We also see that* $\begin{pmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\end{pmatrix} = \frac{\partial f}{\partial(x,y)} = f'(\boldsymbol{x})$ *is acting as the true derivative of* $f = f(x, y)$*. Therefore it is called the Gradient of* $f$ *or* $\nabla f = \mathrm{grad} f$

*The determinant,* $\det \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}$ *is called the Jacobian*

**Definition 17.** *Directional Derivative of* $f$ *in the direction of the non-zero unit vector* $\boldsymbol{u} = (u, v)$ *at* $\boldsymbol{a} = (a, b)$ *is* $D_{\boldsymbol{u}}f(a, b) = \lim_{\Delta t\to 0} \frac{f(a+u\Delta t,b+v\Delta t)-f(a,b)}{\Delta t}$

**Theorem 45.** $f \in \mathcal{C}^1, \nabla f(a, b) \neq \mathbf{0}$
1. $D_{\boldsymbol{u}}(a, b) = \frac{\partial f}{\partial x}(a, b)u + \frac{\partial f}{\partial y}(a, b)v = \nabla f(a, b)\boldsymbol{u}$
2. $\max_{\boldsymbol{u}} D_{\boldsymbol{u}}f(a, b) = D_{\nabla \hat{f(a,b)}}f(a, b) = ||\nabla f(a, b)||$
3. $\min_{\boldsymbol{u}} D_{\boldsymbol{u}}f(a, b) = D_{-\nabla \hat{f(a,b)}}f(a, b) = -||\nabla f(a, b)||$

**Theorem 46.** *For the surface* $f = f(x, y) \in \mathcal{C}^1$ *at* $(a, b)$
1. *Normal vector :* $\boldsymbol{n}(a, b) = (f_x(a, b), f_y(a, b), -1) = (\nabla f(a, b), -1)$
2. *Equation of the Tangent Plane:*
$$z = f(a, b)+f_x(a, b)(x-a)+f_y(a, b)(y-b) = \nabla f(a, b)\begin{pmatrix}x-a\\ y-b\end{pmatrix} = f(\boldsymbol{a})+\nabla f(\boldsymbol{a})(\boldsymbol{x}-\boldsymbol{a})$$

**Example 28.** *Let* $f(x, y) = x^4 + y^4 - x^2 - y^2 + 1$*. At* $(1, 2)$ *find*
1. *Direction at which the function is increasing most rapidly*
2. *Directional derivative in that direction*
3. *Equation of the tangent plane*

**Example 29.** *Assume that all functions are* $\mathcal{C}^1$
1. *Show that if* $x = x(u, v), y = y(u, v), u = u(r, s), v = v(r, s)$ *then* $\frac{\partial(x,y)}{\partial(r,s)} = \frac{\partial(x,y)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(r,s)}$
2. *Show that if* $u = f(x, y), v = g(x, y)$ *then a functional relation of the form* $h(u, v) = 0$ *exists iff the Jacobian is identically zero.*

**Definition 18.** *Higher Order Derivatives*

$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial x^2}$

$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial y\partial x}$

$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial^2 f}{\partial x\partial y}$

$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = \frac{\partial^2 f}{\partial y^2}$

**Note 14.**
1. *We write* $f \in \mathcal{C}^2$ *to mean that* $f_{xx}, f_{xy}, f_{yx}, f_{yy} \in C$
2. *In a similar manner we write* $f \in \mathcal{C}^n$ *to mean that all the* $n$ *th order partial derivatives are continuous. There are* $2^n$ *of them.*
3. *There are* ${}^nC_m = \frac{n!}{m!(n-m)!}$, $n$ *th order partial derivatives that contains* $x$,$m$ *times.*

**Example 30.** *Let* $f(x, y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & ,(x, y)= (0, 0)\\ 0 & ,(x, y) = (0, 0)\end{cases}$

*Show that* $f_{xy}(0, 0) \neq f_{yx}(0, 0)$

**Theorem 47.** $f \in \mathcal{C}^2 \Rightarrow f_{xy}, f_{yx} \in \mathcal{C} \Rightarrow f_{xy} = f_{yx}$

*Proof.* Let $\phi(y) = f(a+\Delta x, y) - f(a,y) = \Delta x f_x(a+\alpha\Delta x, y)$
Then $\phi(b+\Delta y)-\phi(b) = f(a+\Delta x, b+\Delta y) - f(a, b+\Delta y) - f(a+\Delta x, b) + f(a,b) = \Delta y \phi'(b+\beta\Delta y) = \Delta y \Delta x f_{xy}(a+\alpha\Delta x, b+\beta\Delta y)$
Let $\psi(x) = f(x, b+\Delta y) - f(x,b) = \Delta y f_y(x, b+\gamma\Delta y)$
Then $\psi(a+\Delta x)-\phi(a) = f(a+\Delta x, b+\Delta y) - f(a+\Delta x, b) - f(a, b+\Delta y) + f(a,b) = \Delta x \psi'(a+\delta\Delta x) = \Delta x \Delta y f_{yx}(a+\delta\Delta x, b+\gamma\Delta y)$
So we have $f_{xy}(a+\alpha\Delta x, b+\beta\Delta y) = f_{yx}(a+\delta\Delta x, b+\gamma\Delta y)$ and letting $(\Delta x, \Delta y) \to (0,0)$ we get $f_{xy}(a,b) = f_{yx}(a,b)$ □

**Example 31.** *If $u = u(x,y) \in \mathcal{C}^2$ then prove that the Laplace operator $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ can be written as $\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$*

**Theorem 48.** *Taylor series of $f : \mathbb{R}^2 \to \mathbb{R}, f \in \mathcal{C}^{n+1}$ at $(a,b)$.*
$f(a+h, b+k) = \sum_{m=0}^{n} \frac{1}{m!} D^m f(a,b) + \frac{1}{(n+1)!} D^{n+1} f(a+\theta h, b+\theta k)$
*where $D = h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}, D^m = D(D^{m-1}), D^0 = 1$ and $\theta \in (0,1)$*

