Performance Comparison of Shewhart Joint Monitoring Schemes for Mean and Variance

A M Razmy and T S G Peiris

Abstract

In quality control, joint monitoring schemes for mean and variance are preferred for situations in which special causes can result in change in both the mean and the variance. Several such joint monitoring schemes are reported in the literature to monitor the mean and variance simultaneously of a normally distributed process. Like in the single monitoring of one variable, in the joint monitoring also, combined Shewhart scheme for mean and variance is preferred by many practitioners because of its simplicity compared to combined cumulative sum (CUSUM) and exponentially weighted moving average (EWMA) schemes. Four such Shewhart combined schemes are proposed by three authors in the literature. In this study, the performances of these four schemes are compared using average run length properties under a common platform. Overall, the Shewhart distance scheme performs best and the poorest performance is observed for Shewhart scheme with rectangular acceptance region.

1. Introduction[1]

The first quality control chart was proposed by Shewhart in 1939 based on sample means [1]. Since then statistical process control has become an essential tool in quality monitoring. Advanced control charts such as cumulative sum (CUSUM) and exponentially weighted moving average (EWMA) were introduced by Page in 1961 and Roberts in 1959 respectively [2,3]. The Shewhart $\bar{x}$ chart is still preferred by many industries because of its simplicity, even though it is less sensitive for the small shifts in mean. Later these Shewhart, CUSUM and EWMA charts were altered to monitor the process variance. Gan emphasized that process monitoring is really a bivariate problem, which should not be dealt with as two separate univariate problems [4]. It is often desirable to monitor the mean and variance simultaneously, since a change in the variance can affect the control limits of the mean chart [5]. For some processes, special causes can result in simultaneous change in both the mean and the variance. For example, in circuit manufacturing, an improperly fixed stencil can result in a shift in both the mean and variance of the thickness of the solder paste printed onto circuit boards [6]. In such cases, simultaneous monitoring of both parameters is a logical approach to process control. Therefore it is more reasonable to combine the mean and variance information in one scheme and look at their behavior jointly [5]. In the control chart literature, several mean and variance control charts under the Shewhart, CUSUM and EWMA schemes were combined to one or two-chart schemes to monitor the mean and variance parameters simultaneously for a normally distributed processes. Gan [7] combined the EWMA charts for joint monitoring. Max EWMA and EWMA-semicircle schemes were proposed by Chen et al in 2001 and 2004 respectively [8, 9]. However combined Shewhart schemes are preferred in the industry because of their simplicity and four such schemes are presented in the literature. Gan discussed two types of Shewhart combined schemes one with rectangular control region (denote this scheme as SS_r) and other with elliptical control region (denote this scheme as SS_e) [4]. Chen and Cheng proposed a joint monitoring scheme called max charting scheme (denote this scheme as SS_m) [10]. Shewhart distance scheme (denote this scheme as SS_d) was proposed by Razmy [11]. In this paper, performances of these four schemes are studied based on the average run length (ARL) properties.

2. Methodology

A common platform is developed to compare the performance of these four joint monitoring schemes based on their out of control ARLs when there is shift in mean, variance or both. Let X_{tj} denote a certain quality characteristic of a process where t is the sample number, j is the j^{th} unit of the sample and j = 1, 2, ..., n. It is assumed that X_{tj} 's are independently and identically normally distributed random variables with mean μ_0 and standard deviation σ_0. In addition, let $\bar{X}_t = \frac{1}{n}\sum_{j=1}^{n} X_{tj}$ be the t^{th} sample mean and $S_t^2 = \frac{1}{n-1}\sum_{j=1}^{n}(X_{tj} - \bar{X}_t)^2$ be the t^{th} sample variance.

In the SS_r Scheme, sample means $\bar{\boldsymbol{X}}_t$ are plotted against the log of the sample variances ($\log S_t^2$) for each sample. A rectangular acceptance region is used for making decisions with upper control limit (UCL) and lower control limit (LCL) separately for mean and variance. The UCL (UCL_{Mr}) and the LCL (LCL_{Mr}) for $\bar{\boldsymbol{X}}_t$ are found using simulations based on desired in-control ARL (ARL_{Mr}). The UCL (UCL_{Vr}) and LCL (LCL_{Vr}) for $\log S_t^2$ are found using simulations based on desired in-control ARL (ARL_V). The resulting in-control ARL when monitoring a process using this rectangular control region would be $\frac{1}{ARL} = \frac{1}{ARL_M} + \frac{1}{ARL_V}$. Figure 1 shows the rectangular acceptance region for SS_r Scheme for a process with in-control mean zero and variance.

