Example 1. *Find the Continued Fraction Expansions for* $\sqrt{2}, \pi, e$ *and the Golden Ratio which is the positive root of* $\phi^2 - \phi - 1 = 0$.

Definition 1. *Set of Real numbers* $\mathbb{R}$ *is a set satisfying*
1.Field Axioms
2.Order Axioms
3.Completeness Axiom

Axiom 1. *Field Axioms.*
$\mathbb{R}$ *is a set with two or more elements and two binary operations + and . on them satisfying the following properties*

1. $\forall a, b \in \mathbb{R}; a + b \in \mathbb{R}$: *closed under addition*
2. $\forall a, b, c \in \mathbb{R}; a + (b + c) = (a + b) + c$: *addition is associative*
3. $\exists 0 \in \mathbb{R}, \forall a \in \mathbb{R}; a + 0 = 0 + a = a$: *additive identity exists*
4. $\forall a \in \mathbb{R}, \exists -a \in \mathbb{R}; a + (-a) = (-a) + a = 0$: *additive inverse exists*
5. $\forall a, b \in \mathbb{R}; a + b = b + a$: *addition is commutative*
6. $\forall a, b \in \mathbb{R}; a.b \in \mathbb{R}$: *closed under multiplication*
7. $\forall a, b, c \in \mathbb{R}; a.(b.c) = (a.b).c$: *multiplication is associative*
8. $\exists 1 \in \mathbb{R} - \{0\}, \forall a \in \mathbb{R}; a.1 = 1.a = a$: *multiplicative identity exists*
9. $\forall a \in \mathbb{R} - \{0\}, \exists a^{-1} \in \mathbb{R}; a.a^{-1} = a^{-1}.a = 1$: *multiplicative inverse exists*
10. $\forall a, b \in \mathbb{R}; a.b = b.a$: *multiplication is commutative*
11. $\forall a, b, c \in \mathbb{R}; a.(b + c) = (a.b) + (a.c)$: *multiplication is distributive over addition*

Definition 2.
$a - b = a + (-b)$: *Subtraction*
If $a \neq 0, \frac{a}{b} = a.b^{-1}$: *Division*

Definition 3.

1. *We write* $1 + 1 = 2, 2 + 1 = 3, 3 + 1 = 4$ *and so on.*
2. *Set of Positive Integers* $\mathbb{Z}^+ = \{1, 2, 3, \cdots\}$
3. *Set of Natural Numbers* $\mathbb{N} = \mathbb{Z}^+ \cup \{0\}$
4. *Set of Negative Integers* $\mathbb{Z}^- = \{-a | a \in \mathbb{Z}^+\}$
5. *Set of Integers* $\mathbb{Z} = \mathbb{Z}^- \cup \{0\} \cup \mathbb{Z}^+$
6. *Set of Rational Numbers* $\mathbb{Q} = \{\frac{p}{q} | p, q \in \mathbb{Z}$ *and* $q \neq 0\}$
7. *Set of Irrational Numbers* $\mathbb{Q}^c = \mathbb{R} - \mathbb{Q}$
8. *If* $a, b \in \mathbb{Z}$ *we say* a *divides* b *or* a *is a factor of* b *and write* $a|b$ *iff* $\frac{b}{a} \in \mathbb{Z}$

9. *$p \in \mathbb{Z}^+ - \{1\}$ is a Prime Number iff 1 and p are its only factors.*

Example 2. *Any set of two or more elements with two binary operations satisfying the fields axioms is called a Field. See if the following are fields*

1. *$\mathbb{R}$ with . and +*
2. *$\mathbb{Z}, \mathbb{Q}, \mathbb{Q}^c$ with + and .*
3. *$\{0, 1, 2\}$ with $\mod 3$ arithmetic*
4. *$\{0, 1, 2, 3\}$ with $\mod 4$ arithmetic*

Theorem 1.

1. *There are initially many prime numbers.*
2. *Every $n \in \mathbb{Z}^+ - \{1\}$ is a prime number or a unique product of prime numbers*
3. *Gaps between prime numbers can be arbitrary large.*
4. *$\{0, 1, 2, \cdots, n-1\}$ is a field iff n is prime.*

Definition 4. *Integer Powers*
If $a \neq 0, a^0 = 1$
If $a \neq 0, n \in \mathbb{Z}^+$ then $a^n = a.a^{n-1}$
If $a \neq 0, n \in \mathbb{Z}^+$ then $a^{-n} = (a^{-1})^n$

Example 3. *Prove the following with $a, b, c \in \mathbb{R}$*

1. *If $a + b = 0$ then $b = -a$*
2. *If $a + c = b + c$ then $a = b$*
3. *$-(a + b) = (-a) + (-b)$*
4. *$-(-a) = a$*
5. *$a.0 = 0$*
6. *$0, 1, -a, a^{-1}$ are unique*
7. *If $a \neq 0$ and $ab = 1$ then $b = a^{-1}$*
8. *If $ac = bc$ and $c \neq 0$ then $a = b$*
9. *If $ab = 0$ then $a = 0$ or $b = 0$*
10. *$-(ab) = (-a)b = a(-b)$*
11. *$(-a)(-b) = ab$*
12. *If $a \neq 0, (a^{-1})^{-1} = a$*
13. *If $a, b \neq 0, (ab)^{-1} = a^{-1}b^{-1}$*
14. *If $a \neq 0$ and $m, n \in \mathbb{Z}$ then $a^m a^n = a^{m+n}$*