**Note 15.** *When $n = 2$ we may write the above as*
$f(\boldsymbol{a}+\boldsymbol{h}) = f(\boldsymbol{a}) + \nabla f(\boldsymbol{a})\boldsymbol{h} + \frac{1}{2}\boldsymbol{h}^T Hf(\boldsymbol{c})\boldsymbol{h}$
*where $\nabla f = (f_x, f_y)$ is the Gradient and $Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$ is the Hessian.*
*and $\boldsymbol{a} = \begin{pmatrix} a \\ b \end{pmatrix}$,$\boldsymbol{h} = \begin{pmatrix} h \\ k \end{pmatrix}$, $\boldsymbol{c} = \boldsymbol{a} + \theta\boldsymbol{h}, \theta \in (0,1)$.*
*Note that the first two terms are the Tangent Plane at $\boldsymbol{a}$.*
*Also $\boldsymbol{h}^T Hf(\boldsymbol{c})\boldsymbol{h} = h^2 f_{xx}(\boldsymbol{c}) + 2hk f_{xy}(\boldsymbol{c}) + k^2 f_{yy}(\boldsymbol{c}) = k^2 f_{xx}(\boldsymbol{c})\left(\left(\frac{h}{k} + \frac{f_{xy}(\boldsymbol{c})}{f_{xx}(\boldsymbol{c})}\right)^2 + \frac{\det Hf(\boldsymbol{c})}{(f_{xx}(\boldsymbol{c}))^2}\right)$*
*if $f_{xx}(\boldsymbol{c}) \neq 0$. Note that $\det Hf = f_{xx}f_{yy} - f_{xy}f_{yx} = f_{xx}f_{yy} - (f_{xy})^2$ when $f \in \mathcal{C}^2$.*
*Also $\mathrm{tr}Hf = f_{xx} + f_{yy}$ is the Trace of the matrix $Hf$.*

*Then*
$f : \mathbb{R}^2 \to \mathbb{R}, f \in \mathcal{C}^2$. $\nabla f(\boldsymbol{a}) = \boldsymbol{0}$.
$\mathrm{tr}Hf(\boldsymbol{a}) > 0$ *and* $\det Hf(\boldsymbol{a}) > 0$
$\Leftrightarrow f_{xx}(\boldsymbol{a}) > 0$ *and* $\det Hf(\boldsymbol{a}) > 0$
$\Rightarrow f_{xx}(\boldsymbol{c}) > 0$ *and* $\det Hf(\boldsymbol{c}) > 0$, *for sufficiently small* $\boldsymbol{h}$
$\Rightarrow f(\boldsymbol{a}+\boldsymbol{h}) - f(\boldsymbol{a}) = \frac{1}{2}\boldsymbol{h}^T Hf(\boldsymbol{c})\boldsymbol{h} > 0$, *for sufficiently small* $\boldsymbol{h}$

**Definition 19.**
*1. $f$ has a relative minimum at $\boldsymbol{a}$ iff $f(\boldsymbol{a}) > f(\boldsymbol{a}+\boldsymbol{h})$ in a nbd of $\boldsymbol{a}$.*
*2. $f$ has a relative maximum at $\boldsymbol{a}$ iff $f(\boldsymbol{a}) < f(\boldsymbol{a}+\boldsymbol{h})$ in a nbd of $\boldsymbol{a}$.*
*3. $f$ has a saddle point at $\boldsymbol{a}$ iff both $f(\boldsymbol{a}) > f(\boldsymbol{a}+\boldsymbol{h})$ and $f(\boldsymbol{a}) < f(\boldsymbol{a}+\boldsymbol{h})$ in a nbd of $\boldsymbol{a}$.*

**Theorem 49.** *$f \in \mathcal{C}^2$ and $\nabla f(\boldsymbol{a}) = \boldsymbol{0}$*
*1. If $\det Hf(\boldsymbol{a}) > 0$ and $\mathrm{tr}Hf(\boldsymbol{a}) > 0$ then $\boldsymbol{a}$ is a local minimum of $f$.*
*2. If $\det Hf(\boldsymbol{a}) > 0$ and $\mathrm{tr}Hf(\boldsymbol{a}) < 0$ then $\boldsymbol{a}$ is a local maximum of $f$.*
*3. If $\det Hf(\boldsymbol{a}) < 0$ then $\boldsymbol{a}$ is a saddle point of $f$.*

**Definition 20.** *$\boldsymbol{a}$ is a Critical Point of $f$ iff $\nabla f(\boldsymbol{a})$ is $\boldsymbol{0}$ of undefined*

**Example 32.** *Find the critical points and determine the nature(max/min/saddle) of them*
*1.* $f(x,y) = x^3 - 12x + y^3 - 27y + 5$
*2.* $f(x,y) = x^4 + y^4 - x^2 - y^2 + 1$
*3.* $f(x,y) = x^4 + y^4$
*4. Propose a method if* $\det Hf = 0$ *when* $\nabla f = \mathbf{0}$

**Theorem 50.** *Implicit Function Theorem*
*Let* $f(x,y) \in \mathcal{C}^1$ *,* $f_y(a,b) \neq 0$ *and* $f(a,b) = c$*. Then there exists a unique function* $y = g(x) \in \mathcal{C}^1$ *defined on a nbd of* $(a,b)$ *with* $g'(x) = -\frac{f_x(x,g(x))}{f_y(x,g(x))}$ *such that* $f(x,g(x)) = c$*.*

*Proof.* WLOG assume $f(a,b) = c = 0$ and $f_y(a,b) > 0$. By continuity there exists a nbd $(a-\delta, a+\delta) \times (b-\delta, b+\delta)$ of $(a,b)$ such that $f_y(x,y) > 0$. Now by MVT $f(a, b+\delta/2) > 0$ and $f(a, b-\delta/2) < 0$. Again by continuity wrt $x$ we have $f(x, b+\delta/2) > 0$ and $f(x, b-\delta/2) < 0$ for $|x-a| < \delta_1 < \delta$. Now by IVT there exists unique $y$ such that $f(x,y) = 0$ for $|y-b| < \delta/2$. When $x = a$, this $y$ is $b$ so $y(a) = b$. Now $|x-a| < \delta_1$ implies $|y-b| < \delta/2$, which proves the continuity of the function $y = g(x)$.
Within a nbd of $(a,b)$ we have $f(x,g(x)) = 0$ and $f(x+\delta x, g(x+\delta x)) = 0$.
Now $f(x+\delta x, g(x+\delta x)) - f(x, g(x+\delta x)) + f(x, g(x+\delta x)) - f(x, g(x)) = 0$
Or $\delta x f_x(x+\alpha\delta x, g(x+\delta x)) + (g(x+\delta x) - g(x)) f_y(x, g(x) + \beta(g(x+\delta x) - g(x))) = 0$
Finally $g'(x) = \lim_{\delta x \to 0} \frac{g(x+\delta x) - g(x)}{\delta x} = -\frac{f_x(x,g(x))}{f_y(x,g(x))} \in \mathcal{C}$ □

**Theorem 51.** *Constrained Optimization/Lagrange Multipliers*
*Let* $f, g \in \mathcal{C}^1$ *and* $\nabla g \neq \mathbf{0}$*. Then the maxima/minima of* $f(x,y)$ *subjected to* $g(x,y) = 0$ *are included in each of*
*1.* $\det \frac{\partial(f,g)}{\partial(x,y)} = 0$ *and* $g(x,y) = 0$
*2.* $\nabla f(x,y) = \lambda \nabla g(x,y)$ *and* $g(x,y) = 0$