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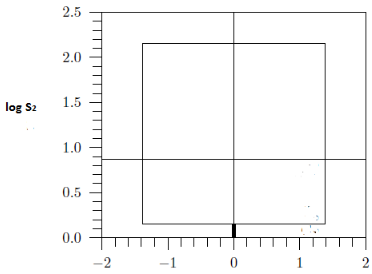

Figure 1: A SS_r Scheme with rectangular acceptance region.

In the SS_e Scheme also, sample means $\bar{X}_t$ are plotted against the log of the sample variances (log S_t^2) for each sample but an elliptical acceptance region is used for making decisions. The UCL (UCL_{Me}) and the LCL (LCL_{Me}) for $\bar{X}_t$ and UCL (UCL_{Ve}) and LCL (LCL_{Ve}) for log S_t^2 are found using simulations based on desired in-control ARL. The Hotelling type statistics T^2 against the sample number t was used for simulation where T_t^2 for a point in which log S_t^2 is greater than $E[\log(S_t^2)]$ is given as

$$\boldsymbol{T_t^2} = \frac{(\bar{X}_t-\mu_0)^2}{UCL_{Me}^2} - \frac{\left[\log S_t^2 - E\left(\log(S_t^2)\right)\right]^2}{\left[UCL_{Ve} - E\left(\log(S_t^2)\right)\right]^2} \text{.........}(1)$$

and T_t^2 for a point in which log S_t^2 is less than or equal to $E[\log(S_t^2)]$ is given as

$$T_t^2 = \frac{(\bar{X}_t-\mu_0)^2}{LCL_{Me}^2} - \frac{\left[\log S_t^2 - E\left(\log(S_t^2)\right)\right]^2}{\left[LCL_{Ve} - E\left(\log(S_t^2)\right)\right]^2} \text{........}(2).$$

Figure 2 shows the elliptical acceptance region for SS_e scheme for a process with in-control mean zero and variance one.

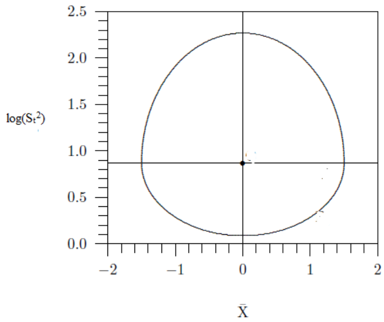

Figure 2: A SS_e Scheme with Elliptical Acceptance Region.

In the SS_m Scheme, Shewhart mean and variance charts are combined to produce one chart. The sample mean for each sample is standardized as

$$U_t = \frac{\bar{X}_t-\mu_0}{\sigma_0/\sqrt{n}} \text{..................}(3).$$

The standard deviation of each sample is transformed as

$$V_t = \Phi^{-1}\left[\mathrm{H}\left(\frac{(\mathrm{n}-1)\mathrm{S}_t^2}{\sigma_0}; \mathrm{n}-1\right)\right] \text{..................}(4)$$

Where,

$$\mathrm{H}\left(\frac{(\mathrm{n}-1)\mathrm{S}_t^2}{\sigma_0}; \mathrm{n}-1\right) = \mathrm{H}(\mathrm{w};\mathrm{v}) = \mathrm{P}(\mathrm{W} \le \mathrm{w}) \ for\ W \sim \chi_v^2, \ ..(5)$$

the chi-square distribution with v degrees of freedom. V_t is the standardized variance. These transformations were originally proposed by Quesenberry [12]. The SS_m Scheme is obtained by plotting the statistics

$$M_t = max[|U_t|, |V_t|] \text{...............................}(6)$$

against the sample number. This scheme will only have UCL (UCL_m) and it is found through simulations based on desired in-control ARLs.

