Example 1. *Find the Continued Fraction Expansions for* $\sqrt{2}, \pi, e$ *and the Golden Ratio which is the positive root of* $\phi^2 - \phi - 1 = 0$.

Definition 1. *Set of Real numbers* $\mathbb{R}$ *is a set satisfying*
1.Field Axioms 2.Order Axioms 3.Completeness Axiom

Axiom 1. *Field Axioms.*
$\mathbb{R}$ *is a non empty set with binary operations* $+$ *and* $.$ *satisfying the following properties*

1. $\forall a, b \in \mathbb{R}; a + b \in \mathbb{R}$: *closed under addition*
2. $\forall a, b, c \in \mathbb{R}; a + (b + c) = (a + b) + c$: *addition is associative*
3. $\exists 0 \in \mathbb{R}, \forall a \in \mathbb{R}; a + 0 = 0 + a = a$: *additive identity exists*
4. $\forall a \in \mathbb{R}, \exists -a \in \mathbb{R}; a + (-a) = (-a) + a = 0$: *additive inverse exists*
5. $\forall a, b \in \mathbb{R}; a + b = b + a$: *addition is commutative*
6. $\forall a, b \in \mathbb{R}; a.b \in \mathbb{R}$: *closed under multiplication*
7. $\forall a, b, c \in \mathbb{R}; a.(b.c) = (a.b).c$: *multiplication is associative*
8. $\exists 1 \in \mathbb{R}, \forall a \in \mathbb{R}; a.1 = 1.a = a$: *multiplicative identity exists*
9. $\mathbb{R} - \{0\} \neq \emptyset$ *and* $\forall a \in \mathbb{R} - \{0\}, \exists a^{-1} \in \mathbb{R}; a.a^{-1} = a^{-1}.a = 1$: *multiplicative inverse exists*
10. $\forall a, b \in \mathbb{R}; a.b = b.a$: *multiplication is commutative*
11. $\forall a, b, c \in \mathbb{R}; a.(b + c) = (a.b) + (a.c)$: *multiplication is distributive over addition*

Definition 2.
$a - b = a + (-b)$: *Subtraction*
If $b \neq 0, \frac{a}{b} = a.b^{-1}$: *Division*

Definition 3.

1. *We write* $1 + 1 = 2, 2 + 1 = 3, 3 + 1 = 4$ *and so on.*
2. *Set of Positive Integers* $\mathbb{Z}^+ = \{1, 2, 3, \cdots\}$
3. *Set of Natural Numbers* $\mathbb{N} = \mathbb{Z}^+ \cup \{0\}$
4. *Set of Negative Integers* $\mathbb{Z}^- = \{-a | a \in \mathbb{Z}^+\}$
5. *Set of Integers* $\mathbb{Z} = \mathbb{Z}^- \cup \{0\} \cup \mathbb{Z}^+$
6. *Set of Rational Numbers* $\mathbb{Q} = \{\frac{p}{q} | p, q \in \mathbb{Z}$ *and* $q \neq 0\}$
7. *Set of Irrational Numbers* $\mathbb{Q}^c = \mathbb{R} - \mathbb{Q}$
8. *If* $a, b \in \mathbb{Z}$ *we say* a *divides* b *or* a *is a factor of* b *and write* $a|b$ *iff* $\frac{b}{a} \in \mathbb{Z}$
9. $p \in \mathbb{Z}^+ - \{1\}$ *is a Prime Number iff* 1 *and* p *are its only factors.*

Example 2. *Any set of two or more elements with two binary operations satisfying the fields axioms is called a Field. See if the following are fields*

1. $\mathbb{R}$ *with* $.$ *and* $+$
2. $\mathbb{Z}, \mathbb{Q}, \mathbb{Q}^c$ *with* $+$ *and* $.$
3. $\{0, 1, 2\}$ *with* $\mod 3$ *arithmetic*
4. $\{0, 1, 2, 3\}$ *with* $\mod 4$ *arithmetic*

Theorem 1.