*Proof.* We have $g \in \mathcal{C}^1$ and WLOG assume $g_y \neq 0$(note that if $g_y = 0$ then $g_x \neq 0$ since $\nabla g \neq \mathbf{0}$, which means we can use $x$ as the independent variable). By Implicit function theorem there exists a function $y = y(x) \in \mathcal{C}^1$ such that $g(x, y(x)) = 0$. By chain rule $0 = g_x \frac{dx}{dx} + g_y \frac{dy}{dx}$ or $\frac{dy}{dx} = -\frac{g_x}{g_y}$. Now since $f \in \mathcal{C}^1$ we have $\frac{df}{dx} = f_x \frac{dx}{dx} + f_y \frac{dy}{dx} = f_x - f_y \frac{g_x}{g_y} = \frac{1}{g_y}(f_x g_y - g_x f_y) = \frac{1}{g_y} \det \frac{\partial(f,g)}{\partial(x,y)} = 0 \Leftarrow \det \frac{\partial(f,g)}{\partial(x,y)} = 0$
Also $\nabla f = \lambda \nabla g \Leftrightarrow f_x = \lambda g_x$ and $f_y = \lambda g_y$ implies $\det \frac{\partial(f,g)}{\partial(x,y)} = \lambda g_x g_y - \lambda g_y g_x = 0$
On the other hand let $\det \frac{\partial(f,g)}{\partial(x,y)} = f_x g_y - g_x f_y = 0$ with $g_y \neq 0$. If $g_x \neq 0$ too then $\frac{f_x}{g_x} = \frac{f_y}{g_y} = \lambda$ means $\nabla f = \lambda \nabla g$. However if $g_x = 0$ then $f_x = 0$ too so $f_x = \lambda g_x$ for any $\lambda$. Since $g_y \neq 0$ we can define $\lambda = \frac{f_y}{g_y}$, which means $\nabla f = \lambda \nabla g$ again. □

**Example 33.**
*1. Find the shortest distance from the point* $(1,0)$ *to the parabola* $y^2 = 4x$*.*
*2. Substitute* $y^2 = 4x$ *or* $x = y^2/4$ *and minimize the distance function in 1. as a*

*function of $y$ or $x$. Explain why we get/can't get the answer in 1.*
*3. Find the absolute maximum/minimum of $x^4 + y^4 - x^2 - y^2 + 1$ on the disk $(x-1)^2 + y^2 \le 4$*
*4. Find the directions of the axes of the ellipse $5x^2 - 6xy + 5y^2 - 4x - 4y - 4 = 0$*

## 9 Least Square Polynomial

**Theorem 52.** *Least Square Line $y = ax + b$ for the points $(x_k, y_k), 1 \le k \le n$ That minimizes $E(a,b) = \sum_{k=1}^n (ax_k + b - y_k)^2$ is given by*

$$\begin{pmatrix} \sum_{k=1}^n x_k^2 & \sum_{k=1}^n x_k \\ \sum_{k=1}^n x_k & \sum 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^n x_k y_k \\ \sum_{k=1}^n y_k \end{pmatrix}$$

*This can also be written as $X^T X \begin{pmatrix} a \\ b \end{pmatrix} = X^T Y$ where*

$$X^T = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{pmatrix} \text{ and } Y^T = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix}$$

**Theorem 53.** *Properties of the Least Square Line*
*With $\overline{x} = \frac{1}{n}\sum_{k=1}^n x_k$, $\overline{y} = \frac{1}{n}\sum_{k=1}^n y_k$, $s_{xx} = \frac{1}{n}\sum_{k=1}^n (x_k - \overline{x})^2 = \frac{1}{n}\sum_{k=1}^n x_k^2 - \overline{x}^2$, and $s_{xy} = \frac{1}{n}\sum_{k=1}^n (x_k - \overline{x})(y_k - \overline{y}) = \frac{1}{n}\sum_{k=1}^n x_k y_k - \overline{x}\,\overline{y}$*
*We have $a = \frac{s_{xy}}{s_{xx}}$*
*Also $(\overline{x}, \overline{y})$ is on the Least square line and therefore $\overline{y} = a\overline{x} + b$ or $b = a\overline{x} - \overline{y}$*
*If $HE = \begin{pmatrix} E_{aa} & E_{ab} \\ E_{ba} & E_{bb} \end{pmatrix}$ is the Hessian of $E(a,b)$ we have $HE = 2X^T X$ and*
$\det HE = E_{aa}E_{bb} - (E_{ab})^2 = 4n^2 s_{xx} > 0$ *when $x_k$ are different and*
$\mathrm{tr}HE = E_{aa} + E_{bb} = 2\sum x_k^2 + 2n > 0$ *so $(a,b)$ is a global minimum.*

**Example 34.** *Let $A = \{(2,1), (3,2), (4,3), (6,4)\}$*
*1. Find the Lest Square Line for A.*
*2. Show that the least square parabola $y = ax^2 + bx + c$ for the data set $(x_k, y_k), k = 1, 2, \cdots, n$ is given by*

$$\begin{pmatrix} \sum x_k^4 & \sum x_k^3 & \sum x_k^2 \\ \sum x_k^3 & \sum x_k^2 & \sum x_k \\ \sum x_k^2 & \sum x_k & \sum 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \sum x_k^2 y_k \\ \sum x_k y_k \\ \sum y_k \end{pmatrix}$$

*3. Find the Least Square Parabola for A*

**Example 35.** *Let $A = \{(1,1), (2,1), (3,2), (4,3)\}$*
*1. Find the least square polynomials of degree 0,1,2,3,4 for A if it is possible.*
*2. Calculate the exact error before and after finding the coefficients in each case.*
*3. Show that we have a unique solution when each $x_k$ is different.*
*4. Fit a least square function of the form $y = ax + bx^3 + cx^4$ for A.*
*5. Fit a least square function of the form $y = ae^x + b\sin x + c\cos x$ for A.*
*6. Find the best combination of functions out of $\{1, x, x^2, e^x, \sin x, \cos x, \log x\}$ if we are looking for a combination of 3 functions.*
*7. Show that the Correlation Coefficient given by $r = \frac{s_{xy}}{\sqrt{s_{xx}s_{yy}}}$ is a measure of the*

*linearity of data in the case of the least square line.*
*8. Suppose we have a 3D date set $B = \{(1,1,1),(2,1,2),(3,2,3),(4,3,4)\}$. Propose a Lagrange-type two variable polynomial and a least square plane.*

**Example 36.** *One method of finding the maximum of a multivariate function $E(a,b,c)$ is called the Steepest Descent Method. Here we start at a given point $(a_0,b_0,c_0)$ and select the direction of the maximum slope at $(a_0,b_0,c_0)$. Then we follow the maximum slope direction till we get the maximum along that direction as a one variable function say at $(a_1,b_1,c_1)$ and we repeat the process. Show that the maximum directions at $(a_0,b_0,c_0)$ and $(a_1,b_1,c_1)$ are perpendicular.*
*The function to minimize for the least square parabola in the earlier question is $E(a,b,c) = 30 - 428a + 1649a^2 - 88b + 630ab + 65b^2 - 20c + 130ac + 30bc + 4c^2$. Write the first two steps of the Steepest Descend Method starting from $(0,0,0)$*