In SS_d scheme the standardized variables U_t and V_t are used. The variables U_t and V_t are independent because $\bar{X}_t$ and S_t^2 are independent and when the process is in-control, both U_t and V_t are standard normal random variables. The SS_d is set up by plotting the statistics D_t against the sample number where,

$$D_t = \sqrt{U_t^2 + V_t^2} \text{.........}(7)$$

and D^2 has a chi-square distribution with 2 degrees of freedom when the process is in-control. The UCL for this scheme (UCL_d) can be found from the equation

$$P(D^2 \le UCL_d^2) = \frac{1}{ARL} \ for\ D^2 \sim \chi_2^2 \text{.......}(8).$$

The control limits of all four schemes discussed above are found for commonly used in control ARLS 250 and 370 by simulation performed in SAS using the normal random number generator RANNOR. The in-control mean and variance are assumed to be $\mu_0 = 0$ and $\sigma_0 = 1$ respectively for easy understanding and comparison purpose. The performances of the four schemes are compared based on out of control ARLs when there is a shift in mean, variance or in both [13]. The shifts in mean and variance investigated are given by $\mu = \mu_0 + \Delta\frac{\sigma_0}{\sqrt{n}}$ and $\sigma = \delta\sigma_0$, where

$\Delta = 0.0, 0.2, 0.4, 0.6, 1.0, 1.5, 3.0$ *and*
$\delta = 0.50, 0.75, 0.95, 1.00, 1.10, 1.25, 1.50, 3.00.$

These Δ and δ are the number of standard deviation shifts in mean and variance respectively. Five normally distributed observations were considered in each sample and all simulations were run in SAS using control limits for in-control ARL of either 250 or 370. For each scheme and each (Δ, δ)combination 1,000,000 runs were performed to estimate the out-of-control ARL. The standard deviations of run length (SDRL) values ensured that SDRL were less than 1% of the estimated ARL. The schemes' behaviors are ranked for easy reference where the scheme giving the lowest out of control ARL for a particular combination of shift in mean and variance is given rank 1 and the scheme giving highest out of control ARL is given rank 4.

3. Results and Discussion

Table 1 presents the chart parameters obtained through simulations for the in control ARLs of 250 and 370. The absolute value of the control limits are always greater for higher in control ARLs for all schemes. When compare to SS_r and SS_e schemes, the control limits are a little large for SS_e scheme for a particular in control ARL because of its elliptical shape acceptance region. SS_m Scheme's control limit is little smaller than SS_d scheme's because the SS_m scheme uses only the maximum of monitoring variables U_t and V_t whereas the SS_d scheme uses the radius of the U_t and V_t plotted on perpendicular axes.

Table 1: Control limits of the Shewhart joint monitoring charting schemes with In-Control ARLs of 250 and 370

Scheme	Control Region	Control Chart Parameters ARL =250	Control Chart Parameters ARL =370
SS_r	Rectangular	UCL_{Mr} = 1.383 LCL_{Mr} = -1.383 UCL_{Vr} = 1.531 LCL_{Vr} = -3.789	UCL_{Mr} = 1.433 LCL_{Mr} = -1.433 UCL_{Vr} = 1.576 LCL_{Vr} = -3.993
SS_e	Elliptical	UCL_{Me} = 1.448 LCL_{Me} = -1.448 UCL_{Ve} = 1.634 LCL_{Ve} = -4.000	UCL_{Me} = 1.500 LCL_{Me} = -1.500 UCL_{Ve} = 1.834 LCL_{Ve} = -4.256
SS_m	Single chart	UCL_m = 3.090	UCL_m = 3.205
SS_d	Circular	UCL_d = 3.323	UCL_d = 3.439

The four schemes discussed detect various magnitudes of shifts in mean and variance based on their sensitivities. The scheme which has the least ARL, when there is a shift in mean or variance is better than other schemes. The out of control ARLs for various magnitudes of shifts in mean and variance are tabulated in the Tables 2 and 3 separately for in-control ARL of 250 and 370. The ranks of the schemes for in control ARLs 250 and 370 are shown in Figure 3 and 4 respectively.