1. *There are infinitely many prime numbers.*
2. *Every* $n \in \mathbb{Z}^+ - \{1\}$ *is a prime number or a unique product of prime numbers*
3. *Gaps between prime numbers can be arbitrary large.*
4. $\{0, 1, 2, \cdots, n-1\}$ *with* $\mod n$ *arithmetic is a field iff* n *is prime.*

Definition 4. *Integer Powers*
If $a \neq 0, a^0 = 1$
If $a \neq 0, n \in \mathbb{Z}^+$ *then* $a^n = a.a^{n-1}$
If $a \neq 0, n \in \mathbb{Z}^+$ *then* $a^{-n} = (a^{-1})^n$

Example 3. *Prove the following with* $a, b, c \in \mathbb{R}$

1. *If* $a + b = 0$ *then* $b = -a$
2. *If* $a + c = b + c$ *then* $a = b$
3. $-(a + b) = (-a) + (-b)$
4. $-(-a) = a$
5. $a.0 = 0$
6. $0, 1, -a, a^{-1}$ *are unique*
7. *If* $a \neq 0$ *and* $ab = 1$ *then* $b = a^{-1}$
8. *If* $ac = bc$ *and* $c \neq 0$ *then* $a = b$
9. *If* $ab = 0$ *then* $a = 0$ *or* $b = 0$
10. $-(ab) = (-a)b = a(-b)$
11. $(-a)(-b) = ab$
12. *If* $a \neq 0, (a^{-1})^{-1} = a$
13. *If* $a, b \neq 0, (ab)^{-1} = a^{-1}b^{-1}$
14. *If* $a \neq 0$ *and* $m, n \in \mathbb{Z}$ *then* $a^m a^n = a^{m+n}$
15. *If* $a, b \neq 0, n \in \mathbb{Z}, (ab)^n = a^n b^n$

Axiom 2. *Order Axioms*
$\mathbb{R}$ *has a Order* $<$ *satisfying the following.*

12. $\forall a, b \in \mathbb{R}$;*exactly one of* $a = b, a < b, b < a$ *holds: Trichotomy*
13. $\forall a, b, c \in \mathbb{R}; a < b$ *and* $b < c$ *implies* $a < c$*: Transitivity*
14. $\forall a, b, c \in \mathbb{R}; a < b$ *implies* $a + c < b + c$*: operations with addition*
15. $\forall a, b \in \mathbb{R}; a < b$ *and* $0 < c$ *implies* $ac < bc$*: operations with multiplication*

Definition 5.
$b > a$ *is same as* $a < b$
$a \leq b$ *means* $a < b$ *or* $a = b$
Above follows that $a \neq b$ *is either* $a < b$ *or* $a > b$.

Definition 6. *Absolute Value* $|a| = a$ *if* $a \geq 0$ *and* $-a$ *if* $a < 0$

Example 4.

1. $\forall a, b \in \mathbb{R}; a < b$ *and* $c < 0$ *implies* $ac > bc$
2. $1 > 0$
3. $a > 0$ *iff* $a^{-1} > 0$
4. *If* $a < b$ *and* $c < d$ *then* $a + c < b + d$
5. *If* $0 < a < b$ *and* $0 < c < d$ *then* $ac < bd$
6. *See if* $|$ *defines an order in* $\mathbb{Z}$
7. $|a| \leq r$ *iff* $-r \leq a \leq r$
8. $a^2 \geq 0$
9. $|ab| = |a||b|$
10. $|a| - |b| \leq |a + b| \leq |a| + |b|$
11. $||a| - |b|| \leq |a - b|$
12. $|a + b|^2 + |a - b|^2 = 2|a|^2 + 2|b|^2$

Definition 7. *Let* A *be a non-empty subset of* $\mathbb{R}$*. Then*

1. *Upper Bound of* A*:* $u \in \mathbb{R}$ *such that* $\forall a \in A; a \leq u$
2. *Bounded Above: An upper bound exists*
3. *Maximum(largest) element of* A*:* $\max A = u \in A$ *and* u *is an upper bound of* A
4. *Lower Bound of* A*:* $\ell \in \mathbb{R}$ *such that* $\forall a \in A; \ell \leq a$
5. *Bounded Below: A lower bound exists*
6. *Minimum(least) element of* A*:* $\min A = \ell \in A$ *and* ℓ *is a lower bound of* A