**Casio 6.** *CASIO fx-991ES*
*MODE 3:STAT 3:_ $+cx^2$ 2 = 3 = 4 = 6 = REPLAY UP 1 = 2 = 3 = 4 = SHIFT STAT(1) 1:Type, 2: Data, 3:Edit, 4:Sum(1:$\sum x^2$, 2:$\sum x$, 4:$\sum y$, 5:$\sum xy$, 6:$\sum x^3$, 7:$\sum x^2y$, 8:$\sum x^4$), 7:Reg(1:A, 2:B, 3:C)*

**Mathematica 7.** $Fit[\{\{2,1\},\{3,2\},\{4,3\},\{6,4\}\},\{x^2,x,1\},x]$

## 10 Ordinary Differential Equations,ODEs

**Definition 21.** *First Order ODE* $\frac{dy}{dx} = f(x,y), y(x_0) = y_0$

**Theorem 54.** *Cauchy-Peano(Exiatence of solutions)*
*If $f \in \mathcal{C}$ then the above ODE has a $\mathcal{C}^1$ solution in a nbd of $x_0$*

**Definition 22.** *$f \in \mathcal{LC}$ in $y$ uniformly in $x$*
*iff* $\exists L > 0 \forall x \forall y_1, y_2; |f(x,y_1) - f(x,y_2)| \le L|y_1 - y_2|$

**Theorem 55.**
*If $\frac{\partial f}{\partial y} \in \mathcal{C}$ in a closed bounded set on $\mathbb{R}^2$ then $f \in \mathcal{LC}$ in $y$ uniformly in $x$.*

**Theorem 56.** *Picard-Lindelof(Uniqueness of solutions)*
*If $f \in \mathcal{C}$ and $f \in \mathcal{LC}$ in $y$ uniformly in $x$. Then the above ODE has a unique $\mathcal{C}^1$ solution in a nbd of $x_0$*

*Proof.* Is based on defining a sequence of functions $y_0(x) = y_0$ and $y_{k+1}(x) = \int_{x_0}^{x} f(t, y_k(t))dt$ and use a variant of Banach-Fixed point theorem as in the case of the Fixed Point method in root finding. □

**Example 37.** *Discuss the Existence and Uniqueness of the following ODEs.*
*1.* $\frac{dy}{dx} = y^3, y(0) = 0$
*2.* $\frac{dy}{dx} = y^{1/3}, y(0) = 0$

**Definition 23.** *Variable Separable:*$f(x,y) = \frac{g(x)}{h(y)}; g,h \in \mathcal{C}; h(y) \neq 0$
$\int h(y)dy = \int g(x)dx$

**Example 38.** $\frac{dy}{dx} = \frac{e^x}{y}, y(0) = \sqrt{2}$

**Solution 1.** *We expect trouble at* $y = 0$.
*Separating the variables and integrate,* $\int ydy = \int e^x dx \Rightarrow \frac{y^2}{2} = e^x + c$
*Substituting boundary conditions:* $\frac{2}{2} = 1 + c \Rightarrow c = 0$. *So* $\frac{y^2}{2} = e^x$ *or* $y = \pm\sqrt{2}e^{x/2}$.
*But the only solution satisfying* $y(0) = \sqrt{2}$ *is* $y = \sqrt{2}e^{x/2}$. $y$ *is never* $0$ *according to this solution and the solution valid for all* $x \in \mathbb{R}$.

**Definition 24.** *Homogeneous:*
$y = vx, v = v(x), f(x, vx) = g(v) \in \mathcal{C}, g(v) \neq v, x \neq 0$
$\frac{dy}{dx} = v + x\frac{dv}{dx} = g(v) \Rightarrow \frac{dv}{dx} = \frac{g(v)-v}{x}$: *Variable Separable*

**Example 39.** $\frac{dy}{dx} = \frac{x^2+y^2}{xy}, y(1) = 1$

**Solution 2.** *We expect trouble at* $xy = 0$. *Put* $y = vx$ *and* $\frac{dy}{dx} = \frac{d(vx)}{dx} = v + x\frac{dv}{dx} = \frac{x^2+v^2x^2}{xvx} = \frac{1+v^2}{v} = \frac{1}{v} + v$ *so* $\frac{dv}{dx} = \frac{1}{v}$. *Separating the variables and integrate:* $\int vdv = \int xdx \Rightarrow \frac{v^2}{2} = x + c$ *or* $\frac{y^2}{2x^2} = x + c$ *or* $y^2 = 2x^2(x + c)$. *Substituting the boundary condition:* $1 = 2(1+c)$ *we get* $c = -\frac{1}{2}$. *Finally* $y^2 = 2x^2(x-\frac{1}{2}) = x^2(2x-1)$. *We see that* $x$ *has to be* $> 1/2$, *so* $y = \pm x\sqrt{2x-1}$. *But the only solution satisfying* $y(1) = 1$ *is* $y = x\sqrt{2x-1}$. *As expected we have trouble at* $y = 0$ *or at* $x = \frac{1}{2}$ *although* $x$ *is never* $0$. *So the solution valid for* $x \in (\frac{1}{2}, \infty)$.

**Definition 25.** *Linear:* $f(x, y) = Q(x) - P(x)y; P, Q \in \mathcal{C}$
*Integrating Factor:* $I(x) = e^{\int P(x)dx} \Rightarrow \frac{d}{dx}(I(x)y) = I(x)Q(x) \Rightarrow y = \frac{1}{I(x)}\int Q(x)I(x)dx$

**Example 40.** $\frac{dy}{dx} + \frac{y}{x} = \log x, y(1) = 1$

**Solution 3.** *We can only have* $x > 0$. *The integrating factor is :* $I = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$. *We don't have to put a constant here since it is equivalent to multiplying the whole ODE by a constant. Now* $x\frac{dy}{dx} + y = \frac{d(xy)}{dx} = x\ln x$. *By integrating both sides:* $xy = \int x\ln xdx = \ln x\frac{x^2}{2} - \int \frac{x^2}{2}\frac{1}{x}dx = \frac{x^2}{2}\ln x - \frac{1}{2}\int xdx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$.
$y = \frac{x}{2}\ln x - \frac{x}{4} + \frac{c}{x}$. *Substituting the boundary conditions:* $1 = 0 - \frac{1}{4} + c$ *or* $c = \frac{5}{4}$.
$y = \frac{x}{2}\ln x - \frac{x}{4} + \frac{5}{4x}$. *The solution is valid for* $x \in (0, \infty)$.