When threre is no shift in mean ($\Delta = 0.0$) and decrease in variance (δ =0.5), the lowest out of control ARL was observed for SS_m scheme and the highest was observed for SS_d scheme. The decrease in variance when the mean is in control is a very favorable condition and the quality control engineers do not want the scheme to signal for out of control. Therefore in this scenario SS_d scheme performs best. When there is no shift in mean and variance, the ARL should be the in control ARL and therefore the rank 1 is given for all schemes. When $(\Delta = 0 , \delta > 1.0)$, $(\Delta = 0.4 , \delta > 1.0)$, $(\Delta = 1.0, \delta > 1.25)$ and $(\Delta = 3.0 , \delta = 0.5 , 3.0))$ combination shifts, the SS_d scheme performs best. When $(\Delta = 0.4 , \delta = 1)$, $(\Delta = 1, \delta \leq 1.25)$ and $(\Delta = 3.0 , \delta = 1.0, 1,25, 1.5))$ combination shifts, the SS_e scheme performs best. If we consider all combination of the shifts, overall the SS_d schemes performs best because it gets the lowest rank total. The SS_e scheme performs next best and the least performance was observed for the SS_r scheme. Therefore one could recommend SS_d scheme for joint monitoring if the shift combination is to be detected quickly and is not exactly known.

References

[1] W.A. Shewhart, "*Statistical methods from the viewpoint of quality control"*, Washington D.C, Graduate School, Department of Agriculture, 1939,. pp. 75.

[2] E.S. Page, (1961), "Cumulative Sum Charts,". *Technometrics*, vol 3, pp. 1–9, 1961.

[3] S.W. Roberts, "Control chart based on geometric moving averages," *Technometrics,* vol. 1, pp. 239-250, 1959.

[4] F.F. Gan, "Joint Monitoring of process mean and variance," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 30, series 7, pp. 4017–4024, 1997.

[5] A.K. McCracken, and S. Chakaborthi, "Control Charts for Joint Monitoring of mean and variance: An overview," *Journal of Quality Technology & Quantitative Management,* vol. 10 series 1, pp. 17–36, 2013.

[6] F.F. Gan, K.W. Ting, and T. Chang,. "Interval charting schemes for joint monitoring of process mean and variance," *Quality and Reliability Engineering International,* vol. 20 series 4, pp. 291 – 303, 2004.

[7] F.F. Gan, "Joint Monitoring of process mean and variance using exponentially weighted moving average control charts," *Technometrics*, vol. 37, pp. 446–453, 1995.

[8] G. Chen, S.W. Cheng, and H.W. Xie, "Monitoring process mean and variability with one EWMA chart," *Journal of Quality Technology*, vol. 33, pp. 223–233, 2001.

[9] G. Chen, S.W. Cheng, and H.W. Xie, " A new EWMA control chart for monitoring both location and dispersion," *Quality Technology & Quantitative Management*, vol 1, pp. 217–231, 2004.

[10] G. Chen, and S.W. Cheng, "Max chart: Combining X-Bar chart and S-chart," *Statistica Sinica*, vol. 8, pp. 263–271, 1998.

[11] A.M. Razmy, "Joint Monitoring of Process Mean and Variance with Shewhart Distance Scheme," *Sri Lankan Journal of Applied Statistics*, vol. 2, pp. 14-26, 2010.

[12] C.P. Quesenberry, "On properties of Q charts variables," *Journal of Quality Technology*, vol. 21, pp. 242–250, 1995.

[13] A.M. Razmy, and T.S.G. Peiris, "A Standard method to Compare the combined Quality Monitoring Schemes using Average Run Length Properties," *Second International Symposium, South Eastern University of Sri Lanka,* 2012, pp. 172-173.

