

RIEMANN INTEGRAL

Example: Use n equal partitions of $[0,1]$ to estimate the “area” under the curve $f(x) = x^2$ using

1. left corner of the intervals
2. right corner of the intervals
3. midpoint of the interval
4. line joining the left and right corners of the interval

Definitions:

P is a **partition** of $[a, b]$ iff it is an ordered set of the form $P = \{x_0, x_1, \dots, x_n\}$ with $x_0 = a, x_n = b$ and $x_{k+1} > x_k$

P^* is a **refinement** of P iff $P^* \supseteq P$

P is a **common refinement** of P_1, P_2 iff $P = P_1 \cup P_2$

$\mathcal{P}[a, b]$ is the set of all partitions of $[a, b]$

Definition: Upper and Lower Riemann Sums $f: [a, b] \rightarrow \mathbb{R}$ is a bounded function, $\Delta x_k = x_{k+1} - x_k$

$U(P, f) = \sum_{k=0}^{n-1} M_k \Delta x_k$ where $M_k = \sup\{f(x) \mid x \in [x_k, x_{k+1}]\}$

$L(P, f) = \sum_{k=0}^{n-1} m_k \Delta x_k$ where $m_k = \inf\{f(x) \mid x \in [x_k, x_{k+1}]\}$

Definition: Upper and Lower Riemann Integrals

$\int_a^b f(x) dx = \inf \{U(P, f) \mid P \in \mathcal{P}[a, b]\}$

$\int_a^b f(x) dx = \sup \{L(P, f) \mid P \in \mathcal{P}[a, b]\}$

Definition:

f is **Riemann Integrable** on $[a, b]$ or $f \in \mathcal{R}[a, b]$ iff $\int_a^b f(x) dx = \overline{\int_a^b f(x) dx}$

Riemann Integral of f is the common value denoted by $\int_a^b f(x) dx$

Theorem: P^* is a refinement of P

1. $L(P, f) \leq L(P^*, f)$
2. $U(P^*, f) \leq U(P, f)$

Theorem: $\int_a^b f(x) dx \leq \overline{\int_a^b f(x) dx}$

Theorem: $f \in \mathcal{R}[a, b]$ iff $\forall \varepsilon > 0 \exists P \in \mathcal{P}[a, b]; U(P, f) - L(P, f) < \varepsilon$

Theorem: If $f \in \mathcal{R}[a, b]$ and $P \in \mathcal{P}[a, b]$ such that $t_i \in [x_{i-1}, x_i]$ then

$\left| \sum_{i=1}^n f(t_i) \Delta x_i - \int_a^b f(x) dx \right| < U(P, f) - L(P, f)$

Theorem: $f \in \mathcal{C}[a, b] \Rightarrow f \in \mathcal{R}[a, b]$

Theorems: $f, g \in \mathcal{R}[a, b]$

1. $f + g \in \mathcal{R}[a, b]$ and $\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$
2. $fg \in \mathcal{R}[a, b]$
3. $|f| \in \mathcal{R}[a, b]$ and $\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$
4. $f \leq g \Rightarrow \int_a^b f(x) dx \leq \int_a^b g(x) dx$
5. $f \leq M \Rightarrow \int_a^b f(x) dx \leq M(b - a)$
6. $c \in [a, b] \Rightarrow f \in \mathcal{R}[a, c], f \in \mathcal{R}[c, b]$ and $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$

Definition: $f(x)$ is **Uniformly continuous** on $I \subset \mathbb{R}$

$\forall \varepsilon > 0, \exists \delta > 0, \forall x_1, x_2 \in I; |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$

Definition: $f(x)$ is **Lipschitz continuous** on $I \subset \mathbb{R}$

$\exists L > 0, \forall x_1, x_2 \in I; |f(x_1) - f(x_2)| \leq L|x_1 - x_2|$

Theorem: Lipschitz continuous \Rightarrow Uniformly continuous \Rightarrow Continuous

Example: Show that $\frac{1}{x}$ is not uniformly continuous on $(0,1]$ but x^2 is.