7. *Supremum of A:* $\sup A$=*least upper bound of A.*
   or equivalently: If u is an upper bound then $\sup A \leq u$
   or equivalently: if $u < \sup A$ *then u is not an upper bound of A.*

8. *Infimum of A:* $\inf A$=*largest lower bound of A.*
   or equivalently: If ℓ *is a lower bound then* $\inf A \geq \ell$
   or equivalently: if $\ell > \inf A$ *then* ℓ *is not a lower bound of A.*

9. *Bounded: bounded above and bounded below*

Axiom 3. *Completeness Axiom.*

1. *Every non-empty subset of* $\mathbb{R}$ *which is bounded above has a supremum.*

2. *Every non-empty subset of* $\mathbb{R}$ *which is bounded below has a infimum*

Definition 8. *Real Intervals,* $a < b$

1. $(a, b) = \{x \in \mathbb{R} | a < x < b\}$*: Open interval*

2. $(a, b] = \{x \in \mathbb{R} | a < x \leq b\}$*: half open/closed interval*

3. $[a, b) = \{x \in \mathbb{R} | a \leq x < b\}$*: half open/closed interval*

4. $[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$*: Closed interval*

Example 5. *Assume that* $A, B \subset \mathbb{R}$ *are non-empty subsets.*

1. *Prove that* $\sup(a, b) = b$ *and* $\inf(a, b) = a$.

2. *Show that* $\mathbb{Z}$ *is unbounded.*

3. *Show that for every* $a \in \mathbb{R}$ *there is* $n \in \mathbb{Z}$ *such that* $n > a$.

4. *Prove the existence of* $\inf$ *using the existence of* $\sup$ *with suitable conditions.*

5. *Show that* $\forall a \in A, \forall b \in B; a < b \Rightarrow \sup A \leq \sup B$.

6. *Show that* $A \subset B \Rightarrow \sup A \leq \sup B$.

7. *Show that* $A \subset B \Rightarrow \inf A \geq \inf B$.

8. *Show that* $\forall \epsilon > 0, \exists a \in A; a + \epsilon > \sup A$

9. *Show that* $\forall \epsilon > 0, \exists a \in A; a - \epsilon < \inf A$

10. *Show that* $\exists a, \forall \epsilon > 0; a < \epsilon \Rightarrow a < \epsilon$

11. *Show that if* $\exists a, \forall \epsilon > 0; 0 \leq a < \epsilon$ *then* $a = 0$

12. *Define* $A + B = \{a + b | a \in A, b \in B\}$. *Show that* $\sup(A + B) = \sup A + \sup B$

13. *Show that there is a rational number and an irrational number between any two real numbers.*

14. *Show that for each* $a \geq 0$ *there exists a unique real number* $x \geq 0$ *such that* $x^2 = a$

Theorem 2. *Let A be a non empty subset of $\mathbb{R}$ which has an upper bound u.*
Then $\forall\epsilon > 0, \exists a \in A; a + \epsilon > u$ iff $u = \sup A$

Theorem 3. *Let A be a non empty subset of $\mathbb{R}$ which has a lower bound ℓ.*
Then $\forall\epsilon > 0, \exists a \in A; a - \epsilon < \ell$ iff $\ell = \inf A$

Axiom 4. *Well Ordering Principle(Completeness Axiom for $\mathbb{Z}$)*

1. *Every non-empty subset of $\mathbb{Z}$ which is bounded above has a maximum.*
2. *Every non-empty subset of $\mathbb{Z}$ which is bounded below has a minimum.*

Theorem 4. *Division Algorithm*
If $a, b \in \mathbb{Z}$ with $b > 0$, there exists unique $q, r \in \mathbb{Z}$ with $0 \leq< r < b$ such that $a = qb+r$

Theorem 5. *Euclidean Algorithm*
Let $a = qb + r$ according to the Division Algorithm, then $\gcd(a, b) = \gcd(b, r)$