Table 2: Average Run Lengths of combined schemes with respect to the process mean ($\mu = \mu_0 + \Delta \frac{\sigma_0}{\sqrt{n}}$) and standard deviation ($\sigma = \delta\sigma_0$). In-Control ARL = 250

Δ	δ	SS_m	SS_d	SS_r	SS_e	Δ	δ	SS_m	SS_d	SS_r	SS_e
0	0.5	68.1	128.9	68.6	93.6	1	0.5	68.4	64.0	68.8	59.8
0	0.75	322.1	451.9	322.2	371.1	1	0.75	174.3	128.6	175.9	136.2
0	0.95	360.9	370.2	365.9	358.7	1	0.95	64.7	65.1	69.0	59.3
0	1	250.2	249.3	251.8	251.1	1	1	49.3	49.8	49.9	44.7
0	1.05	160.5	156.8	160.4	164.8	1	1.05	38.0	37.3	38.3	34.1
0	1.1	101.6	97.0	102.5	105.7	1	1.1	29.8	28.7	30.0	26.2
0	1.25	31.0	28.1	31.1	32.9	1	1.25	15.1	13.6	12.6	13.4
0	1.5	8.3	7.4	8.3	8.8	1	1.5	6.2	5.4	6.2	5.9
0	3	1.2	1.2	1.2	1.3	1	3	1.2	1.2	1.2	1.2
0.2	0.5	68.3	124.7	69.0	92.0	1.5	0.5	65.1	27.9	65.8	32.1
0.2	0.75	316.3	424.9	318.5	356.0	1.5	0.75	49.9	39.2	50.3	41.6
0.2	0.95	324.5	331.5	328.9	317.6	1.5	0.95	20.5	21.9	20.7	19.6
0.2	1	226.1	223.9	227.2	223.0	1.5	1	17.2	18.2	17.4	16.2
0.2	1.05	147.3	143.0	148.9	147.0	1.5	1.05	14.8	15.2	14.8	13.5
0.2	1.1	94.8	89.7	95.9	96.4	1.5	1.1	12.6	12.6	12.8	11.3
0.2	1.25	29.8	27.1	29.9	31.1	1.5	1.25	8.4	7.7	8.5	7.2
0.2	1.5	8.2	7.3	8.3	8.7	1.5	1.5	4.6	4.0	4.6	4.1
0.2	3	1.2	1.2	1.2	1.3	1.5	3	1.2	1.2	1.2	1.2
0.4	0.5	68.6	114.0	68.8	87.1	3	0.5	2.3	1.8	2.3	1.9
0.4	0.75	307.3	364.5	310.8	320.6	3	0.75	2.2	2.3	2.2	2.3
0.4	0.95	245.3	244.8	247.9	231.7	3	0.95	2.2	2.3	2.2	2.2
0.4	1	171.9	170.2	174.0	162.6	3	1	2.2	2.3	2.2	2.2
0.4	1.05	116.5	111.3	116.7	110.8	3	1.05	2.2	2.2	2.2	2.1
0.4	1.1	78.1	73.4	78.9	75.1	3	1.1	2.1	2.2	2.1	2.1
0.4	1.25	26.9	24.3	27.1	27.1	3	1.25	2.1	2.0	2.0	1.9
0.4	1.5	7.9	7.0	7.9	8.1	3	1.5	1.9	1.8	1.9	1.7
0.4	3	1.2	1.2	1.2	1.3	3	3	1.2	1.1	1.2	1.2
0.6	0.5	68.6	99.6	69.2	79.6						
0.6	0.75	282.4	279.3	285.5	264.3						
0.6	0.95	165.2	162.8	167.0	151.4						
0.6	1	117.7	115.9	118.8	108.4						
0.6	1.05	83.1	79.9	84.2	76.5						
0.6	1.1	59.0	55.4	59.4	54.3						
0.6	1.25	23.1	20.6	23.1	22.0						
0.6	1.5	7.4	6.5	7.4	7.4						
0.6	3	1.2	1.2	1.2	1.2						

Table 3: Average run lengths of combined schemes with respect to the process mean ($\mu = \mu_0 + \Delta \frac{\sigma_0}{\sqrt{n}}$) and standard deviation ($\sigma = \delta\sigma_0$). In-Control ARL = 370

