

3

Synthesis of analogue circuits

We have studied how (network) functions, systems of differential equations, matrices and other mathematical abstractions can represent physical reality. We will now look at this from the other side – can we construct physical objects that correspond to a given abstraction?

We will first look at the relatively simple problem of constructing passive circuits that correspond to a given network function. By this time, you are of course aware that the function itself should have certain characteristics if it is to correspond to a realisable network. In the special but all-important case of the design of filters to meet specific requirements, we have to go one step backwards to obtain a system function that ensures such compliance, before we can obtain a realisation. We will see how the so-called “modern filter design” techniques accomplish this task through mathematical approximations of the desired performance characteristics. These techniques, even though traditionally called “modern”, have been about for more than a century!

We will then look at a few standard methods for the realisation of functional blocks such as filters using active circuits.

The treatment up to this has been limited to linear circuits. We have seen that the state space approach allows us to model non-linear systems as well as linear systems. As far as circuits are concerned, we still need mechanisms for the modelling of non-linear elements, even the simplest of them; the diode. We will now step two steps back and investigate some elementary methods for the modelling of non-linear elements.

We also need at least some idea about the sensitivity of network characteristics to changes in component values. Tellegan’s Theorem is an unexpected result that allows us to investigate the sensitivity of circuit performance to parameter changes quite economically. We will look at how this helps us in the design of robust systems.

3.1 Passive circuits

We already know that a network can be represented by a network function. We have also studied how this can be done. We are now going to explore how a network can be synthesised, given its network function. We will first look at one port networks, defined by either impedance or admittance functions and later go on to investigate two port networks.

Let us recall some of the common properties of passive RLC functions.

- They are real, rational functions.
- They do not have poles (or zeros) on the right-half s-plane, nor do they have multiple order poles on the $j\omega$ axis.
- The resulting matrices are symmetric.

Let us now look at each of these types of networks. The LC network functions have these additional properties:

- They are simple, that is there are no higher order poles or zeros.
- All poles and zeros lie on the $j\omega$ axis.
- Poles and zeros alternate.
- The origin and infinity are always critical frequencies, that is, there will be either a pole or a zero at both the origin and at infinity.
- The multiplicative constant is positive.

The properties of RC network functions are:

- The poles and zeros of an RC driving point function lie on the non-positive real axis.
- They are simple.
- Poles and zeros alternate.
- The slopes of impedance functions are negative, those of admittance functions are positive.

Foster and Cauer first proposed the realisation of these networks in various canonical forms, in the 1920s. We will now examine each of these forms. There are four canonical forms, relating to the realisation of LC, RC and RL networks as follows:

	1 st Foster form	2 nd Foster form	1 st Cauer form	2 nd Cauer form
Partial fraction expansion of impedance function	✓			
Partial fraction expansion of admittance function		✓		
Continued fraction expansion about the point at infinity			✓	
Continued fraction expansion about the origin				✓

Considering the pole-zero properties of LC network functions, we can assume that the impedance function will have a zero or pole at the origin and pairs of complex conjugate poles and zeros. The admittance function will have a pole or zero at the origin. These functions may be expanded as partial fractions to yield realisations of Foster-form networks. They may also be subjected to continued fraction expansion to yield Cauer-form networks.

We will consider realisations of each of these forms.

3.1.1 Realisation of LC driving point functions

First Foster form – Impedance functions of LC networks

$$Z(s) = \frac{H(s^2 + \omega_1^2)(s^2 + \omega_3^2) \dots (s^2 + \omega_m^2)}{(s^2 + \omega_2^2)(s^2 + \omega_4^2) \dots (s^2 + \omega_r^2)}$$

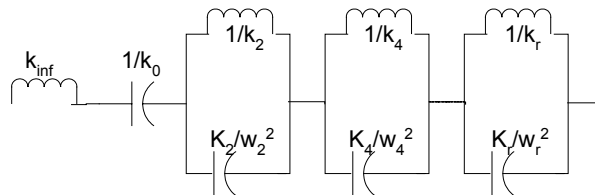
where $0 \leq \omega_1^2 \leq \omega_2^2 \leq \omega_3^2 \leq \dots$

and $m = r \pm 1$

By partial fraction expansion we can write:

$$Z(s) = k_\infty s + \frac{k_0}{s} + \sum_{p=2,4,\dots}^r \frac{k_p s}{s^2 + \omega_p^2}, \quad k_\infty, k_0 \geq 0, k_p > 0$$

This corresponds to the following circuit realisation:



Let us now consider a few examples.