**Definition 26.** *Bournoulli:* $f(x, y) = Q(x)y^n - P(x)y; P, Q \in \mathcal{C}$
$z = y^{1-n} \Rightarrow \frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx} = (1-n)y^{-n}(Q(x)y^n - P(x)y) = (1-n)(Q(x) - P(x)z)$
*So we have* $\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x)$:*Linear*

**Example 41.** $\frac{dy}{dx} + \frac{y}{x} = y^3\ln x, y(1) = 1$

**Solution 4.** *We must have* $x > 0$. *Put* $z = y^{1-3} = y^{-2}$. *We arrive at* $\frac{dz}{dx} + (-2)\frac{1}{x}z = (-2)\ln x$. *Integrating factor* $I = e^{\int \frac{-2}{x}}dx = e^{-2\ln x} = x^{-2}$. *So* $\frac{d(x^{-2}z)}{dx} = -2x^{-2}\ln x$ *or* $x^{-2}z = -2\int x^{-2}\ln xdx = -2(\ln x\frac{x^{-1}}{-1} - \int \frac{x^{-1}}{-1}\frac{1}{x}dx = \frac{2}{x}\ln x - 2\int x^{-2}dx = \frac{2}{x}\ln x + \frac{2}{x} + c$
*Now* $y^{-2} = 2x\ln x + 2x + cx^2$. *With the boundary condition* $y(1) = 1$ *we have* $1 = 0 + 2 + c$ *or* $c = -1$ *and* $y^{-2} = 2x(\ln x + 1) - x^2$ *finally* $y = \frac{1}{\sqrt{2x(\ln x + 1 - \frac{x}{2})}}$, *-ve square root does not agree with the boundary condition.* $x \in (0, \alpha)$ *where* $\alpha$ *is the root of* $\ln x + 1 - \frac{x}{2} = 0$ *which is* $0.463922.....$

**Definition 27.** *Exact:*
$M(x,y)+N(x,y)\frac{dy}{dx}=0; N(x,y)\neq 0; M,N\in\mathcal{C}^1; \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}\in\mathcal{C}$
*Then there exists* $f$ *such that* $\frac{\partial f}{\partial x}=M$ *and* $\frac{\partial f}{\partial y}=N$ *(see proof)*
$\Rightarrow 0=M+N\frac{dy}{dx}=\frac{\partial f}{\partial x}\frac{dx}{dx}+\frac{\partial f}{\partial y}\frac{dy}{dx}=\frac{df}{dx}\Rightarrow f=c$*:constant*

*Proof.* Let $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}=P(x,y)\in\mathcal{C}$.
Then $M(x,y)=\int P(x,y)dy$ and $N(x,y)=\int P(x,y)dx$
Now $\int M(x,y)dx$
$=\int\int P(x,y)dydx$
$=\int\int P(x,y)dxdy,$ by Fubini's Theorem for Double Integrals since$P\in\mathcal{C}$
$=\int N(x,y)dy=f(x,y), say$ Then $\frac{\partial f}{\partial x}=M$ and $\frac{\partial f}{\partial y}=N$.
Also note that $\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial M}{\partial y}=P$ and $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial N}{\partial x}=P$.
So $\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial^2 f}{\partial y\partial x}$ automatically. □

**Example 42.**
*1.* $(y^2+x^2)+(2xy+y)\frac{dy}{dx}=0, y(1)=1$
*2.* $(3x^2+6xy^2)+(6x^2y+4y^3)\frac{dy}{dx}=0$

**Solution 5.** $\frac{dy}{dx}=-\frac{x^2+y^2}{y(2x+1)}$ *so we expect trouble when* $y=0$ *and* $x=-\frac{1}{2}$.
*Here* $M=x^2+y^2=\frac{\partial f}{\partial x}$ *so* $f=\int_{y\quad constant}(x^2+y^2)dx=\frac{x^3}{3}+y^2x+c(y)$. *Here the constant* $c(y)$ *is a function of* $y$ *since we kept* $y$ *constant in the integration. Now we have* $\frac{\partial f}{\partial y}=0+2yx+c'(y)=N=2xy+y$ *so* $c'(y)=y$ *and* $c(y)=\frac{y^2}{2}+c_1$. *So* $f=\frac{x^3}{3}+y^2x+\frac{y^2}{2}+c_1=c_2$ *or* $y^2(x+\frac{1}{2})+\frac{x^3}{3}=c_2-c_1=a$. *With the boundary conditions* $y(1)=1$ *we have* $1(1+\frac{1}{2})+\frac{1}{3}=a$ *or* $a=\frac{11}{6}$. *Now* $y^2(x+\frac{1}{2})=\frac{11}{6}-\frac{x^3}{3}$ *or* $y^2=\frac{11-2x^3}{3(2x+1)}$ *or* $y=\sqrt{\frac{11-2x^3}{3(2x+1)}}$, *-ve square root does nor match the boundary condition. As expected we have trouble at* $x=-\frac{1}{2}$ *and* $y=0$. *The latter is when* $x^3=\frac{11}{2}$ *or when* $x=\sqrt[3]{\frac{11}{2}}$. *So the range of* $x$ *is* $(-\frac{1}{2},\sqrt[3]{\frac{11}{2}})$.
*Note that we could have got the answer as follows if we knew that the above ODE is exact.* $0=x^2+y^2+(2xy+y)\frac{dy}{dx}$ *implies* $\int 0dx=\int x^2dx+\int(y^2dx+2xydy)+\int ydy=\frac{x^3}{3}+\int\frac{d(y^2x)}{dx}dx+\frac{y^2}{2}=\frac{x^3}{3}+y^2x+\frac{y^2}{2}=c$.

**Definition 28.** *Reducible to Exact:*
$M(x,y)+N(x,y)\frac{dy}{dx}=0, M,N\in\mathcal{C}^1, \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}\in\mathcal{C}$
*If* $\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)/N=g(x)$ *is a function of* $x$ *alone,*
*define* $I(x)=e^{\int g(x)dx}$ *so* $\frac{\partial(NI)}{\partial x}=I\frac{\partial N}{\partial x}+NIg(x)=I\frac{\partial M}{\partial y}=\frac{\partial(MI)}{\partial y}$*: Exact*
*If* $\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)/M=h(y)$ *is a function of* $y$ *alone,*
*define* $J(y)=e^{\int h(y)dy}$ *so* $\frac{\partial(MJ)}{\partial y}=J\frac{\partial M}{\partial y}+MJh(y)=J\frac{\partial N}{\partial x}=\frac{\partial(NJ)}{\partial x}$*: Exact*