15. *If* $a, b \neq 0, n \in \mathbb{Z}, (ab)^n = a^n b^n$

Axiom 2. *Order Axioms*
$\mathbb{R}$ *has a Order* $<$ *satisfying the following.*

12. $\forall a, b \in \mathbb{R}$;*exactly one of* $a = b, a < b, b < a$ *holds: Trichotomy*

13. $\forall a, b, c \in \mathbb{R}; a < b$ *and* $b < c$ *implies* $a < c$*: Transitivity*

14. $\forall a, b, c \in \mathbb{R}; a < b$ *implies* $a + c < b + c$*: operations with addition*

15. $\forall a, b \in \mathbb{R}; a < b$ *and* $0 < c$ *implies* $ac < bc$*: operations with multiplication*

Definition 5.
$b > a$ *is same as* $a < b$
$a \leq b$ *means* $a < b$ *or* $a = b$
Above follows that $a \neq b$ *is either* $a < b$ *or* $a > b$.

Definition 6. *Absolute Value* $|a| = a$ *if* $a \geq 0$ *and* $-a$ *if* $a < 0$

Example 4.

1. $\forall a, b \in \mathbb{R}; a < b$ *and* $c < 0$ *implies* $ac > bc$
2. $1 > 0$
3. $a > 0$ *iff* $a^{-1} > 0$
4. *If* $a < b$ *and* $c < d$ *then* $a + c < b + d$
5. *If* $0 < a < b$ *and* $0 < c < d$ *then* $ac < bd$
6. *See if* $|$ *defines an order in* $\mathbb{Z}$
7. $|a| \leq r$ *iff* $-r \leq a \leq r$
8. $a^2 \geq 0$
9. $|ab| = |a||b|$
10. $|a| - |b| \leq |a + b| \leq |a| + |b|$
11. $||a| - |b|| \leq |a - b|$
12. $|a + b|^2 + |a - b|^2 = 2|a|^2 + 2|b|^2$

Definition 7. *Let* A *be a non-empty subset of* $\mathbb{R}$*. Then*

1. *Upper Bound of A:* $u \in \mathbb{R}$ *such that* $\forall a \in A; a \leq u$
2. *Bounded Above: An upper bound exists*
3. *Maximum(largest) element of A:* $\max A = u \in A$ *and* u *is an upper bound of* A
4. *Lower Bound of A:* $\ell \in \mathbb{R}$ *such that* $\forall a \in A; \ell \leq a$
5. *Bounded Below: A lower bound exists*

6. *Minimum(least) element of A:* $\min A = \ell \in A$ *and* ℓ *is a lower bound of* A
7. *Supremum of A:* $\sup A$*=least upper bound of* A.
8. *Infimum of A:* $\inf A$*=largest lower bound of* A.
9. *Bounded: bounded above and bounded below*

Axiom 3. *Completeness Axiom.*

16. *Every non-empty subset of* $\mathbb{R}$ *which is bounded above has a supremum.*
17. *Every non-empty subset of* $\mathbb{R}$ *which is bounded below has a infimum*

Definition 8. *Real Intervals,* $a < b$

1. $(a, b) = \{x \in \mathbb{R} | a < x < b\}$*: Open interval*
2. $(a, b] = \{x \in \mathbb{R} | a < x \leq b\}$*: half open/closed interval*
3. $[a, b) = \{x \in \mathbb{R} | a \leq x < b\}$*: half open/closed interval*
4. $[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$*: Closed interval*

Example 5. *Assume that* $A, B \subset \mathbb{R}$ *are non-empty subsets which are bounded above*

1. *Prove that* $\sup(a, b) = b$ *and* $\inf(a, b) = a$.
2. *Which of the following sets have the completeness axiom property* $\mathbb{Z}, \mathbb{Q}, \mathbb{Q}^c$
3. *Show that* $\mathbb{Z}$ *is unbounded.*
4. *Show that for every* $a \in \mathbb{R}$ *there is* $n \in \mathbb{Z}$ *such that* $n > a$.
5. *Show that for given* $a, b \in \mathbb{R}$ *with* $b > a$*, there exists* $n \in \mathbb{Z}$ *such that* $na > b$
6. *Show that for* $a, b \in \mathbb{Z}^+$ *such that* $a < b$*, there exists unique* $x, y \in \mathbb{Z}^+$ *such that* $b = xa + y$ *with* $0 \leq y < a$
7. *Prove the existence of* $\inf$ *using the existence of* $\sup$ *with suitable conditions.*
8. *Suppose we have* $\forall a \in A, \forall b \in B; a < b$. *Show that* $\sup A \leq \sup B$.
9. *Show that* $\forall \epsilon > 0, \exists a \in A; a + \epsilon > \sup A$
10. *Show that* $\forall \epsilon > 0, \exists a \in A; a - \epsilon < \inf A$
11. *Show that if* $\exists a, \forall \epsilon > 0; 0 \leq a < \epsilon$ *then* $a = 0$
12. *Define* $A + B = \{a + b | a \in A, b \in B\}$. *Show that* $\sup(A + B) = \sup A + \sup B$
13. *Show that there is a rational number and an irrational number between any two real numbers.*
14. *Show that for each* $a \geq 0$ *there exists a unique real number* $x \geq 0$ *such that* $x^2 = a$