Example 1

We will consider the following simple driving point impedance function, where $\omega_1 > 0$ and $m = r - 1$:

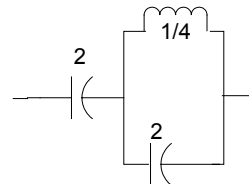
$$Z(s) = \frac{s^2 + 1}{s(s^2 + 2)}$$

By partial fraction expansion we get: This corresponds to the circuit shown:

$$Z(s) = \frac{A}{s} + \frac{Bs}{s^2 + 2} = \frac{A(s^2 + 2) + Bs^2}{s(s^2 + 2)}$$

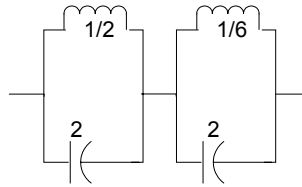
$$\therefore A + B = 1, \quad 2A = 1$$

$$A = \frac{1}{2}, B = \frac{1}{2}$$



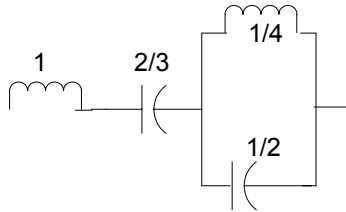
Example 2 $\omega_1 = 0$ and $m = r - 1$:

$$\begin{aligned} Z(s) &= \frac{(s^2+0)(s^2+2)}{s(s^2+1)(s^2+3)} = \frac{s(s^2+2)}{(s^2+1)(s^2+3)} \\ &= \frac{As}{s^2+1} + \frac{Bs}{s^2+3} = \frac{s^3(A+B) + s(3A+B)}{(s^2+1)(s^2+3)} \\ \therefore A+B &= 1, \quad 3A+B = 2 \\ \therefore A &= \frac{1}{2}, \quad B = \frac{1}{2} \\ \therefore Z(s) &= \frac{s}{2(s^2+1)} + \frac{s}{2(s^2+3)} \end{aligned}$$



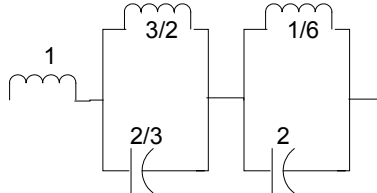
Example 3 $\omega_1 > 0$ and $m = r + 1$:

$$Z(s) = \frac{(s^2+1)(s^2+3)}{s(s^2+2)} = s + \frac{3}{2s} + \frac{s}{2(s^2+2)}$$



Example 4

$$\begin{aligned} Z(s) &= \frac{(s^2+0)(s^2+2)(s^2+4)}{s(s^2+1)(s^2+3)} \\ &= \frac{s(s^2+2)(s^2+4)}{(s^2+1)(s^2+3)} = s + \frac{3s}{2(s^2+1)} + \frac{s}{2(s^2+3)} \end{aligned}$$



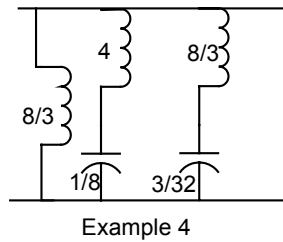
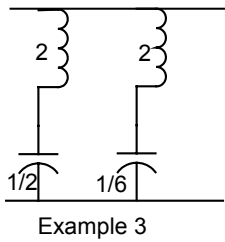
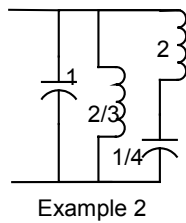
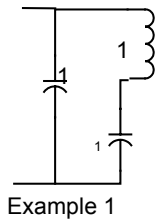
Second Foster form – Admittance functions of LC networks

We will consider the same examples as before:

1. $Z(s) = \frac{s^2+1}{s(s^2+2)} \Rightarrow Y(s) = \frac{s(s^2+2)}{s^2+1} = s + \frac{s}{s^2+1}$
2. $Z(s) = \frac{(s^2+0)(s^2+2)}{s(s^2+1)(s^2+3)} \Rightarrow Y(s) = \frac{s(s^2+1)(s^2+3)}{(s^2+0)(s^2+2)} = s + \frac{3}{2s} + \frac{s}{2(s^2+2)}$
3. $Z(s) = \frac{(s^2+1)(s^2+3)}{s(s^2+2)} \Rightarrow Y(s) = \frac{s(s^2+2)}{(s^2+1)(s^2+3)} = \frac{s}{2(s^2+1)} + \frac{s}{2(s^2+3)}$
- 4.