**Example 43.**
*1.* $(x^3+y^3)-xy^2\frac{dy}{dx}=0$
*2.* $y-(2x+y)\frac{dy}{dx}$

**Solution 6.** $\frac{dy}{dx} = \frac{x^3+y^3}{xy^2}$. *Again we expect trouble when $x = 0$ or $y = 0$. Here $M = x^3 + y^3$ and $\frac{\partial M}{\partial y} = 3y^2$. Also $N = -xy^2$ and $\frac{\partial N}{\partial x} = -y^2$, so $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. But $(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})/N = (3y^2 + y^2)/(-xy^2) = -\frac{4}{x} = g(x)$ is a function of $x$ only. Now $I = e^{\int -\frac{4}{x} dx} = e^{-4\ln x} = x^{-4}$. This means that the original ODE is lagging a factor of $x^{-4}$ to be an Exact ODE. Multiplying by it we arrive at: $x^{-1} + y^3x^{-4} - x^{-3}y^2\frac{dy}{dx} = 0$. Multiplying by $-3$ we see that $-3x^{-1} - 3x^{-4}y^3 + x^{-4}3y^2\frac{dy}{dx} = 0$ or $-3\frac{d}{dx}(\ln x) + \frac{d}{dx}(x^{-3}y^3) = 0$. Finally we have $-3\ln x + x^{-3}y^3 = c$. With the boundary conditions $y(1) = 1$ we have $0 + 1 = c$ or $c = 1$ and $y^3 = x^3(1 + 3\ln x)$ or $y = x\sqrt[3]{1 + 3\ln x}$. We need $x > 0$ for $\ln x$ to be defined and need $1 + 3\ln x \neq 0$ or $x \neq e^{-1/3}$ for $\frac{dy}{dx}$ to exist. The latter corresponds to $y \neq 0$. So the range of $x$ is $(e^{-1/3}, \infty)$.*

**Definition 29.** *Second Order ODE:* $\frac{d^2y}{dx^2} = f(x, y, \frac{dy}{dx}), y(x_0) = y_0, y'(x_0) = y_0'$

**Definition 30.** *Second Order Linear ODE:*
$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x), y(x_0) = y_0, y'(x_0) = y_0'$
$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$ *is called the Homogeneous Equation and*
$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$ *is called the Non-Homogeneous Equation*

**Theorem 57.**
*1. If $p, q, r \in \mathcal{C}$ then the above ODE has a unique solution in a nbd of $x_0$*
*2. The solution can be expressed as $y = y_c + y_p$*
*3. The Complimentary/Fundamental Solution $y_c$ is the solution for the corresponding Homogeneous Equation.*
*4. It can be expressed as $y_c = au(x) + bv(x)$ where $a, b$ are constants and $u, v$ are the Fundamental Solutions which are Linearly Independent, i.e.*
$\forall(a, b)[\forall x(au(x) + bv(x) = 0) \Rightarrow (a, b) = (0, 0)]$
*5. Note that $u, v$ satisfy the Homogeneous equation on their own so is their Linear Combination $au(x) + bv(x)$*
*6. Particular Solution $y_p$ is the solution for the corresponding Non-Homogeneous Equation. It does not have an arbitrary constants as in the case of $y_c$.*
*7. None of $u(x), v(x), y_p(x)$ are unique. We can have $u(x)$ in $v(x)$ and wise-versa in a way that the new ones are linearly independent. Also we can have parts of $u(x)$ and $v(x)$ in $y_p(x)$*

**Definition 31.** *Second Order Linear ODE with constant coefficients:*
$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = r(x); p, q$ *are real numbers*

**Theorem 58.**
*Method 1*
*By substituting $y = ce^{\alpha x}$ in the Homogeneous Equation we arrive at the Characteristic Equation $\alpha^2 + p\alpha + q = (\alpha - a)(\alpha - b) = 0$. There are three possibilities*
*1. $a, b \in \mathbb{R}, a \neq b \Rightarrow y_c = ce^{ax} + de^{bx}, u(x) = e^{ax}, v(x) = e^{bx}$*
*2. $a, b \in \mathbb{R}, a = b \Rightarrow u(x) = e^{ax}$. To get the other solution we let $y = xe^{ax}$ and confirm that it satisfy the Homogeneous Equation. So $v(x) = xe^{ax}$ and $y_c = ce^{ax} + dxe^{ax}$*
*3. $a, b \in \mathbb{C}, a = a_1 + ia_2 = \bar{b} \Rightarrow y_c = ce^{a_1x}\sin a_2x + de^{a_1x}\cos a_2x, u(x) = e^{a_1x}\sin a_2x, v(x) = e^{a_1x}\cos a_2x$*

*The Particular Solution $y_p$ may be obtained by assuming a solution and confirming it.*

*Method 2*
*If* $\alpha^2+p\alpha+q=(\alpha-a)(\alpha-b)=\alpha(\alpha-b)-a(\alpha-b)=\alpha(\alpha-a)-b(\alpha-a)$
*We can write* $\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy=\frac{d}{dx}(\frac{dy}{dx}-ay)-b(\frac{dy}{dx}-ay)=\frac{dz}{dx}-bz=r(x)$ *and* $\frac{dy}{dx}-ay=z$ *or alternatively*
$\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy=\frac{d}{dx}(\frac{dy}{dx}-by)-a(\frac{dy}{dx}-by)=\frac{dz}{dx}-az=r(x)$ *and* $\frac{dy}{dx}-by=z$
*In either case we get two First Order Linear ODEs which can be solved to find* $z$ *and then* $y$.

**Definition 32.** *Reducible to Second Order Linear ODE with constant coefficients:*
$x^2\frac{d^2y}{dx^2}+px\frac{dy}{dx}+qy=r(x); p,q$ *are real numbers*
*Let* $x=e^z\Rightarrow x\frac{dy}{dx}=\frac{dy}{dz}\Rightarrow x\frac{d^2y}{dx^2}+\frac{dy}{dx}=\frac{d^2y}{dz^2}\frac{1}{x}\Rightarrow x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}-\frac{dy}{dz}$ *so*
$x^2\frac{d^2y}{dx^2}+xp\frac{dy}{dx}+qy=r(x)\Rightarrow\frac{d^2y}{dz^2}+(p-1)\frac{dy}{dx}+qy=r(e^z)$: *Second Order Linear ODE with constant coefficients*

**Definition 33.** *Wronskian*
$W(u,v)(x)=uv'-vu'=\det\begin{pmatrix}u & v\\ u' & v'\end{pmatrix}$

**Theorem 59.** *If $u,v$ are Fundamental Solutions to the Homogeneous Equation, then*
*1.* $W'+p(x)W=0$
*2.* $W(u,v)(x)=ce^{-\int p(x)dx}=W(u,v)(x_0)e^{-\int_{x_0}^{x}p(t)dt}$
*2.* $\forall x, W(u,v)=0$ *or* $\forall x, W(u,v)\neq 0$
*3.* $\exists x, W(u,v)(x)\neq 0\Leftrightarrow u,v$ *are Linearly Independent.*

*Proof.* Let $W\neq 0$. Then if $\forall x, au(x)+bv(x)=0$ implies $\forall x, au'(x)+bv'(x)=0$.
Or equivalently $\forall x, \begin{pmatrix}u(x) & v(x)\\ u'(x) & v'(x)\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$.
Since $W=uv'-vu'\neq 0$, this implies that $a=b=0$, so $u,v$ are linearly independent.
On the other hand let $W=uv'-vu'=0$. This means $\frac{v'}{v}=\frac{u'}{u}$ or $\ln v=\ln u+\ln c$ or $v=cu$ or $1.v+(-c)u=0$ but clearly $1\neq 0$ and therefore $u,v$ are linearly dependent.
Ot in other words, If $u,v$ are linearly independent then $W\neq 0$. □

**Example 44.**
*1. Assume $v(x)=a(x)u(x)$ and derive a method of finding $a(x)$. Are the results same as above?*
*1. We know that* $u(x)=e^{ax}$ *is a solution to* $\frac{d^2y}{dx^2}-2a\frac{dy}{dx}+a^2y=0$. *Show that* $v(x)=xe^{ax}$ *is the other Linearly Independent Solution.*
*2. Consider the ODE:* $x^2y''+x(x+1)y'-y=0$. *If* $u(x)=\frac{e^{-x}}{x}$ *is a solution, use Wronskian to find the other linearly independent solution* $v(x)$.