Definition 9.
Ordered Pair $(x, y) = \{\{x\}, \{x, y\}\}$
Cartesian Product between two sets A, B*:* $A \times B = \{(x, y) | x \in A, y \in B\}$

Definition 10. *Relation. Let A, B be non-empty.*

- *Then a Relation $P : A \to B$ is a non-empty subset of $A \times B$*
- *We write any of $P : x \mapsto y, x \xrightarrow{P} y, xPy, {}_xP_y$ to mean $(x, y) \in P$*
- *A is called the Domain or* $\mathrm{dom}P$
- *B is called the Co-domain or* $\mathrm{codom}P$
- *$\{y|(x, y) \in P\}$ is called the Range or* $\mathrm{ran}P$.
- *$\{x|(x, y) \in P\}$ is called the Pre-range or* $\mathrm{preran}P$
- *P is One-many iff $\exists x \in A, \exists y_1, y_2 \in B; (x, y_1), (x, y_2) \in P \wedge y_1 \neq y_2$*
- *This implies that P is not one-many iff $\forall x \in A, \forall y_1, y_2 \in B; (x, y_1), (x, y_2) \in P \Rightarrow y_1 = y_2$*
- *P is Many-one iff $\exists x_1, x_2 \in A, \exists y \in B; (x_1, y), (x_2, y) \in P \wedge x_1 \neq x_2$*
- *This implies that P is not many-one iff $\forall x_1, x_2 \in A, \forall y \in B; (x_1, y), (x_2, y) \in P \Rightarrow x_1 = x_2$*
- *P is Many-many iff it is one-many and many one.*
- *P is One-one(Injection) iff it is not one-many and not many-one.*
- *P is Onto(Surjection) iff $\mathrm{ran}P = B$. Note that P is onto iff $\forall y \in B \exists x \in A; (x, y) \in P$*
- *P is a Bijection iff it is one-one and onto*
- *$P^{-1} : B \to A$ defined by $P^{-1} = \{(y, x)|(x, y) \in P\}$ is the Inverse relation of P*
- *If $Q : B \to C$ is also a relation,we define the Composite relation $Q \circ P : A \to C$ as $Q \circ P = \{(x, z)|(x, y) \in P \wedge (y, z) \in Q, y \in \mathrm{ran}P \cap \mathrm{preran}Q\}$. So we need to ensure that $\mathrm{ran}P \cap \mathrm{preran}Q \neq \emptyset$.*

Definition 11. *Function. Let A, B be non-empty sets.*

- *Then the function $f : A \to B$ is a relation which is not one-many and $A = \mathrm{preran}f$. This allows us to write $(x, y) \in f$ as $f(x) = y$*
- *The inverse function f^{-1} is the inverse relation $f^{-1} : B \to A$ which is also a function. This requires f to be a bijection.*
- *Let $g : B \to C$ is also a function. Now the requirement $\mathrm{ran}f \cap \mathrm{preran}g \neq \emptyset$ is for the composite relation $g \circ f : A \to C$ is automatically satisfied. It can be easily shown that this is also a function. Being a functions we can simply write $(g \circ f)(x) = g(f(x))$ for all $x \in A$*
- *When A, B subsets of $\mathbb{R}$ we say that f is a real valued function.*

Definition 12. *Let $f, g : A \to B$ be real valued functions. We define*

1. $(f + g)(x) = f(x) + g(x)$ *for all* $x \in A$
2. $(f - g)(x) = f(x) - g(x)$ *for all* $x \in A$
3. $(fg)(x) = f(x)g(x)$ *for all* $x \in A$
4. $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$ *for all* $x \in A$ *if* $g(x) \neq 0$

Definition 13. *Countable set*
A set A is said to be Countable iff there exists a one-one function $f : A \to \mathbb{Z}^+$
A set is Uncountable iff it is not countable.

Example 6.