Δ	δ	SS_m	SS_d	SS_r	SS_e	Δ	δ	SS_m	SS_d	SS_r	SS_e
0	0.5	99.2	192.7	101.0	129.6	1	0.5	99.2	95.7	100.1	83.6
0	0.75	475.6	686.7	480.3	520.1	1	0.75	267.4	192.5	270.5	199.2
0	0.95	547.8	564.0	551.7	526.3	1	0.95	89.0	90.4	89.6	81.7
0	1	368.9	370.9	370.3	370.7	1	1	66.1	67.5	66.3	59.3
0	1.05	229.4	224.1	229.1	235.4	1	1.05	50.0	49.5	50.5	45.4
0	1.1	141.7	133.6	141.4	145.3	1	1.1	38.1	37.0	38.1	33.1
0	1.25	39.5	35.9	39.8	42.3	1	1.25	18.6	16.4	18.5	16.4
0	1.5	9.7	8.6	9.7	10.6	1	1.5	7.1	6.2	7.2	6.8
0	3	1.3	1.2	1.3	1.3	1	3	1.3	1.2	1.3	1.3
0.2	0.5	100.1	187.6	100.9	127.6	1.5	0.5	96.0	41.2	97.3	45.6
0.2	0.75	469.4	653.1	480.4	506.9	1.5	0.75	73.5	56.0	73.2	60.2
0.2	0.95	491.7	500.8	493.9	466.7	1.5	0.95	26.8	28.7	26.6	25.5
0.2	1	331.6	330.2	335.0	321.4	1.5	1	22.1	23.4	22.0	20.3
0.2	1.05	210.0	203.1	207.4	209.1	1.5	1.05	18.3	19.0	18.5	16.8
0.2	1.1	130.7	124.4	130.7	132.5	1.5	1.1	15.7	15.5	15.7	13.7
0.2	1.25	38.2	34.3	37.9	40.0	1.5	1.25	10.0	9.1	10.0	8.5
0.2	1.5	9.6	8.5	9.5	10.2	1.5	1.5	5.2	4.5	5.2	4.6
0.2	3	1.3	1.2	1.3	1.3	1.5	3	1.2	1.2	1.2	1.2
0.4	0.5	99.4	172.3	100.9	121.9	3	0.5	2.9	2.1	2.9	2.3
0.4	0.75	455.9	555.1	461.4	458.1	3	0.75	2.5	2.7	2.5	2.6
0.4	0.95	365.0	363.3	367.0	335.4	3	0.95	2.4	2.6	2.4	2.5
0.4	1	249.6	244.1	252.9	229.2	3	1	2.4	2.6	2.4	2.4
0.4	1.05	162.9	156.7	163.2	152.8	3	1.05	2.4	2.5	2.3	2.3
0.4	1.1	106.3	100.1	107.3	102.8	3	1.1	2.3	2.4	2.3	2.3
0.4	1.25	34.4	30.6	34.3	34.4	3	1.25	2.3	2.2	2.3	2.1
0.4	1.5	9.2	8.1	9.2	9.5	3	1.5	2.1	1.9	2.1	1.9
0.4	3	1.3	1.2	1.3	1.3	3	3	1.2	1.1	1.2	1.2
0.6	0.5	99.6	150.3	101.3	109.6						
0.6	0.75	423.2	424.7	431.5	377.5						
0.6	0.95	239.3	237.0	239.9	217.1						
0.6	1	166.2	164.6	167.2	150.0						
0.6	1.05	113.8	110.6	114.2	103.7						
0.6	1.1	78.7	74.6	78.5	72.2						
0.6	1.25	29.0	25.6	29.1	26.9						
0.6	1.5	8.6	7.6	8.6	8.6						
0.6	3	1.3	1.2	1.3	1.3						