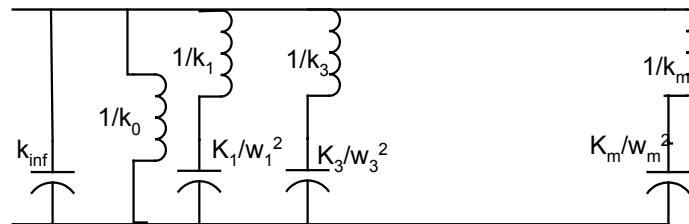
$$\begin{aligned} Z(s) &= \frac{(s^2+0)(s^2+2)(s^2+4)}{s(s^2+1)(s^2+3)} \Rightarrow \\ Y(s) &= \frac{s(s^2+1)(s^2+3)}{(s^2+0)(s^2+2)(s^2+4)} = \frac{3}{8s} + \frac{s}{4(s^2+2)} + \frac{3s}{8(s^2+4)} \end{aligned}$$

The realisations of the four functions are as shown below:



We can obtain a general realisation for the admittance function $Y(s)$ as follows:

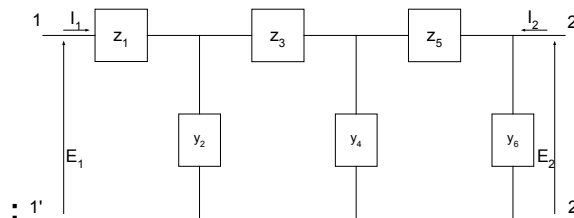
$$Y(s) = k_{\infty}s + \frac{k_0}{s} + \sum_{p=1,3,\dots}^m \frac{k_p s}{s^2 + \omega_p^2}, \quad k_{\infty}, k_0 \geq 0, k_p > 0$$



Realisation of LC network functions using First Cauer form

The first Cauer form is generated by the continued fraction expansion of a reactance function about the point at infinity.

Consider a ladder network as shown [Note that the series arms are impedances while the shunt arms are admittances]



To compute the driving point impedance $Z = \frac{E_1}{I_1}$, we start at the other end:

The impedance of the last term $z_6 = \frac{1}{y_6}$

The impedance of the last two arms is $z_5 + \frac{1}{y_6}$

The impedance of the last three arms is $\frac{1}{y_4 + \frac{1}{z_5 + \frac{1}{y_6}}}$

Proceeding in this manner, we can write

$$Z = z_1 + \frac{1}{y_2 + \frac{1}{z_3 + \frac{1}{y_4 + \frac{1}{z_5 + \frac{1}{y_6}}}}}$$

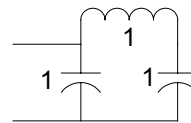
The driving point admittance may also be written in a similar manner, where the first term is an admittance, the next an impedance, etc. We will now consider each of the four examples used to illustrate the Foster forms.

Example 1 $Z(s) = \frac{s^2 + 1}{s(s^2 + 2)}$, $Y(s) = \frac{s(s^2 + 2)}{s^2 + 1}$

Starting with the impedance function $Z(s)$ that has a zero at infinity, we proceed as follows:

$$Z(s) = \frac{1}{\frac{s^3 + 2s}{s^2 + 1}} = \frac{1}{s + \frac{s}{s^2 + 1}} = \frac{1}{s + \frac{1}{\frac{s^2 + 1}{s}}} = \frac{1}{s + \frac{1}{s + \frac{1}{s}}}$$

Either of these will give us the following circuit realisation:



If we use the admittance function, we get:

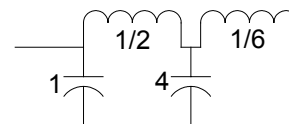
$$Y(s) = \frac{s^3 + 2s}{s^2 + 1} = s + \frac{s}{s^2 + 1} = s + \frac{1}{\frac{s^2 + 1}{s}} = s + \frac{1}{s + \frac{1}{s}}$$