1. *Evaluate* $(\sqrt{x})^2, \sqrt{x^2}, \sin(\sin^{-1} x), \sin^{-1}(\sin x)$
2. *Find the maximal domain and range of* $f(x) = x^2$ *and define the inverse functions* $\sqrt{(\cdot)}$ *and* $-\sqrt{(\cdot)}$.
3. *Do the above for* exp, sin, cos, tan *functions.*
4. *Let* $f(x) = x + \frac{1}{x}$. *Find the range and domain. Show that the function is not one-one. Restrict the domain and find an inverse function.*
5. *Let* $f : A \to B$ *be a bijection. Show that* $(f \circ f^{-1})(y) = y$ *for all* $y \in B$ *and* $(f^{-1} \circ f)(x) = x$ *for all* $x \in A$.
6. *Let* $f(x) = \frac{1-x}{1+x}$ *and* $g(x) = 4x(1 - x)$ *with suitable domains. Find* $f \circ g$ *and* $g \circ f$.
7. *Let F be the set of onto functions $f : A \to A$. Is F under the composition operations $\circ$ form a Group(structure similar to $\mathbb{R}$ with $+$)?*
8. *Let* $f : A \to B$ *and* $C, D \subseteq A$ *and let* $f(C) = \{f(x)|x \in C\}$. *Show that* $f(C \cup D) = f(C) \cup f(D)$ *and* $f(C \cap D) \subseteq f(C) \cap f(D)$
9. *Show that the composition of two one-one functions is one-one and the composition of two onto functions is onto.*
10. *Show that the integers and rational numbers are countable, but irrationals are uncountable.*
11. *Show that a subset of a countable set is countable and that a superset of an uncountable set is uncountable.*

Definition 14. *Limit.* $a, L \in \mathbb{R}$
$\lim_{x\to a} f(x) = L \Leftrightarrow \forall \epsilon > 0 \exists \delta > 0 \forall x, 0 < |x-a| < \delta \Rightarrow |f(x) - L| < \epsilon$

Example 7. *Show that* $\lim_{x\to 2} 2x+3 = 7$, $\lim_{x\to 2} x^2 = 4, \lim_{x\to 2} \frac{x-1}{2x+1} = \frac{1}{5}$.

Example 8. *Prove the following with* $\lim_{x\to a} f(x) = L \in \mathbb{R}, \lim_{x\to a} g(x) = M \in \mathbb{R}$

1. $\lim_{x\to a} f(x) + g(x) = L + M$
2. $\lim_{x\to a} f(x) - g(x) = L - M$
3. $\lim_{x\to a} f(x)g(x) = LM$
4. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ *provided* $M \neq 0$
5. $\lim_{x\to b} f(g(x)) = L$ *provided that* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to b} g(x) = a$ *and that* $g(x) \neq a$ *for* $0 < |x-b| < \delta$ *for some* δ.

Definition 15. *Similarly limits correspond to any of the following combinations can be defined. Here* $\delta, \epsilon, N, M > 0$

$$\begin{array}{cc}
x \to a : a-\delta < x < a+\delta, x \neq a & f(x) \to L : L-\epsilon < f(x) < L+\epsilon \\
x \to a^+ : a < x < a+\delta & f(x) \to L^+ : L \leq f(x) < L+\epsilon \\
x \to a^- : a-\delta < x < a & f(x) \to L^- : L-\epsilon < f(x) \leq L \\
x \to \infty : x > N & f(x) \to \infty : f(x) > M \\
x \to -\infty : x < -N & f(x) \to -\infty : f(x) < -M
\end{array}$$

Theorem 2.
$\lim_{x\to a} f(x) = L \Leftrightarrow \lim_{x\to a^-} f(x) = L \wedge \lim_{x\to a^+} f(x) = L$
$\lim_{x\to a} f(x) = L^- \vee \lim_{x\to a} f(x) = L^+ \Rightarrow \lim_{x\to a} f(x) = L$

Example 9. *Suppose* $\lim_{x\to a} f(x) = \infty, \lim_{x\to a} g(x) = \infty$.
Show that $\lim_{x\to a} f(x) + g(x) = \infty$. *We can agree to write* $\infty + \infty = \infty$. *In the same way justify the following notation with* $b \in \mathbb{R}$

1. $\infty + b = \infty$
2. $b\infty = \infty$ *if* $b > 0$
3. $b\infty = -\infty$ *if* $b < 0$
4. $\infty\infty = \infty$
5. $\frac{a}{\infty} = 0$
6. *Show that you can't have a consistent notation for* $\infty - \infty, 0\infty, \frac{\infty}{\infty}, \frac{0}{0}$. *Hence these are called indeterminate forms(together with* $0^0, \infty^0, 1^\infty$*).*

Theorem 3. *Well-known limits*
1. $\lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}$
2. $\lim_{x\to 0} \frac{\sin x}{x} = 1$
3. $\lim_{x\to \infty} \left(1 + \frac{a}{x}\right)^x = e^a$

Definition 16. *Monotonic Functions*
f is Increasing on $A \subset \mathbb{R}$ iff $\forall x_1, x_2 \in A, x_2 > x_1 \Rightarrow f(x_2) \geq f(x_1)$
f is Decreasing on $A \subset \mathbb{R}$ iff $\forall x_1, x_2 \in A, x_2 > x_1 \Rightarrow f(x_2) \leq f(x_1)$
f is Increasing at a iff $\exists \delta > 0 \forall x, (-\delta < x - a \Rightarrow f(x) \leq f(a)) \wedge (x - a < \delta \Rightarrow f(a) \leq f(x))$
f is Decreasing at a iff $\exists \delta > 0 \forall x, (-\delta < x - a \Rightarrow f(x) \geq f(a)) \wedge (x - a < \delta \Rightarrow f(a) \geq f(x))$
Strict means no $=$
Monotonic is either increasing or decreasing.