Theorem: Fundamental Theorem of Calculus

If $f \in \mathcal{R}[a, b]$ and there is a differentiable function F such that $F' = f$ then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Theorem: Second Fundamental Theorem of Calculus

If $f \in \mathcal{R}[a, b]$ and $x \in [a, b]$ and $F(x) = \int_a^x f(x)dx$ then

1. F is continuous on $[a, b]$.
2. If f is continuous at a point $x_0 \in [a, b]$ then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Theorem: Integration by Parts

F, G differentiable on $[a, b]$, $F' = f \in \mathcal{R}[a, b]$ and $G' = g \in \mathcal{R}[a, b]$ then

$$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx$$

Theorem: Change of Variable

g has continuous derivative g' on $[c, d]$. f is continuous on $g([c, d])$ and let $F(x) = \int_{g(c)}^x f(t)dt, x \in g([c, d])$. Then for each $x \in [c, d]$, $\int_c^x f(g(t))g'(t)dt$ exists and has value $F(g(x))$.

Theorem: Mean Value Theorem for Integrals

$f \in \mathcal{R}[a, b]$ with $m \leq f \leq M$. Then $\exists c \in [a, b]$ such that $\int_a^b f(x)dx = c(b - a)$.

If also $f \in \mathcal{C}[a, b]$ then $\exists x_0 \in (a, b)$ such that $\int_a^b f(x)dx = f(x_0)(b - a)$.

Definition: Improper Integrals of the first kind

Suppose $\int_a^b f(x)dx$ exists for each $b \geq a$.

If $\lim_{b \rightarrow \infty} \int_a^b f(x)dx$ exists and equal to $I \in \mathbb{R}$ we say that $\int_a^\infty f(x)dx$ converges and has value I

Otherwise we say that $\int_a^\infty f(x)dx$ diverges

Definition: Improper Integrals

$$\int_a^\infty f(x)dx = \lim_{b \rightarrow \infty} \int_a^b f(x)dx, f: [a, \infty) \rightarrow \mathbb{R}$$

$$\int_{-\infty}^b f(x)dx = \lim_{a \rightarrow -\infty} \int_a^b f(x)dx, f: (-\infty, b] \rightarrow \mathbb{R}$$

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx, f: (-\infty, \infty) \rightarrow \mathbb{R}, c \in \mathbb{R}$$

$$\int_{a^+}^b f(x)dx = \lim_{t \rightarrow a^+} \int_t^b f(x)dx, f: (a, b] \rightarrow \mathbb{R}$$

$$\int_a^{b^-} f(x)dx = \lim_{t \rightarrow b^-} \int_a^t f(x)dx, f: [a, b) \rightarrow \mathbb{R}$$

$$\int_a^b f(x)dx = \int_a^{c^-} f(x)dx + \int_{c^+}^b f(x)dx, f: [a, c) \cup (c, b] \rightarrow \mathbb{R}, c \in (a, b)$$

Example: Find $\int_{-1}^1 \frac{1}{x^2} dx$ if it exists

Example:

Prove that if f is bounded above and increasing, then $\lim_{x \rightarrow \infty} f(x)$ is existing and finite

Prove that $\int_a^\infty |f(x)|dx$ converges $\Rightarrow \int_a^\infty f(x)dx$ converges

Prove that if $|f(x)| \leq Me^{ax}$, then the **Laplace Transform** of $f(x)$, $\bar{f}(s) = \int_0^\infty e^{-sx} f(x)dx$ exists for all $s > a$.