Example 6. *Find $\gcd(63, 12)$ using the Euclidean Algorithm.*

Example 7. *Use the continued fraction expansion of $\sqrt{2}$ and show that $\mathbb{Q}$ and $\mathbb{Q}^c$ does not possess the Completeness Axiom Property*

Definition 9.
Ordered Pair $(x, y) = \{\{x\}, \{x, y\}\}$
Cartesian Product between two sets A, B: $A \times B = \{(x, y)|x \in A, y \in B\}$

Example 8.
Show that iff $(a, b) = (c, d)$ then $a = c$ and $b = d$.
Identify $<$ with $\mathbb{R}$ and $|$ with $\mathbb{Z}$ as relations.

Definition 10. *Relation. Let A, B be non-empty.*

- *Then a Relation $P : A \to B$ is a non-empty subset of $A \times B$*
- *We write any of $P : x \mapsto y, x \xrightarrow{P} y, xPy, {}_xP_y$ to mean $(x, y) \in P$*
- *A is called the Domain or $\mathrm{dom}P$*
- *B is called the Co-domain or $\mathrm{codom}P$*
- *$\{y|(x, y) \in P\}$ is called the Range or $\mathrm{ran}P$.*
- *$\{x|(x, y) \in P\}$ is called the Pre-range or $\mathrm{preran}P$*
- *P is One-many iff $\exists x \in A, \exists y_1, y_2 \in B; (x, y_1), (x, y_2) \in P \wedge y_1 \neq y_2$*
- *This implies that P is not one-many iff $\forall x \in A, \forall y_1, y_2 \in B; (x, y_1), (x, y_2) \in P \Rightarrow y_1 = y_2$*
- *P is Many-one iff $\exists x_1, x_2 \in A, \exists y \in B; (x_1, y), (x_2, y) \in P \wedge x_1 \neq x_2$*
- *This implies that P is not many-one iff $\forall x_1, x_2 \in A, \forall y \in B; (x_1, y), (x_2, y) \in P \Rightarrow x_1 = x_2$*
- *P is Many-many iff it is one-many and many-one.*

- *P is One-one(Injection) iff it is not one-many and not many-one.*
- *P is Everywhere-defined iff* $\text{dom}P = \text{preran}P$*. This is same as* $\forall x \in A \exists y \in B; (x, y) \in P$*.*
- *P is Onto(Surjection) iff* $\text{codom}P = \text{ran}P$*. This is same as* $\forall y \in B \exists x \in A; (x, y) \in P$*.*
- *P is a Bijection iff it is one-one and onto*
- *If* $P : A \to B$ *and* $Q : B \to C$ *are relations with* $\text{ran}P = \text{dom}Q = S$ *,we define the Composite relation*
  $Q \circ P : A \to C$ *as* $Q \circ P = \{(x, z)|(x, y) \in P \wedge (y, z) \in Q, y \in S\}$*. Note that* $\text{dom}(Q \circ P) = \text{dom}P$ *and* $\text{ran}(Q \circ P) = \text{ran}Q$
- *The Inverse relation of* $P : A \to B$ *is the relation* $P^{-1} : B \to A$ *defined by* $P^{-1} = \{(y, x)|(x, y) \in P\}$*.*
- *Note that* $\text{ran}P = \text{dom}P^{-1}$ *and* $\text{ran}P^{-1} = \text{dom}P$ *so the following compositions are allowed.* $P^{-1} \circ P = I = \{(x, x)|x \in \text{dom}P\}$ *and* $P \circ P^{-1} = I = \{(y, y)|y \in \text{ran}P\}$*.*
- *I is called the Identity relation and we can simply write* $P \circ P^{-1} = P^{-1} \circ P = I$*.*