Example 2 $Z(s) = \frac{s(s^2 + 2)}{(s^2 + 1)(s^2 + 3)}$, $Y(s) = \frac{(s^2 + 1)(s^2 + 3)}{s(s^2 + 2)}$

$$Y(s) = \frac{s^4 + 4s^2 + 3}{s^3 + 2s} = s + \frac{2s^2 + 3}{s^3 + 2s} = s + \frac{1}{\frac{s^3 + 2s}{2s^2 + 3}} = s + \frac{1}{s/2 + \frac{s/2}{2s^2 + 3}}$$

$$= s + \frac{1}{s/2 + \frac{1}{\frac{2s^2 + 3}{s/2}}} = s + \frac{1}{s/2 + \frac{1}{4s + \frac{3}{s/2}}}$$

This leads to the realisation:

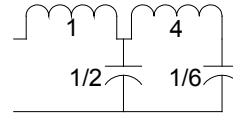


Example 3 $Z(s) = \frac{(s^2 + 1)(s^2 + 3)}{s(s^2 + 2)}$, $Y(s) = \frac{s(s^2 + 2)}{(s^2 + 1)(s^2 + 3)}$

Note that this is the reciprocal of the function considered in example 2.

$$Z(s) = \frac{s^4 + 4s^2 + 3}{s^3 + 2s} = s + \frac{1}{s/2 + \frac{1}{4s + \frac{3}{s/2}}}$$

This leads to the realisation:

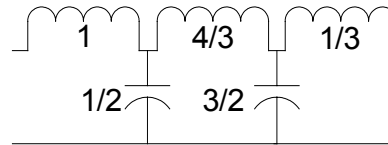


Example 4 $Z(s) = \frac{s(s^2 + 2)(s^2 + 4)}{(s^2 + 1)(s^2 + 3)}$, $Y(s) = \frac{(s^2 + 1)(s^2 + 3)}{s(s^2 + 2)(s^2 + 4)}$

$$Z(s) = \frac{s^5 + 6s^3 + 8s}{s^4 + 4s^2 + 3} = s + \frac{1}{\frac{s^4 + 4s^2 + 3}{2s^3 + 5s}} = s + \frac{1}{s/2 + \frac{1}{\frac{2s^3 + 5s}{3s^2/2 + 3}}}$$

$$= s + \frac{1}{s/2 + \frac{1}{4s/3 + \frac{1}{3s^2/2 + 3}}} = s + \frac{1}{s/2 + \frac{1}{4s/3 + \frac{1}{3s/2 + \frac{1}{s/3}}}}$$

It can be realised as:

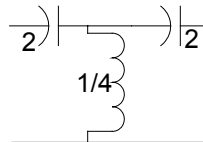


Realisation of LC network functions using Second Caueer form

The second Caueer form is generated by the continued fraction expansion of a reactance function about the origin. This too results in a realisation where the series arms are impedances while the shunt arms are admittances. We will consider the same examples as before. However, in this realisation the series arms contain capacitors while the shunt arms have inductors.

Example 1 $Z(s) = \frac{s^2 + 1}{s(s^2 + 2)}$, $Y(s) = \frac{s(s^2 + 2)}{s^2 + 1}$

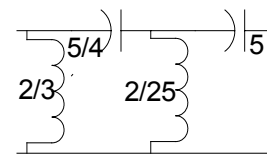
$$Z(s) = \frac{1 + s^2}{2s + s^3} = \frac{1}{2s} + \frac{1}{\frac{4}{s} + 2s} = \frac{1}{2s} + \frac{1}{s + \frac{1}{\frac{1}{s}}}$$



Example 2 $Z(s) = \frac{s(s^2 + 2)}{(s^2 + 1)(s^2 + 3)}$, $Y(s) = \frac{(s^2 + 1)(s^2 + 3)}{s(s^2 + 2)}$

$$Y(s) = \frac{3 + 4s^2}{2s + s^3} + s^4 = \frac{3}{2s} + \frac{(5/2)s^2 + s^4}{2s + s^3} = \frac{3}{2s} + \frac{1}{\frac{2s + s^3}{(5/2)s^2 + s^4}} = \frac{3}{2s} + \frac{1}{\frac{4/5}{s} + \frac{(1/5)s}{(5/2) + s}}$$

$$= \frac{3}{2s} + \frac{1}{\frac{4/5}{s} + \frac{1}{\frac{(5/2) + s}{(1/5)s}}} = \frac{3}{2s} + \frac{1}{\frac{4/5}{s} + \frac{1}{\frac{25/2}{s} + 5s}} = \frac{3}{2s} + \frac{1}{\frac{4/5}{s} + \frac{1}{\frac{25/2}{s} + \frac{1}{1/5}}}$$