Theorem: Comparison Test

Assume that the proper integral $\int_a^b f(x)dx$ exists for each $b \geq a$ and suppose that $0 \leq f(x) \leq g(x)$

for all $x \geq a$, then $\int_a^\infty g(x)dx$ converges $\Rightarrow \int_a^\infty f(x)dx$ converges

Theorem: Limit Comparison Test

Assume both proper integrals $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ exist for each $b \geq a$, where $f(x) \geq 0$ and $g(x) > 0$

If $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = c$, then

1. $c \neq 0, \infty \Rightarrow \int_a^\infty f(x)dx$ converges $\Leftrightarrow \int_a^\infty g(x)dx$ converges
2. $c = 0$ and $\int_a^\infty g(x)dx$ converges $\Rightarrow \int_a^\infty f(x)dx$ converges
3. $c = \infty$ and $\int_a^\infty g(x)dx$ diverges $\Rightarrow \int_a^\infty f(x)dx$ diverges

Note: There are similar comparison tests for other improper integrals

Example: Gamma Function is defined by $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$. Show that

1. $\Gamma(x)$ exists for all $x > 0$
2. $\Gamma(x) = (x - 1)\Gamma(x - 1)$
3. $\Gamma(n) = (n - 1)!$ for integer $n \geq 1$
4. we can use 2. to define $\Gamma(x)$ for $x < 0$
5. $\Gamma(x)$ does not exist for $x = 0, -1, -2, -3, \dots$
6. Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and $\int_0^\infty e^{-t^2} dt = \sqrt{\pi}$ using $\Gamma(x)\Gamma(1 - x) = \frac{\pi}{\sin \pi x}$
7. Use the formula for the the n dimensional ball $V_n(r) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} r^n$ to find volumes of 2,3,4,5 dimensional balls
8. Use the fact that $\Gamma(x + 1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x$ asymptotically as $t \rightarrow \infty$ to find 10! approximately
9. What is $-\Gamma'(1)$? It is called the Euler Constant γ and no one knows if it is rational or irrational!

Prove that the **Beta function** $B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} dt$ exists for all $x, y > 0$. It can be shown that

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

MULTIVARIATE CALCULUS**Definition: Function of two variables** $f: A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ **Example:** Draw the graphs of the following functions/surfaces

1. $f(x, y) = x^2 + y^2$
2. $f(x, y) = \sqrt{x^2 + y^2}$
3. $\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 1$

Definition: Limit

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L \Leftrightarrow \forall \varepsilon > 0 \exists \delta > 0, 0 < d((x, y), (a, b)) < \delta \Rightarrow |f(x, y) - L| < \varepsilon$$

Note: Matric
 $0 < d((x, y), (a, b)) < \delta$ is a region around and excluding (a, b) . Some options for the matric d are

1. $\sqrt{(x-a)^2 + (y-b)^2}$
2. $|x-a| + |y-b|$
3. $\max\{|x-a|, |y-b|\}$

We will use the first matric. One can show that they are equivalent, what is needed is a region around (a, b) .**Example:** Use the definition to show that $\lim_{(x,y) \rightarrow (1,2)} x^2 y = 6$ **Example:** Investigate the existence of the limit, $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ for the following functions

1. $f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & , (x, y) \neq (0,0) \\ 0 & , (x, y) = (0,0) \end{cases}$
2. $f(x, y) = \begin{cases} \frac{x^2 y^2}{x^2 y^2 + (x-y)^2} & , (x, y) \neq (0,0) \\ 0 & , (x, y) = (0,0) \end{cases}$
3. $f(x, y) = \begin{cases} x \sin \frac{1}{y} & , (x, y) \neq (0,0) \\ 0 & , (x, y) = (0,0) \end{cases}$

Theorem:If $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$, $\lim_{x \rightarrow a} f(x, y)$ and $\lim_{y \rightarrow b} f(x, y)$ exists then

$$\lim_{x \rightarrow a} \lim_{y \rightarrow b} f(x, y) = \lim_{y \rightarrow b} \lim_{x \rightarrow a} f(x, y) = L.$$