Theorem 4. *Monotone Convergence Theorem(for real valued functions)*
Suppose f is bounded above and increasing on $\mathbb{R}$. Then $\lim_{x\to\infty} f(x) = \sup\{f(x)\}$.
Suppose f is bounded below and decreasing on $\mathbb{R}$. Then $\lim_{x\to\infty} f(x) = \inf\{f(x)\}$.

Theorem 5. *Sandwich/Squeeze Theorem*
Suppose $f(x) \leq g(x) \leq h(x)$ for $0 < |x - a| < \delta$ and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$. Then $\lim_{x\to a} g(x) = L$.

Example 10.

1. *See if the above two theorems can be generalized for other cases of limits.*
2. *Use the inequality $\sin\theta\cos\theta < \theta < \tan\theta$ valid for $0 < \theta < \frac{\pi}{2}$ to show that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$*
3. *Show that $\lim_{x\to\infty} \sin x$ does not exist and deduce that $\lim_{x\to 0^+} \sin\frac{1}{x}$ does not exist.*
4. *given $a > 0$ prove that $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$ for $n \in \mathbb{Q}$*
5. *Let $p(x) = x^{13} + 17x^{12} - 10x^{11} + 1$. Prove that $\lim_{x\to\infty} p(x)^{1/13} - x = \frac{17}{13}$*
6. *Prove that $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1$*
7. *Show that $\lim_{x\to 0} x \sin\frac{1}{x}$ exists.*
8. *Show that $\lim_{x\to 0} e^{1/x}$ does not exist.*
9. *Let $f(x) = 1$ when $x \in \mathbb{Q}$ and $f(x) = 0$ when $x \in \mathbb{R} - \mathbb{Q}$. Prove that $\lim_{x\to 0} f(x)$ does not exist but $\lim_{x\to 0} x f(x)$ exists.*

Definition 17. *Continuity*
f is Continuous at a iff $\lim_{x\to a} f(x) = f(a)$
Being a limit it has two sides. $f(a^+) = \lim_{x\to a^+} f(x)$ and $f(a^-) = \lim_{x\to a^-} f(x)$
f is Right Continuous at a iff $f(a^+) = f(a)$
and f is Left Continuous at a iff $f(a^-) = f(a)$
Therefore f is continuous at a iff $f(a^-) = f(a) = f(a^+)$
f is continuous on $A \subset \mathbb{R}$ iff f is continuous at each $a \in A$. At the boundary of A this means the left or right continuity as desired.
We write $f \in \mathcal{C}(A)$ to mean this where $\mathcal{C}(A)$ is the set of continuous functions on A.
When the set is understood from the context we simply write $f \in \mathcal{C}$.
We can ignore the concepts of left and right continuity and talk only about continuity if we make sure that f is continuous on a larger open interval.

Theorem 6. *f is continuous at a iff*
$\forall\epsilon > 0\exists\delta > 0\forall x, 0 < |x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$ *iff*
$\forall\epsilon > 0\exists\delta > 0\forall x, |x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$

Example 11.

1. *Define $f(0)$ so that $f(x) = x\sin\frac{1}{x}, x \neq 0$ is continuous at 0*
2. *Show that $\sin x$ is a continuous function.*
3. *Let $\lim_{x\to a} f(x) = L, \lim_{x\to b} g(x) = a$ and that f is continuous at a. Then show that $\lim_{x\to b}(f\circ g)(x) = L$*
4. *Suppose that f is continuous at a and $f(a) > 0$. Show that $f(x) > 0$ for all $x \in (a-\delta, a+\delta)$ for some $\delta > 0$.*

Theorem 7. *Intermediate Value Theorem*
Suppose f is continuous on $[a, b]$ and $f(a) \neq f(b)$ and that u is strictly between $f(a)$ and $f(b)$(i.e $f(a) < u < f(b)$ or $f(b) < u < f(a)$). Then there exists $c \in (a, b)$ such that $f(c) = u$.

Proof 1. *Using the Completeness axiom on the set $B = \{x \in [a, b] | f(x) < u\}$ when $f(a) < u < f(b)$*

Example 12.

1. *Show that there is a real root of $x = e^{-x}$ on $[0, 1]$.*
2. *Find intervals that contains the real roots of $p(x) = x^{13} + 17x^{12} - 10x^{11} + 1$*

Theorem 8. *Boundedness Theorem*
Suppose f is continuous on $[a, b]$. Then f is bounded on $[a, b]$.

Theorem 9. *Extremum Value Theorem*
Suppose f is continuous on $[a, b]$. Then f has a maximum and a minimum on $[a, b]$.

Proof 2. *Uses the Bolzano-Weistrass Theorem(which will be discussed under sequences)*

Definition 18. *Differentiability*
f is Differentiable at a iff $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \in \mathbb{R}$
In that case we call this limit the Derivative of f at a and write $f'(a)$
Being a limit it has two sides.
f is Right Diffentiable at a iff $\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} \in \mathbb{R}$ and in that case we write $f'_+(a)$ for the limit and call it the Right Derivative of f at a.
f is Left Diffentiable at a iff $\lim_{x\to a^-} \frac{f(x)-f(a)}{x-a} \in \mathbb{R}$ and in that case we write $f'_-(a)$ for the limit and call it the Left Derivative of f at a.
Therefore f is differentiable at a iff $f'_-(a) = f'_+(a) = f'(a) \in \mathbb{R}$
f is differenatiable on $A \subset \mathbb{R}$ iff f is differantiable at each $a \in A$. At the boundary of A this means the left or right differenatiability as desired.
We write $f \in \mathcal{D}(A)$ to mean this where $\mathcal{D}(A)$ is the set of differentiable functions on A.