**Theorem 60.** *If $u,v$ are fundamental solutions to the Homogeneous Equation then*
*1. Particular Solution can be expressed as* $y_p=a(x)u(x)+b(x)v(x)$
*2.* $\begin{pmatrix}u & v\\ u' & v'\end{pmatrix}\begin{pmatrix}a'\\ b'\end{pmatrix}=\begin{pmatrix}0\\ r\end{pmatrix}$

*3.* $a' = \frac{-rv}{W} = \frac{1}{W}\det\begin{pmatrix} 0 & v \\ r & v' \end{pmatrix}$ *and* $b' = \frac{ru}{W} = \frac{1}{W}\det\begin{pmatrix} u & 0 \\ u' & r \end{pmatrix}$

**Theorem 61.** *To solve* $y'' + p(x)y' + q(x)y = r(x), y(x_0) = y_0, y'(x_0) = y'_0$
$y(x) = Au(x) + Bv(x) + y_p(x)$ *and* $y'(x) = Au'(x) + Bv'(x) + y'_p(x)$
*So* $\begin{pmatrix} u(x_0) & v(x_0) \\ u'(x_0) & v'(x_0) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} y(x_0) - y_p(x_0) \\ y'(x_0) - y'_p(x_0) \end{pmatrix}$

*Or* $\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{W(x_0)}\begin{pmatrix} v'(x_0) & -v(x_0) \\ -u'(x_0) & u(x_0) \end{pmatrix}\begin{pmatrix} y_0 - y_p(x_0) \\ y'_0 - y'_p(x_0) \end{pmatrix}$

**Example 45.** *Solve*
*1.* $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \sin x$
*2.* $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$
*3.* $\frac{d^2y}{dx^2} + y = x\ln x$

**Solution 7.** *We will solve* $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \sin x$ *in all the 3 methods we discussed. We will use the integration by parts results,* $\int e^{ax}\sin bx dx = \frac{e^{ax}}{a^2+b^2}(-b\cos bx + a\sin bx) + c$ *and* $\int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$

*Method 1:We write* $\frac{d}{dx}(\frac{dy}{dx} - 2y) - 3(\frac{dy}{dx} - 2y) = \sin x$. *Let* $z = \frac{dy}{dx} - 2y$ *so* $\frac{dz}{dx} - 3z = \sin x$. *Here* $I = e^{\int -3dx} = e^{-3x}$. *Now* $\frac{d}{dx}(e^{-3x}z) = e^{-3x}\sin x$ *or* $e^{-3x}z = \int e^{-3x}\sin x dx = \frac{e^{-3x}}{(-3)^2+1^2}(-1\cos x - 3\sin x) + A = -\frac{e^{-3x}}{10}(\cos x + 3\sin x) + A$ *so* $z = -\frac{1}{10}(\cos x + 3\sin x) + Ae^{3x} = \frac{dy}{dx} - 2y$.
*Now* $I = e^{\int -2dx} = e^{-2x}$ *and* $\frac{d}{dx}(e^{-2x}y) = -\frac{e^{-2x}}{10}(\cos x + 3\sin x) + Ae^x$. *So* $e^{-2x}y = -\frac{1}{10}\int e^{-2x}\cos x dx - \frac{3}{10}\int e^{-2x}\sin x dx + A\int e^x dx = -\frac{1}{10}\frac{e^{-2x}}{(-2)^2+1^2}(-2\cos x + 1\sin x) - \frac{3}{10}\frac{e^{-2x}}{(-2)^2+1^2}(-1\cos x - 2\sin x) + Ae^x + B = \frac{e^{-2x}}{50}(5\cos x + 5\sin x) + Ae^x + B$. *Finally* $y = \frac{1}{10}(\cos x + \sin x) + Ae^{3x} + Be^{2x}$

*Method 2: If we put* $y = Ae^{\alpha}x$ *in* $y'' - 5y' + 6y = 0$ *we get* $0 = A\alpha^2 e^{\alpha x} - 5A\alpha e^{\alpha x} + 6Ae^{\alpha x} = Ae^{\alpha x}(\alpha^2 - 5\alpha + 6) = Ae^{\alpha x}(\alpha - 2)(\alpha - 3)$ *so* $\alpha = 2, 3$.
*So* $y = Ae^{3x} + Be^{2x}$ *is the solution. (Note that if the two roots are coinciding this cannot be the solution, instead one can show that* $Ae^{\alpha x} + Bxe^{\alpha x}$ *is the solution).*
*To get the complete solution for* $y'' - 5y' + 6y = \sin x$ *we assume* $y = a\sin x + b\cos x$ *(it may not be possible to guess the solution like this always). So we get* $\sin x = 6(a\sin x + b\cos x) - 5(a\cos x - b\sin x) + (-a\sin x - b\cos x) = (5a + 5b)\sin x + (5b - 5a)\cos x$ *so we have* $5a + 5b = 1$ *and* $5b - 5a = 0$ *or* $a = b$. *So the solution is* $a = b = \frac{1}{10}$. *Which means* $y = \frac{1}{10}(\sin x + \cos x)$ *is the solution. Finally the complete solution is* $y = Ae^{3x} + Be^{2x} + \frac{1}{10}(\sin x + \cos x)$.