Example:Use the above theorem to prove that $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ is not existing for $f(x, y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2} & , (x, y) \neq (0,0) \\ 0 & , (x, y) = (0,0) \end{cases}$.Prove by definition that if $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ along $y = x$ and $y = 2x$ are different, then the limit is not existing.**Definition: Continuity** of $f (f \in \mathcal{C})$ at (a, b)

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b)$$

Definition: Partial derivatives

$$f_x(a, b) = f_1(a, b) = \frac{\partial f}{\partial x}(a, b) = \lim_{x \rightarrow a} \frac{f(x, b) - f(a, b)}{x - a} = \lim_{\Delta x \rightarrow 0} \frac{f(a + \Delta x, b) - f(a, b)}{\Delta x}$$

$$f_y(a, b) = f_2(a, b) = \frac{\partial f}{\partial y}(a, b) = \lim_{y \rightarrow b} \frac{f(a, y) - f(a, b)}{y - b} = \lim_{\Delta y \rightarrow 0} \frac{f(a, b + \Delta y) - f(a, b)}{\Delta y}$$

Definition: $f \in \mathcal{C}^1 \Leftrightarrow f_x \in \mathcal{C}$ and $f_y \in \mathcal{C}$ **Theorem: Mean Value**

1. f_x and f_y exists
2. $\mathbb{D} = \{(x, y) | (x - a)^2 + (y - b)^2 < \delta^2\} \subset A$
3. $\Delta x^2 + \Delta y^2 < \delta^2$

Then

1. $f(a + \Delta x, b + \Delta y) - f(a, b) = \Delta x f_x(a + \theta \Delta x, b) + \Delta y f_y(a + \Delta x, b + \alpha \Delta y)$
2. $0 < \theta, \alpha < 1$

Definition: Differentiability of f ($f \in \mathcal{D}$) at (a, b)

- f_x and f_y exists at (a, b)
- $f(a + \Delta x, b + \Delta y) - f(a, b) = \Delta x f_x(a, b) + \Delta y f_y(a, b) + \Delta x \phi(\Delta x, \Delta y) + \Delta y \psi(\Delta x, \Delta y)$ for all $\Delta x^2 + \Delta y^2 < \delta^2$
- $\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \phi(\Delta x, \Delta y) = \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \psi(\Delta x, \Delta y) = 0$

Theorem: $f \in \mathcal{C}^1 \Rightarrow f \in \mathcal{D} \Rightarrow f \in \mathcal{C}$

Example: Let $f(x, y) = g(\sqrt{x^2 + y^2}), g(x) = x \sin \frac{1}{x}, g(0) = 0$. Show that $f \in \mathcal{D}$ but $f \notin \mathcal{C}^1$

Definition: Higher order derivatives

$$\begin{aligned} f_{xx} &= (f_x)_x = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} \\ f_{xy} &= (f_x)_y = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} \\ f_{yx} &= (f_y)_x = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} \\ f_{yy} &= (f_y)_y = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} \text{ and so on} \end{aligned}$$

Note:

- We write $f \in \mathcal{C}^2$ to mean $f_{xx}, f_{xy}, f_{yx}, f_{yy} \in \mathcal{C}$
- In a similar manner we write $f \in \mathcal{C}^n$ to mean that all the n th order partial derivatives are continuous. There are 2^n of them.
- There are $\binom{n}{m} = \frac{n!}{m!(n-m)!}$, n th order partial derivatives that contains x, m times.