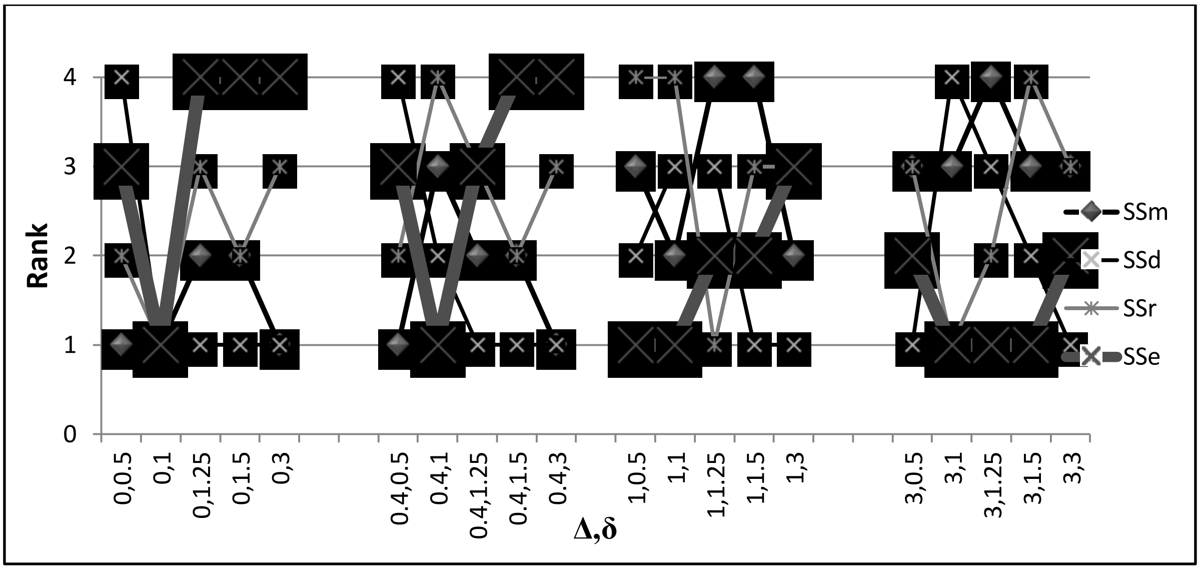

Figure 3: Rank comparisons for different shifts in mean and variance for in control ARL 250

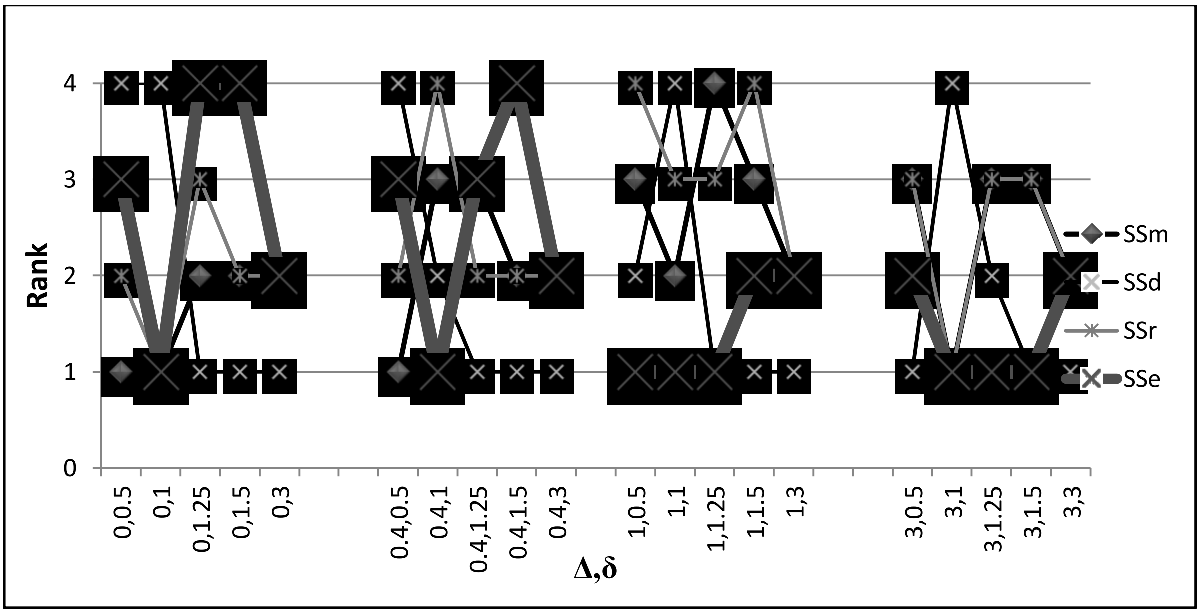

Figure 4: Rank comparisons for different shifts in mean and variance for in control ARL 370