Definition 11. *Function. Let* A, B *be non-empty sets.*

- *Then we define the relation* $f : A \to B$ *as a a function when f is everywhere-defined and not one-many. This allows us to write* $(x, y) \in f$ *as* $f(x) = y$
- *Let* $f : A \to B$ *is a onto function and* $g : B \to C$ *is also a function. Now* $\text{ran}f = B = \text{dom}g$*, so the composite relation* $g \circ f : A \to C$ *is allowed and it is also a function. Being a functions we can simply write* $(g \circ f)(x) = g(f(x))$ *for all* $x \in A$*.*
- *When f is a bijection, the inverse relation*f^{-1} *is also a function and a bijection. We have* $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x, \forall x \in A$ *and* $(f \circ f^{-1})(y) = f(f^{-1}(y)) = f(x) = y, \forall y \in B$*. If I is the identity function given by* $I(x) = x$ *we can simply write* $f \circ f^{-1} = f^{-1} \circ f = I$
- *When* A, B *subsets of* $\mathbb{R}$ *we say that f is a real valued function.*

Definition 12. *Let* $f, g : A \to B$ *be real valued functions. We define*

1. $(f + g)(x) = f(x) + g(x)$ *for all* $x \in A$
2. $(f - g)(x) = f(x) - g(x)$ *for all* $x \in A$
3. $(fg)(x) = f(x)g(x)$ *for all* $x \in A$
4. $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$ *for all* $x \in A$ *if* $g(x) \neq 0$

Definition 13. *Countable set*
A set A is said to be Countable iff there exists a one-one function $f : A \to \mathbb{Z}^+$
A set is Uncountable iff it is not countable.

Example 9.

1. *Evaluate* $(\sqrt{x})^2, \sqrt{x^2}, \sin(\sin^{-1} x), \sin^{-1}(\sin x)$
2. *Find the maximal domain and range of* $f(x) = x^2$ *and define the inverse functions* $\sqrt{(\cdot)}$ *and* $-\sqrt{(\cdot)}$.
3. *Do the above for* $\exp, \sin, \cos, \tan$ *functions.*
4. *Let* $f(x) = x + \frac{1}{x}$. *Find the range and domain. Show that the function is not one-one. Restrict the domain and find an inverse function.*
5. *Let* $f : A \to B$ *be a bijection. Show that* $(f \circ f^{-1})(y) = y$ *for all* $y \in B$ *and* $(f^{-1} \circ f)(x) = x$ *for all* $x \in A$.
6. *Let* $f(x) = \frac{1-x}{1+x}$ *and* $g(x) = 4x(1-x)$ *with suitable domains. Find* $f \circ g$ *and* $g \circ f$.
7. *Let* F *be the set of onto functions* $f : A \to A$. *Is* F *under the composition operations* $\circ$ *form a Group(structure similar to* $\mathbb{R}$ *with* $+$*)?*
8. *Let* $f : A \to B$ *and* $C, D \subseteq A$ *and let* $f(C) = \{f(x) | x \in C\}$. *Show that* $f(C \cup D) = f(C) \cup f(D)$ *and* $f(C \cap D) \subseteq f(C) \cap f(D)$
9. *Show that the composition of two one-one functions is one-one and the composition of two onto functions is onto.*
10. *Show that the integers and rational numbers are countable, but irrationals are uncountable.*
11. *Show that a subset of a countable set is countable and that a superset of an uncountable set is uncountable.*

Definition 14. *Limit.* $a, L \in \mathbb{R}$
$\lim_{x\to a} f(x) = L \Leftrightarrow \forall \epsilon > 0 \exists \delta > 0 \forall x, 0 < |x-a| < \delta \Rightarrow |f(x) - L| < \epsilon$

Example 10. *Show that* $\lim_{x\to 2} 2x + 3 = 7$, $\lim_{x\to 2} x^2 = 4, \lim_{x\to 2} \frac{x-1}{2x+1} = \frac{1}{5}$.

Example 11. *Prove the following with* $\lim_{x\to a} f(x) = L \in \mathbb{R}, \lim_{x\to a} g(x) = M \in \mathbb{R}$

1. $\lim_{x\to a} f(x) + g(x) = L + M$
2. $\lim_{x\to a} f(x) - g(x) = L - M$
3. $\lim_{x\to a} f(x)g(x) = LM$
4. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ *provided* $M \neq 0$
5. $\lim_{x\to b} f(g(x)) = L$ *provided that* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to b} g(x) = a$ *and that* $g(x) \neq a$ *for* $0 < |x-b| < \delta$ *for some* δ.