Example: Let

$$f(x, y) = \begin{cases} xy \frac{x^2 - y^2}{x^2 + y^2} & , (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$$

Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

Theorem: $f \in \mathcal{C}^2 \Rightarrow f_{xy} = f_{yx}$

Example: If $u = u(x, y) \in \mathcal{C}^2$ then prove that the **Laplace operator** $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ becomes

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \text{ when } x = r \cos \theta, y = r \sin \theta.$$

Theorem: Chain rule

- $f = f(x, y), y = y(t), x = x(t)$ all in \mathcal{C}^1 . Then $\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$
- $f = f(x, y), y = y(u, v), x = x(u, v)$ all in \mathcal{C}^1 . Then $\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u}$ and $\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}$

Note: The above may be written as

$$\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \right) \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \frac{\partial f}{\partial(x,y)} \frac{\partial(x,y)}{\partial t} \text{ and } \frac{\partial f}{\partial(u,v)} = \left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \right) \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{\partial f}{\partial(x,y)} \frac{\partial(x,y)}{\partial(u,v)}$$

The determinant, $\det \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ is called the **Jacobian** or J

With $\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\underline{u} = \begin{pmatrix} u \\ v \end{pmatrix}$, the above may also be written as

$$(f \circ \underline{x})'(t) = (f' \circ \underline{x})(t) \underline{x}'(t) \text{ and } (f \circ \underline{x})'(\underline{u}) = (f' \circ \underline{x})(\underline{u}) \underline{x}'(\underline{u})$$

We also see that $\left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \right) = \frac{\partial f}{\partial(x,y)} = f'(\underline{x})$ is acting as the true first derivative of $f = f(x, y)$. Therefore it is also called $\nabla f = \text{grad} f$ or the **Gradient** of f .

Example: Assume all functions are C^1

Show that if $x = x(u, v), y = y(u, v), u = u(r, s), v = v(r, s)$ then $\frac{\partial(x,y)}{\partial(r,s)} = \frac{\partial(x,y)}{\partial(u,v)} \frac{\partial(u,v)}{\partial(r,s)}$.

Show that if $u = f(x, y), v = g(x, y)$ then a functional relation of the form $h(u, v) = 0$ exists iff $\det \frac{\partial(u,v)}{\partial(x,y)} \equiv 0$.

Definition: Directional Derivative of f in the direction of the unit vector $\underline{u} = (u, v)$ at (a, b) .

$$D_{\underline{u}}f(a, b) = \lim_{\Delta t \rightarrow 0} \frac{f(a+u\Delta t, b+v\Delta t) - f(a, b)}{\Delta t}$$

Theorem: $f \in C^1, \nabla f(a, b) \neq \underline{0}$

1. $D_{\underline{u}}f(a, b) = \frac{\partial f}{\partial x}(a, b)u + \frac{\partial f}{\partial y}(a, b)v = \nabla f(a, b) \cdot \underline{u}$
2. $\max_{\underline{u}} D_{\underline{u}}f(a, b) = D_{\nabla f(a,b)}f(a, b) = \|\nabla f(a, b)\|$
3. $\min_{\underline{u}} D_{\underline{u}}f(a, b) = D_{-\nabla f(a,b)}f(a, b) = -\|\nabla f(a, b)\|$

Theorem: Normal vector to a surface at (a, b)

$$\underline{n}(a, b) = (f_x(a, b), f_y(a, b), -1) = (\nabla f(a, b), -1)$$

Proof: Let $\underline{r} = \underline{r}(t) = (x(t), y(t), z(t)) \in C^1$ be a curve on the surface of $z = f(x, y) \in C^1$

and $\underline{r}(t_0) = (x(t_0), y(t_0), z(t_0)) = (a, b, f(a, b))$.

Note that $\underline{r}'(t_0) = (x'(t_0), y'(t_0), z'(t_0))$ is the tangent vector to the curve at (a, b) .

Now $\underline{n}(a, b) \cdot \underline{r}'(t_0) = f_x(a, b)x'(t_0) + f_y(a, b)y'(t_0) - z'(t_0) = \frac{df}{dt}(t_0) - z'(t_0) = 0$

ie $\underline{n}(a, b) = (f_x(a, b), f_y(a, b), -1) = (\nabla f(a, b), -1)$ is a vector perpendicular to the surface $z = f(x, y)$ at (a, b) .