When the set is understood from the context we simply write $f \in \mathcal{D}$.
We can ignore the concepts of left and right differentiability and talk only about differantiability if we make sure that f is differantiable on a larger open interval.
When the derivative is also continuous we write $f \in \mathcal{C}^1$ where $\mathcal{C}^1$ is the set of continuously differentiable functions.

Theorem 10. *Differantiability implies Continuity*
If f is differantiable at a then f is continuous at a
If f is right differantiable at a then f is right continuous at a
If f is left differantiable at a then f is left continuous at a

Theorem 11. *Let f, g be differantiable functions, then*

1. $(f+g)' = f' + g'$
2. $(f-g)' = f' - g'$
3. $(fg)' = fg' + f'g$
4. $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$ *when* $g(x) \neq 0$
5. $(f \circ g)' = (f' \circ g)g'$

Example 13.

1. *Determine which of the functions $|x|, x + |x|, x|x|$ are differentiable*
2. *Define $f(0)$ so that $f(x) = x^2 \sin \frac{1}{x}$ is continuous at 0. Also show that then f is differeantiable at 0.*
3. *Let f be differentiable at a. Show that if $f'(a) > 0$ then f is increasing at a and if $f'(a) < 0$ then f is decreasing at a.*
4. *Show that the right derivative of $\sqrt{x}$ does not exist(not right differenatible) at 0.*
5. *Let f be differenatiable. Show that iff $\lim_{x\to\infty} f'(x) = \infty$ then $\lim_{x\to\infty} f(x) = \infty$*
6. *Let f be differenatiable. Show that iff $\lim_{x\to\infty} f'(x) = 0$ then $\lim_{x\to\infty} f(x) = L \in \mathbb{R}$*
7. *(Darboux's Theorem)Show that the Intermediate Value Theorem holds for f' even without the derivative being continuous. i.e. Let f be differantiable on $[a, b]$ with $f'(a) \neq f'(b)$ and u is strictly between $f'(a)$ and $f'(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = u$. Use the function $g(x) = ux - f(x)$.*

Definition 19.
f has a Local Maximum at a iff $\exists\delta > 0 \forall x, |x - a| < \delta \Rightarrow f(x) \leq f(a)$
f has a Local Minimum at a iff $\exists\delta > 0 \forall x, |x - a| < \delta \Rightarrow f(x) \geq f(a)$
Extremum is either a minimum or a maximum.
a is a Critical Point of f iff it is not differentiable at a or $f'(a) = 0$.

Theorem 12. *Let f be differantiable. If a is a local extremum then $f'(a) = 0$.*

Theorem 13. *Rolle's Theorem*
Let f be continuous on $[a, b]$ and differeantiable on (a, b) and $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Proof 3. *Using the extremum value theorem and that $f'(x) = 0$ at extremums.*

Theorem 14. *Mean Value Theorem*
Let f be continuous on $[a, b]$ and differeantiable on (a, b). Then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Theorem 15. *Cauchy Mean Value Theorem*
Let f, g be continuous on $[a, b]$ and differeantiable on (a, b) and $g'(x) \neq 0$. Then there exists $c \in (a, b)$ such that $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$.

Proof 4. *Use the Rolle's theorem on $h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(x)-g(a)}(g(x) - g(a))$*

Definition 20. *Second and higher order derivatives*
$\lim_{x\to a} \frac{f'(x)-f(a)}{x-a} = f''(a)$ *is the Second Derivative when the limit exists and finite*
$\lim_{x\to a} \frac{f^{(n-1)}(x)-f^{(n-1)}(a)}{x-a} = f^{(n)}(a)$ *is the nth Derivative, when the limit exists and finite*
We write $f \in \mathcal{D}^n$ to mean that the $f^{(n)}$ is existing and write $f \in \mathcal{C}^n$ to mean that $f^{(n)}$ is existing and continuous.
We write $f^{(0)}(x)$ for $f(x)$

Theorem 16. *Let f'' be exists on (a, b) and $x, c \in (a, b)$. Then there exists ζ between c and x such that $f(x) = f(c) + f'(c)(x - c) + \frac{f''(\zeta)}{2}(x - c)^2$*

Theorem 17. *Second Derivative Test*
Let f'' exists and continuous(i.e. $f \in \mathcal{C}^2$) and $f'(c) = 0$.
If $f''(c) > 0$ then c is a local minimum. If $f''(c) < 0$ then c is a local maximum.

Theorem 18. *Taylor Polynomial*
Let $f^{(m+1)}$ exists on (a, b) and $x, c \in (a, b)$. Then there exists ζ between c and x such that $f(x) = \sum_{n=0}^{m} \frac{f^{(n)}(c)}{n!}(x - c)^n + \frac{f^{(m+1)}(\zeta)}{(m+1)!}(x - c)^{m+1}$. The sum is called the Taylor Polynomial $T_m(x)$ and the last term is called the Remainder $R_m(x)$.