Definition 15. *Similarly limits correspond to any of the following combinations can be defined. Here* $\delta, \epsilon, N, M > 0$

$$\begin{array}{cc} x \to a : a-\delta < x < a+\delta, x \neq a & f(x) \to L : L-\epsilon < f(x) < L+\epsilon \\ x \to a^+ : a < x < a+\delta & f(x) \to L^+ : L \leq f(x) < L+\epsilon \\ x \to a^- : a-\delta < x < a & f(x) \to L^- : L-\epsilon < f(x) \leq L \\ x \to \infty : x > N & f(x) \to \infty : f(x) > M \\ x \to -\infty : x < -N & f(x) \to -\infty : f(x) < -M \end{array}$$

Theorem 6.
$\lim_{x\to a} f(x) = L \Leftrightarrow \lim_{x\to a^-} f(x) = L \wedge \lim_{x\to a^+} f(x) = L$
$\lim_{x\to a} f(x) = L^- \vee \lim_{x\to a} f(x) = L^+ \Rightarrow \lim_{x\to a} f(x) = L$

Example 12. *Suppose* $\lim_{x\to a} f(x) = \infty, \lim_{x\to a} g(x) = \infty$.
Show that $\lim_{x\to a} f(x) + g(x) = \infty$. *We can agree to write* $\infty + \infty = \infty$. *In the same way justify the following notation with* $b \in \mathbb{R}$

1. $\infty + b = \infty$
2. $b\infty = \infty$ *if* $b > 0$
3. $b\infty = -\infty$ *if* $b < 0$
4. $\infty\infty = \infty$
5. $\frac{a}{\infty} = 0$
6. *Show that you can't have a consistent notation for* $\infty - \infty, 0\infty, \frac{\infty}{\infty}, \frac{0}{0}$. *Hence these are called indeterminate forms(together with* $0^0, \infty^0, 1^\infty$*).*

Theorem 7. *Well-known limits*
1. $\lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}$
2. $\lim_{x\to 0} \frac{\sin x}{x} = 1$
3. $\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x = e^a$

Definition 16. *Monotonic Functions*
f *is Increasing on* $A \subset \mathbb{R}$ *iff* $\forall x_1, x_2 \in A, x_2 > x_1 \Rightarrow f(x_2) \geq f(x_1)$
f *is Decreasing on* $A \subset \mathbb{R}$ *iff* $\forall x_1, x_2 \in A, x_2 > x_1 \Rightarrow f(x_2) \leq f(x_1)$
f *is Increasing at* a *iff* $\exists\delta > 0 \forall x, (-\delta < x - a \Rightarrow f(x) \leq f(a)) \wedge (x - a < \delta \Rightarrow f(a) \leq f(x))$
f *is Decreasing at* a *iff* $\exists\delta > 0 \forall x, (-\delta < x - a \Rightarrow f(x) \geq f(a)) \wedge (x - a < \delta \Rightarrow f(a) \geq f(x))$
Strict means no $=$
Monotonic is either increasing or decreasing.

Theorem 8. *Monotone Convergence Theorem(for real valued functions)*
Suppose f *is bounded above and increasing on* $\mathbb{R}$. *Then* $\lim_{x\to\infty} f(x) = \sup\{f(x)\}$.
Suppose f *is bounded below and decreasing on* $\mathbb{R}$. *Then* $\lim_{x\to\infty} f(x) = \inf\{f(x)\}$.

Theorem 9. *Sandwich/Squeeze Theorem*
Suppose $f(x) \leq g(x) \leq h(x)$ *for* $0 < |x - a| < \delta$ *and* $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$.
Then $\lim_{x\to a} g(x) = L$.

Example 13.