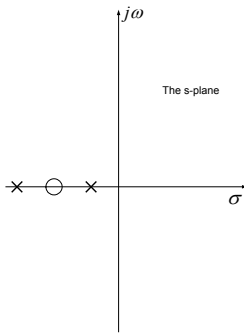


Please work out the other two examples.

3.1.2 RC Driving point functions

We will start with the properties of RC driving point functions that we are already familiar with. We have noted that:

- The poles and zeros are simple.
- They all lie on the non-positive σ - axis.
- Poles and zeros alternate.
- The critical frequency of the smallest magnitude is a pole.



A typical pole-zero plot on the s-plane is as shown. With these properties, the form of the impedance function may be represented by:

$$Z(s) = \frac{H(s-s_2)(s-s_4)\dots(s-s_m)}{(s-s_1)(s-s_3)\dots(s-s_n)}$$

As the smallest critical frequency corresponds to a pole and as poles and zeros alternate,

$$0 \leq s_1 < s_2 < s_3 < \dots$$

The following table gives the possible forms of the expansion of such a function.

First Foster form: Expansion of $Z(s)$	$Z(s) = k_0 + \frac{k_1}{s-s_1} + \frac{k_3}{s-s_3} \dots + \frac{k_n}{s-s_n}$
Second Foster form: Expansion of $Y(s)$	$Y(s) = k_\infty s + k_0 + \frac{k_2 s}{s-s_2} + \frac{k_4 s}{s-s_4} + \dots + \frac{k_m s}{s-s_m}$
First Cauer form: Continued fraction expansion about infinity	$Z(s) = a_1 + \frac{1}{b_2 s + \frac{1}{a_3 + \frac{1}{b_4 s + \frac{1}{a_5 + \dots}}}}$
Second Cauer form: Continued fraction expansion about the origin.	$Z(s) = \frac{1}{a_1 s} + \frac{1}{b_2 + \frac{1}{a_3 s + \frac{1}{b_4 + \frac{1}{a_5 s + \dots}}}}$

There are four possible combinations that we need to study. We will consider one example from each of these:

	$m = n - 1$ (Order of numerator is one less than that of denominator)	$m = n$ (Both numerator and denominator are of the same order)
$s_1 = 0$	Example 1 $Z(s) = \frac{(s+1)}{s(s+2)}$	Example 2 $Z(s) = \frac{(s+1)(s+3)}{s(s+2)}$
$s_1 > 0$	Example 3 $Z(s) = \frac{(s+2)}{(s+1)(s+3)}$	Example 4 $Z(s) = \frac{(s+2)(s+4)}{(s+1)(s+3)}$

First Foster form:

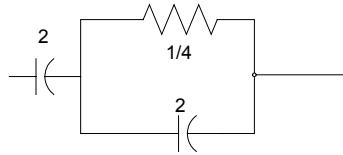
Example 1

$$Z(s) = \frac{(s+1)}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2} = \frac{A(s+2) + Bs}{s(s+2)}$$