Proof 5. *Use Cauchy mean value theorem on $F(t) = \sum_{n=0}^{m} \frac{f^{(n)}(t)}{n!}(x - t)^n$ and $G(t) = (x - t)^{m+1}$ keeping x fixed.*

Example 14.

1. *Show that if f is differentiable on (a, b) and $f'(x) \geq 0$ then f is increasing on (a, b).*
2. *Show that $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.*
3. *Show that $\frac{x-1}{x} < \ln x < x - 1$ for all $x > 1$*
4. *Let $f : [0, \infty) \to \mathbb{R}$ be differantiable on $(0, \infty)$ and assume that $\lim_{x\to\infty} f'(x) = b$. Show that $\lim_{x\to\infty} \frac{f(x)}{x} = b$*

5. *Let f, g be differentiable on $\mathbb{R}$ and $f(0) = g(0)$ and $f'(x) \leq g'(x)$ for all $x \geq 0$. Show that $f(x) \leq g(x)$ for all $x \geq 0$.*
6. *Draw a rough sketch of the graph $f(x) = x - \frac{1}{x}$*
7. *Write down the Taylor's Polynomial and Remainder for $e^x, \sin x, \cos x, \ln(1+x), \tan^{-1} x$ at $x = 0$.*

Theorem 19. *L'Hopital Rule*
1. $f(a) = g(a) = 0$
2. *f, g are continuous on $[a, a+\delta)$*
3. *f, g are differentiable on $(a, a+\delta)$*
4. *$g' \neq 0$ on $(a, a+\delta)$*
5. $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$

then $\lim_{x\to a^+} \frac{f(x)}{f(x)} = L$

Note 1. *The above is valid*
1. *When $L = \pm\infty$*
2. *$\lim f(x) = g(x) = 0$ for any limit with conditions satisfied on the associated region.*
3. *$\lim f(x) = g(x) = \infty$ for any limit with conditions satisfied on the associated region.*

Example 15. *Evaluate the following limits*
1. $\lim_{x\to\frac{1}{2}} \frac{\ln 2x}{2x-1}$
2. $\lim_{x\to 0} \frac{x^2 \sin\frac{1}{x}}{\sin x}$
3. $\lim_{x\to 0^+} \frac{\ln x}{\cot \pi x}$
4. $\lim_{x\to 0} \frac{\sec x - 1}{x \sin 2x}$
5. $\lim_{x\to\infty} \frac{\frac{\pi}{2} - \tan^{-1} x}{x^{-1}}$
6. $\lim_{x\to 0^-} \frac{1-\sec x}{x^3}$
7. $\lim_{x\to\frac{\pi}{4}} (\tan x)^{\tan 2x}$
8. $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$
9. $\lim_{x\to 0} \left(\frac{1}{x^2} - \cot^2 x\right)$
10. $\lim_{x\to 0^+} (\sin x)^x$
11. *Prove the other versions of the L'Hopital rule.*

Definition 21. *Sequance*
A sequence $u(n)$ is a function $u : \mathbb{Z}^+ \to \mathbb{R}$
We usually write the image of $n : u(n)$ as u_n.

Definition 22. *Convegence*
A sequence u_n is converging iff $\lim_{n\to\infty} u_n = L \in \mathbb{R}$
ie.iff $\exists L \in \mathbb{R} \forall \epsilon > 0 \exists N \in \mathbb{Z}^+ \forall n, n > N \Rightarrow |u_n - L| < \epsilon$
u_n is diverging iff it is not converging.

Theorem 20.
Every sequence which is increasing and bounded above is converging.
Every sequence which is decreasing and bounded below is converging.
Every converging sequence is bounded.

Example 16. *Check the convergence of the following sequences*
1. $u_n = (-1)^n$
2. $u_{n+1} = \frac{k}{1+u_n}, k > 0, u_1 > 0$
3. $u_{n+1} = \frac{1}{2}(u_n + \frac{a}{u_n}), a > 0$
4. $u_{n+2} = \frac{1}{2}(u_n + u_{n+1}), x_1 = a, x_2 = b$

Definition 23. *Sub Sequance*
Let $u : \mathbb{Z}^+ \to \mathbb{R}$ *be a sequence and* $v : \mathbb{Z}^+ \to \mathbb{Z}^+$ *be an increasing sequence. Then* $u \circ v : \mathbb{Z}^+ \to \mathbb{R}$ *is a subsequence of* u_n

Theorem 21.
Every sequence has a monotone(either increasing or decreasing) subsequence.

Theorem 22. *Bolzano-Weistrass*
Every bounded sequence has a converging subsequance.

Proof 6.
Using the above theorem and the fact that bounded monotone sequences converge.