1. *See if the above two theorems can be generalized for other cases of limits.*
2. *Use the inequality* $\sin\theta\cos\theta < \theta < \tan\theta$ *valid for* $0 < \theta < \frac{\pi}{2}$ *to show that* $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$
3. *Show that* $\lim_{x\to\infty}\sin x$ *does not exist and deduce that* $\lim_{x\to 0^+}\sin\frac{1}{x}$ *does not exist.*
4. *given* $a > 0$ *prove that* $\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}$ *for* $n\in\mathbb{Q}$
5. *Let* $p(x) = x^{13} + 17x^{12} - 10x^{11} + 1$. *Prove that* $\lim_{x\to\infty} p(x)^{1/13} - x = \frac{17}{13}$
6. *Prove that* $\lim_{x\to 0}\frac{\sin^{-1}x}{x} = 1$
7. *Show that* $\lim_{x\to 0} x\sin\frac{1}{x}$ *exists.*
8. *Show that* $\lim_{x\to 0} e^{1/x}$ *does not exist.*
9. *Let* $f(x) = 1$ *when* $x\in\mathbb{Q}$ *and* $f(x) = 0$ *when* $x\in\mathbb{R}-\mathbb{Q}$. *Prove that* $\lim_{x\to 0} f(x)$ *does not exist but* $\lim_{x\to 0} xf(x)$ *exists.*

Definition 17. *Continuity*
f *is Continuous at* a *iff* $\lim_{x\to a} f(x) = f(a)$
Being a limit it has two sides. $f(a^+) = \lim_{x\to a^+} f(x)$ *and* $f(a^-) = \lim_{x\to a^-} f(x)$
f *is Right Continuous at* a *iff* $f(a^+) = f(a)$
and f *is Left Continuous at* a *iff* $f(a^-) = f(a)$
Therefore f *is continuous at* a *iff* $f(a^-) = f(a) = f(a^+)$
f *is continuous on* $A\subset\mathbb{R}$ *iff* f *is continuous at each* $a\in A$. *At the boundary of* A *this means the left or right continuity as desired.*
We write $f\in\mathcal{C}(A)$ *to mean this where* $\mathcal{C}(A)$ *is the set of continuous functions on* A.
When the set is understood from the context we simply write $f\in\mathcal{C}$.
We can ignore the concepts of left and right continuity and talk only about continuity if we make sure that f *is continuous on a larger open interval.*

Theorem 10. f *is continuous at* a *iff*
$\forall\epsilon > 0\exists\delta > 0\forall x, 0 < |x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$ *iff*
$\forall\epsilon > 0\exists\delta > 0\forall x, |x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$

Example 14.

1. *Define* $f(0)$ *so that* $f(x) = x\sin\frac{1}{x}, x\neq 0$ *is continuous at* 0
2. *Show that* $\sin x$ *is a continuous function.*
3. *Let* $\lim_{x\to a} f(x) = L, \lim_{x\to b} g(x) = a$ *and that* f *is continuous at* a. *Then show that* $\lim_{x\to b}(f\circ g)(x) = L$
4. *Suppose that* f *is continuous at* a *and* $f(a) > 0$. *Show that* $f(x) > 0$ *for all* $x\in(a-\delta, a+\delta)$ *for some* $\delta > 0$.

Theorem 11. *Intermediate Value Theorem*
Suppose f *is continuous on* $[a,b]$ *and* $f(a)\neq f(b)$ *and that* u *is strictly between* $f(a)$ *and* $f(b)$*(i.e* $f(a) < u < f(b)$ *or* $f(b) < u < f(a)$*). Then there exists* $c\in(a,b)$ *such that* $f(c) = u$.

Proof 1. *Using the Completeness axiom on the set* $B = \{x \in [a,b] | f(x) < u\}$ *when* $f(a) < u < f(b)$

Example 15.

1. *Show that there is a real root of* $x = e^{-x}$ *on* $[0,1]$.
2. *Find intervals that contains the real roots of* $p(x) = x^{13} + 17x^{12} - 10x^{11} + 1$

Theorem 12. *Boundedness Theorem*
Suppose f *is continuous on* $[a,b]$. *Then* f *is bounded on* $[a,b]$.

Theorem 13. *Extremum Value Theorem*
Suppose f *is continuous on* $[a,b]$. *Then* f *has a maximum and a minimum on* $[a,b]$.