Theorem: Equation of the **tangent plane** to the surface $z = f(x, y) \in C^1$ at (a, b) .

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) = \nabla f(a, b) \begin{pmatrix} x - a \\ y - b \end{pmatrix} = \nabla f(a, b) \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix} \right)$$

Example: Let $f(x, y) = x^4 + y^4 - x^2 - y^2 + 1$. At the point $(1, 2)$ find

1. Direction in which the function increases most rapidly
2. Directional derivative in that direction
3. Equation of the tangent plane.

Theorem: Taylor's expansion for one variable $f: I \in \mathbb{R} \rightarrow \mathbb{R}$

If $f \in C^{n+1}$ and $a, a + h \in I$

$$\text{then } f(a + h) = \sum_{m=0}^n \frac{1}{m!} \frac{d^m f}{dx^m}(a) h^m + \frac{1}{(n+1)!} \frac{d^{n+1} f}{dx^{n+1}}(c) h^{n+1}$$

where c is between a and $a + h$.

Note: We can also write the above as

If $f \in C^{n+1}$ and $a + th \in I$ for all $t \in [0, 1]$

$$\text{Then } f(a + h) = \sum_{m=0}^n \frac{1}{m!} \left(h \frac{d}{dx} \right)^m f(a) + \frac{1}{(n+1)!} \left(h \frac{d}{dx} \right)^{n+1} f(c)$$

for some $c = a + \theta h$ with $\theta \in (0, 1)$.

We agree to use the notation $\left(h \frac{d}{dx} \right)^m f(a) \equiv h^m \frac{d^m f}{dx^m}(a)$

Note: The first two terms are the equation of the tangent line.

Proof: Use generalized mean value theorem on

$$F(t) = \sum_{m=0}^n \frac{1}{m!} f^{(m)}(t)(x - t)^m \text{ and } G(t) = (x - t)^{n+1}$$

Example: When $n = 1$

$$f(a + h) = f(a) + \frac{1}{1!} f'(a)h + \frac{1}{2!} f''(c)h^2$$

Example: Write the Taylor's expansion for $f(x) = e^x$ at $a = 0$.

Example: Derive the second derivative test to find the extrema of $f(x)$. What to do when $f''(a) = 0$?

Theorem: Taylor's for two variables $f: A \subset \mathbb{R}^2 \rightarrow \mathbb{R}$

$f \in C^{n+1}$ and $(a + th, b + tk) \in A$ for all $t \in [0,1]$

Then $f(a + h, b + k) = \sum_{m=0}^n \frac{1}{m!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^m f(a, b) + \frac{1}{(m+1)!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^{m+1} f(\mathbf{c})$

for some $\mathbf{c} = (a + \theta h, b + \theta k)$ with $\theta \in (0,1)$.

Proof: Use Taylor's expansion for $F(t) = f(a + th, b + tk)$

Example: When $n = 1$

$$\begin{aligned} & f(a + h, b + k) \\ &= \sum_{m=0}^1 \frac{1}{m!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^m f(a, b) + \frac{1}{(1+1)!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^{1+1} f(a + \theta h, b + \theta k) \\ &= f(a, b) + \frac{1}{1!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) f(a, b) + \frac{1}{2!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^2 f(\mathbf{c}) \\ &= f(a, b) + \frac{1}{1!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) f(a, b) + \frac{1}{2!} \left(h^2 \frac{\partial^2}{\partial x^2} + 2hk \frac{\partial^2}{\partial x \partial y} + h^2 \frac{\partial^2}{\partial y^2} \right) f(\mathbf{c}) \\ &= f(a, b) + f_x(a, b)h + f_y(a, b)k + \frac{1}{2!} (f_{xx}(\mathbf{c})h^2 + 2f_{xy}(\mathbf{c})hk + f_{yy}(\mathbf{c})k^2) \\ &= f(a, b) + (f_x(a, b) \quad f_y(a, b)) \begin{pmatrix} h \\ k \end{pmatrix} + \frac{1}{2!} (h \quad k) \begin{pmatrix} f_{xx}(\mathbf{c}) & f_{xy}(\mathbf{c}) \\ f_{yx}(\mathbf{c}) & f_{yy}(\mathbf{c}) \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} \\ &= f(a, b) + \nabla f(a, b) \begin{pmatrix} h \\ k \end{pmatrix} + \frac{1}{2!} (h \quad k) Hf(\mathbf{c}) \begin{pmatrix} h \\ k \end{pmatrix} \\ &= f(a, b) + \frac{1}{1!} f'(a, b) \begin{pmatrix} h \\ k \end{pmatrix} + \frac{1}{2!} (h \quad k) f''(\mathbf{c}) \begin{pmatrix} h \\ k \end{pmatrix} \end{aligned}$$