*Method 3: Using Wronskian(This method works even when the coefficients are not constants). This needs one solution, say* $u(x) = e^{2x}$ *to* $y'' - 5y' + 6y = 0$ *to start with. To get the other linearly independent solution* $v(x)$ *we solve* $W' + p(x)W = W' - 5W = 0$. *So* $\int \frac{W'}{W}dx = \int 5dx$ *or* $\ln W = 5x + \ln c$ *or* $W = ce^{5x}$. *But* $W =$

$uv' - vu' = e^{2x}v' - v2e^{2x} = ce^{5x}$. *We have* $v' - 2v = ce^{3x}$. *Here* $I = e^{\int -2dx} = e^{-2x}$ *so* $\frac{d}{dx}(e^{-2x}v) = ce^{3x}e^{-2x}$ *or* $e^{-2x}v = c\int e^x dx = ce^x + c_1$ *or* $v = ce^{3x} + c_1e^{2x}$. *We can let* $v(x) = e^{3x}$. *This means that the complete solution to* $y'' - 5y' + 6y = 0$ *is* $y = Au(x) + Bv(x) = Ae^{2x} + Be^{3x}$.
*Now to get the complete solution for* $y'' - 5y' + 6y = \sin x$, *we assume* $y = a(x)u(x) + b(x)v(x)$ *where* $a(x), b(x)$ *are functions to be determined. We have shown above that* $a'(x) = \frac{-v(x)r(x)}{W}$ *and* $b(x) = \frac{u(x)r(x)}{W}$. *Here* $r(x) = \sin x$ *and* $W = uv' - vu' = e^{2x}3e^{3x} - e^{3x}2e^{2x} = e^{5x}$. *Now* $a'(x) = \frac{-e^{3x}\sin x}{e^{5x}} = -e^{-2x}\sin x$ *or* $a(x) = -\int e^{-2x}\sin x dx = -\frac{e^{-2x}}{(-2)^2+1^2}(-1\cos x - 2\sin x) + A = \frac{e^{-2x}}{5}(\cos x + 2\sin x) + A$.
*Also* $b'(x) = \frac{e^{2x}\sin x}{e^{5x}} = e^{-3x}\sin x$ *or* $b(x) = \int e^{-3x}\sin x dx = \frac{e^{-3x}}{(-3)^2+1^2}(-1\cos x - 3\sin x) + B = -\frac{e^{-3x}}{10}(\cos x + 3\sin x) + B$. *Now the final solution is* $y = (\frac{e^{-2x}}{5}(\cos x + 2\sin x) + A)e^{2x} + (-\frac{e^{-3x}}{10}(\cos x + 3\sin x) + B)e^{3x} = \frac{1}{5}(\cos x + 2\sin x) + Ae^{2x} - \frac{1}{10}(\cos x + 3\sin x) + Be^{3x} = Ae^{2x} + Be^{3x} + \frac{1}{10}(\cos x + \sin x)$

**Note 16.**
*1. Note that the Wroskian is* $W = e^{5x} \neq 0$ *for any* $x$, *confirming the* $u(x) = e^{2x}$ *and* $v(x) = e^{3x}$ *are linearly independent, i.e. one is not a scalar multiplication of the other.*
*2. Note that it is correct to take linearly independent linear combination of* $e^{2x}$ *and* $e^{3x}$ *as* $u(x), v(x)$. *For example* $u(x) = 5e^{2x} + 7e^{3x}$ *and* $v(x) = e^{2x} - 4e^{3x}$.
*3. Any combination of* $e^{2x}$ *and* $e^{3x}$ *can be in* $y_p$. *For example* $y_p = \frac{1}{10}(\cos x + \sin x) + 6e^{2x} - e^{3x}$.
*4. To solve* $y'' - 5y' + 6y = \sin x, y(0) = 0, y'(0) = 0$.
*We know* $y(x) = Ae^{2x} + Be^{3x} + \frac{1}{10}(\cos x + \sin x)$ *so* $y(0) = A + B + \frac{1}{10} = 0$ *and* $y'(x) = 2Ae^{2x} + 3Be^{3x} + \frac{1}{10}(-\sin x + \cos x)$ *so* $y'(0) = 2A + 3B + \frac{1}{10} = 0$
*Then we have* $B = \frac{1}{10}$ *and* $A = -\frac{2}{10}$.
*Therefore the final solution is* $y = -\frac{1}{5}e^{2x} + \frac{1}{10}e^{3x} + \frac{1}{10}(\cos x + \sin x)$.
*5.Note that the Wronskian at* $0$ *is* $W(0) = e^{5(0)} = 1 \neq 0$ *has enabled us to find the coefficients* $A, B$ *in the above solution.*

**Example 46.**
*1. Show that* $u(x) = x$ *is a solution to* $y'' - xy' + y = 0$.
*2. Find the other linearly independent solution* $v(x)$ *and express it interms of well-known function and the complex error function given by* $\mathrm{erfi}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{t^2}dt$.
*3. Solve* $y'' - xy' + y = 1$ *and express the solution by well-known function and the error function.*
*4. Solve* $y'' - xy' + y = 1, y(1) = 2, y'(1) = 3$.

**Example 47.**
*Each of the following are well-known differential equations with a parameter* $n$ *which is a positive integer. Find the intervals of* $x$ *within which the solution exist. Select one* $n \geq 2$ *and show that the* $u(x)$ *given below is actually a solution. Also find the other linearly independent solution* $v(x)$.
*1. Legendre ODE:* $(1 - x^2)y'' - 2xy' + n(n+1)y = 0$;
*Legendre Polynomials:* $u(x) = P_n(x) = \frac{d^n}{dx^n}[(x^2 - 1)^n]$

*2. Laguerre ODE:* $xy'' + (1-x)y' + ny = 0$;
*Laguerre Polynomials:* $u(x) = L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}[e^{-x}x^n]$
*Also try to find solutions directly by assuming power series of the form* $u(x) = \sum_{k=0}^{\infty} a_k x^k$ *and finding* $a_k$. *What is the radius of convergence?*

**Example 48.** *Consider the ODE:* $y' + y = x, y(0) = 0$.
*1. Find the solution analytically and* $y(1)$ *in decimal.*
*2. Consider a Numerical Solution by the Euler's Method. Let* $h$ *be the width of a subdivision of* $[0,1], x_k = kh$ *and* $y_k$ *is an approximation to* $y(x_k)$. *Consider the 1st order Taylor Series expansion of* $y(x_k + h)$ *at* $x_k$ *and show that the formula* $y_(k+1) = y_k + h(kh - y_k)$ *can be used for generating* $y_k$. *Use this formula to approximate the value of* $y(1)$ *when* $h = 0.1$.
*3. With the same setting as above consider the 2nd order Taylor Series expansion of* $y(x_k+h)$ *at* $x_k$ *and show that the formula* $y_(k+1) = y_k + h(kh - y_k) + \frac{h^2}{2}(1-(kh-y_k))$ *can be used for generating* $y_k$. *Use this formula to approximate* $y(1)$ *when* $h = 0.1$.

**Mathematica 8.**
$\mathrm{DSolve}[\{\mathrm{y}''[\mathrm{x}] + \mathrm{p}[\mathrm{x}]\mathrm{y}'[\mathrm{x}] + \mathrm{q}[\mathrm{x}]\mathrm{y}[\mathrm{x}] == \mathrm{r}[\mathrm{x}], \mathrm{y}[\mathrm{a}] == \mathrm{b}, \mathrm{y}'[\mathrm{a}] == \mathrm{c}\}, \mathrm{x}, \mathrm{y}[\mathrm{x}]]$