Proof 2. *Uses the Bolzano-Weistrass Theorem(which will be discussed under sequences)*

Definition 18. *Differentiability*
f *is Differentiable at* a *iff* $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \in \mathbb{R}$
In that case we call this limit the Derivative of f *at* a *and write* $f'(a)$
Being a limit it has two sides.
f *is Right Diffentiable at* a *iff* $\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} \in \mathbb{R}$ *and in that case we write* $f'_+(a)$ *for the limit and call it the Right Derivative of* f *at* a.
f *is Left Diffentiable at* a *iff* $\lim_{x\to a^-} \frac{f(x)-f(a)}{x-a} \in \mathbb{R}$ *and in that case we write* $f'_-(a)$ *for the limit and call it the Left Derivative of* f *at* a.
Therefore f *is differentiable at* a *iff* $f'_-(a) = f'_+(a) = f'(a) \in \mathbb{R}$
f *is differenatiable on* $A \subset \mathbb{R}$ *iff* f *is differantiable at each* $a \in A$. *At the boundary of* A *this means the left or right differenatiability as desired.*
We write $f \in \mathcal{D}(A)$ *to mean this where* $\mathcal{D}(A)$ *is the set of differentiable functions on* A.
When the set is understood from the context we simply write $f \in \mathcal{D}$.
We can ignore the concepts of left and right differentiability and talk only about differantiability if we make sure that f *is differantiable on a larger open interval.*
When the derivative is also continuous we write $f \in \mathcal{C}^1$ *where* $\mathcal{C}^1$ *is the set of continuously differentiable functions.*

Theorem 14. *Differantiability implies Continuity*
If f *is differantiable at* a *then* f *is continuous at* a
If f *is right differantiable at* a *then* f *is right continuous at* a
If f *is left differantiable at* a *then* f *is left continuous at* a

Theorem 15. *Let* f, g *be differantiable functions, then*

1. $(f+g)' = f' + g'$
2. $(f-g)' = f' - g'$
3. $(fg)' = fg' + f'g$
4. $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$ *when* $g(x) \neq 0$
5. $(f \circ g)' = (f' \circ g)g'$

Example 16.

1. *Determine which of the functions* $|x|, x + |x|, x|x|$ *are differentiable*
2. *Define* $f(0)$ *so that* $f(x) = x^2 \sin \frac{1}{x}$ *is continuous at* 0*. Also show that then* f *is differeantiable at* 0*.*
3. *Let* f *be differentiable at* a*. Show that if* $f'(a) > 0$ *then* f *is increasing at* a *and if* $f'(a) < 0$ *then* f *is decreasing at* a*.*
4. *Show that the right derivative of* $\sqrt{x}$ *does not exist(not right differenatible) at* 0*.*
5. *Let* f *be differenatiable. Show that iff* $\lim_{x\to\infty} f'(x) = \infty$ *then* $\lim_{x\to\infty} f(x) = \infty$
6. *Let* f *be differenatiable. Show that iff* $\lim_{x\to\infty} f'(x) = 0$ *then* $\lim_{x\to\infty} f(x) = L \in \mathbb{R}$
7. *(Darboux's Theorem)Show that the Intermediate Value Theorem holds for* f' *even without the derivative being continuous. i.e. Let* f *be differantiable on* $[a, b]$ *with* $f'(a) \neq f'(b)$ *and* u *is strictly between* $f'(a)$ *and* $f'(b)$*. Then there exists* $c \in (a, b)$ *such that* $f'(c) = u$*. Use the function* $g(x) = ux - f(x)$*.*

Definition 19.
f has a Local Maximum at a iff $\exists\delta > 0 \forall x, |x - a| < \delta \Rightarrow f(x) \leq f(a)$
f has a Local Minimum at a iff $\exists\delta > 0 \forall x, |x - a| < \delta \Rightarrow f(x) \geq f(a)$
Extremum is either a minimum or a maximum.
a is a Critical Point of f iff it is not differentiable at a or $f'(a) = 0$*.*

Theorem 16. *Let f be differantiable. If a is a local extremum then* $f'(a) = 0$*.*