Note: The first two terms are the equation of the tangent plane.

Definition: $f'' = Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$: **Hessian** of f

$\det Hf = f_{xx}f_{yy} - f_{xy}^2$: determinant

$\text{tr} Hf = f_{xx} + f_{yy}$: **trace**

Note: $\det Hf > 0$ and $f_{xx} > 0 (< 0) \Rightarrow f_{yy} > 0 (< 0) \Rightarrow \text{tr} Hf > 0 (< 0)$

Example:

Write the Taylor's expansion for $f(x, y) = e^{xy}$ and $f(x, y) = \sin(\sin x + xe^y)$ at $(a, b) = (0, 0)$.

Get the same answer by applying multiple one variable Taylor series expansions at 0.

Definition: (a, b) is a **critical point** of $f \in C^1 \Leftrightarrow \nabla f(a, b) = \mathbf{0}$ or f is not defined

Definition:

1. f has a **relative maximum** at $(a, b) \Leftrightarrow f(a, b) \geq f(a + h, b + k)$ in a neighbourhood of (a, b)
2. f has a **relative minimum** at $(a, b) \Leftrightarrow f(a, b) \leq f(a + h, b + k)$ in a neighbourhood of (a, b)
3. f has a **saddle point** at $(a, b) \Leftrightarrow f$ is both above and below its tangent plane at (a, b) .

Theorem: $f \in C^1$ and (a, b) is a relative maximum/minimum/saddle point of $f \Rightarrow \nabla f(a, b) = \mathbf{0}$

Theorem: $f \in C^2$ and $\nabla f(a, b) = \mathbf{0}$ then

1. $\det Hf(a, b) > 0$ and $\text{tr} Hf(a, b) > 0$ then (a, b) is a relative minimum
2. $\det Hf(a, b) > 0$ and $\text{tr} Hf(a, b) < 0$ then (a, b) is a relative maximum
3. $\det Hf(a, b) < 0$ then (a, b) is a saddle point
4. $\det Hf(a, b) = 0$ inconclusive (why?)

Example: Find the critical points and determine the nature of them (relative maxima/minima/saddle points).

$$f(x, y) = x^3 - 12x + y^3 - 27y + 5$$

$$f(x, y) = x^4 + y^4 - x^2 - y^2 + 1$$

$$f(x, y) = x^4 + y^4$$

Example: Propose a method to determine the nature of critical points when $\det Hf = 0$.

Theorem: Lagrange Multipliers

If $f, g \in C^1$ and $\nabla g \neq \mathbf{0}$ then the maxima/minima of $f(x, y)$ subjected to $g(x, y) = 0$ are included in the set of solutions of $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$.

Example:

Find the shortest distance from the point $(1, 0)$ to the parabola $y^2 = 4x$.

Find the directions of the axes of the ellipse $5x^2 - 6xy + 5y^2 - 4x - 4y - 4 = 0$.

Find the absolute maximum/minimum of $f(x, y) = x^4 + y^4 - x^2 - y^2 + 1$ on the closed disk $(x - 1)^2 + y^2 \leq 4$.