Example 17. *Find converging subsequances of the following sequences*
1. $u_n = (-1)^n$
2. $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \cdots$
3. $u_n = \sin n$

Definition 24. *Cauchy Sequance*
A sequence u_n *is Cauchy iff* $\forall \epsilon > 0 \exists N \in \mathbb{Z}^+ \forall m, n; m, n > N \Rightarrow |u_n - u_m| < \epsilon$

Theorem 23.
Every converging sequence is Cauchy.
Every Cauchy sequence is converging

Definition 25. *Series*
Given a sequence u_n*, a series is the sequence* $s_m = \sum_{n=1}^{m} u_n$
We usually write $\sum_{n=1}^{\infty} u_n$ *to mean this series.*

Theorem 24. *If* $\sum_{n=1}^{\infty} u_n$ *is converging then* $\lim_{n\to\infty} u_n = 0$.
Or equivalentyl if $\lim_{n\to\infty} u_n \neq 0$ *then* $\sum_{n=1}^{\infty} u_n$ *is diverging.*

Theorem 25. *Comparison Test*
Let $0 < u_n < v_n$*. Then*
If $\sum_{n=1}^{\infty} v_n$ *is converging then* $\sum_{n=1}^{\infty} u_n$ *is converging.*
or equivalently If $\sum_{n=1}^{\infty} u_n$ *is diverging then* $\sum_{n=1}^{\infty} v_n$ *is diverging.*

Theorem 26. *Limit Comparison Test*
Let $0 < u_n < v_n$ *and* $\lim_{n\to\infty} \frac{u_n}{v_n} = R$.
1. $R \in (0, \infty)$, $\sum_{n=1}^{\infty} u_n$ *is converging* $\Leftrightarrow \sum_{n=1}^{\infty} v_n$ *is converging*
2. $R = 0$ *and* $\sum_{n=1}^{\infty} v_n$ *is converging* $\Rightarrow \sum_{n=1}^{\infty} u_n$ *is converging*
3. $R = \infty$ *and* $\sum_{n=1}^{\infty} v_n$ *is diverging* $\Rightarrow \sum_{n=1}^{\infty} u_n$ *is diverging*

Theorem 27. *Integral Test*
Let $u : [1, \infty) \to (0, \infty)$ *be a decreasing function which is integrable.*
Then $\sum_{n=1}^{\infty} u(n)$ *converging iff* $\int_1^{\infty} u(x)dx \in \mathbb{R}$

Theorem 28. *$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is converging iff $p > 1$.*

Definition 26. *Absolute convergence*
A series $\sum_{n=1}^{\infty} u_n$ is Absolutely Converging iff $\sum_{m=1}^{\infty} |u_n|$ is converging.

Theorem 29. *absolutely converging $\Rightarrow$ converging*

Theorem 30. *Ratio Test*
Consider the series $\sum_{n=1}^{\infty} u_n$ with $u_n \neq 0$. Let $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = R$.
1. If $R < 1$ then the series is converging absolutely
2. If $R > 1$ then the series is diverging

Theorem 31. *Root Test*
Consider the series $\sum_{n=1}^{\infty} u_n$. Let $\lim_{n\to\infty} |u_n|^{\frac{1}{n}} = R$.
1. If $R < 1$ then the series is converging absolutely
2. If $R > 1$ then the series is diverging

Definition 27. *Alternating series*
A series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ is an alternating series provided $u_n > 0$

Theorem 32. *Let $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ be an alternating series.*
If u_n is decreasing and $\lim_{x\to\infty} u_n = 0$ then the series is converging.

Example 18. *Check the convergence of the following series*
1. $\sum_{n=1}^{\infty} (-1)^n$ *2.* $\sum_{n=1}^{\infty} \frac{2}{n^2+3}$ *3.* $\sum_{n=1}^{\infty} \frac{n}{n^2+3}$ *4.* $\sum_{n=1}^{\infty} \frac{3n-2}{n(n+1)(n+2)}$ *5.* $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$
6. $\sum_{n=1}^{\infty} (\frac{n}{n+1})^{n^2}$ *7.* $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$ *8.* $\sum_{n=1}^{\infty} \cos\frac{1}{n}$ *9.* $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$
10. $\sum_{n=1}^{\infty} \frac{\sin n}{n}$ *11.* $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

Definition 28. *Power Series*
Is a series of the form $\sum_{n=0}^{\infty} a_n(x-c)^n$ where x is a variable and c is a constant.
$\sup\{r | \textit{series converges for} |x-c| < r\}$ *is called the Radius of convergence R.*
$(c-R, c+R)$ is called the Range of Convergence.
We can use the ratio test or the root test to find R

Theorem 33. *Taylor Series*
Let f be infinitely many times differentiable on (a, b) and $x, c \in (a, b)$. If the Remainder $R_m(x) \to 0$ as $m \to \infty$ for $x \in (c-R, c+R) \subset (a, b)$ we have $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$ on the same interval and we call this the Taylor Series of f at c. Taylor Series can be used to define the corresponding functions.

Example 19. *Find the Taylor Series and the radius of convergence of the following functions at 0*
1. e^x
2. $\sin x$
3. $\cos x$
4. $\ln(1+x)$
5. $\tan^{-1} x$
6. $x^3 + x^2 + 1$
7. $f(x) = e^{-\frac{1}{x^2}}$ *with* $f(0) = 0$