$$\therefore A + B = 1, \quad 2A = 1$$

$$\therefore A = 1/2, \quad B = 1/2$$

$$Z(s) = \frac{1/2}{s} + \frac{1/2}{s+2}$$



Example 2

$$Z(s) = \frac{(s+1)(s+3)}{s(s+2)} = \frac{s^2 + 4s + 3}{s^2 + 2s}$$

$$= 1 + \frac{2s+3}{s(s+2)} = 1 + \frac{A}{s} + \frac{B}{s+2}$$

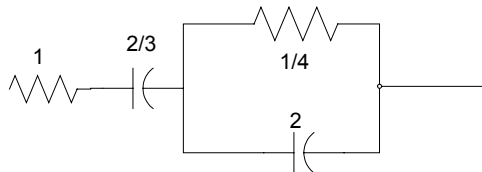
$$= 1 + \frac{A(s+2) + Bs}{s(s+2)}$$

$$A + B = 2$$

$$2A = 3$$

$$A = 3/2, \quad B = 1/2$$

$$Z(s) = 1 + \frac{3/2}{s} + \frac{1/2}{s+2}$$



Example 3

$$Z(s) = \frac{(s+2)}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$$

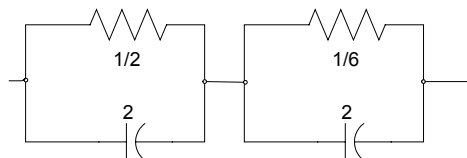
$$\frac{A(s+3) + B(s+1)}{(s+1)(s+3)}$$

$$A + B = 1$$

$$3A + B = 2$$

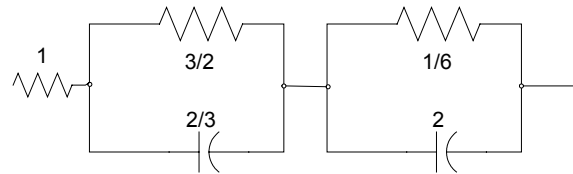
$$A = 1/2, \quad B = 1/2$$

$$Z(s) = \frac{1/2}{s+1} + \frac{1/2}{s+3}$$



Example 4

$$\begin{aligned} Z(s) &= \frac{(s+2)(s+4)}{(s+1)(s+3)} = \frac{s^2 + 6s + 8}{s^2 + 4s + 3} \\ &= 1 + \frac{2s+5}{(s+1)(s+3)} = 1 + \frac{A}{s+1} + \frac{B}{s+3} \\ &= 1 + \frac{A(s+3) + B(s+1)}{(s+1)(s+3)} \end{aligned}$$



$$A + B = 2$$

$$3A + B = 5$$

$$A = 3/2, B = 1/2$$

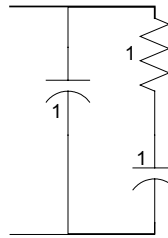
$$Z(s) = 1 + \frac{3/2}{(s+1)} + \frac{1/2}{(s+3)}$$

Second Foster form:

Example 1

$$Z(s) = \frac{(s+1)}{s(s+2)}$$

$$Y(s) = \frac{s(s+2)}{(s+1)} = \frac{s^2 + 2s}{s+1} = s + \frac{s}{s+1}$$



Example 2

$$Z(s) = \frac{(s+1)(s+3)}{s(s+2)}$$

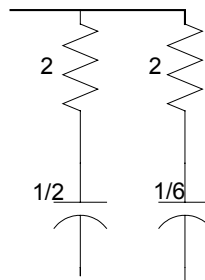
$$\begin{aligned} Y(s) &= \frac{s(s+2)}{(s+1)(s+3)} = \frac{As}{s+1} + \frac{Bs}{s+3} \\ &= \frac{As(s+3) + Bs(s+1)}{(s+1)(s+3)} \end{aligned}$$

$$A + B = 1$$

$$3A + B = 2$$

$$A = 1/2, B = 1/2$$

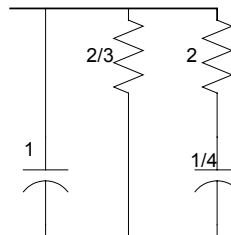
$$Y(s) = \frac{s/2}{s+1} + \frac{s/2}{s+3}$$



Example 3

$$Z(s) = \frac{(s+2)}{(s+1)(s+3)}$$

$$\begin{aligned} Y(s) &= \frac{(s+1)(s+3)}{(s+2)} = \frac{s^2 + 4s + 3}{s+2} \\ &= s + \frac{2s+3}{s+2} = s + \frac{3}{2} + \frac{s/2}{s+2} \end{aligned}$$



Example 4

$$Z(s) = \frac{(s+2)(s+4)}{(s+1)(s+3)}$$

$$Y(s) = \frac{(s+1)(s+3)}{(s+2)(s+4)} = \frac{3}{8} + \frac{7s/4 + 5s^2/8}{(s+2)(s+4)}$$

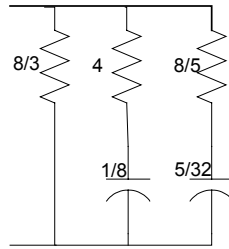
$$= \frac{3}{8} + \frac{As}{s+2} + \frac{Bs}{s+4} = \frac{3}{8} + \frac{As(s+4) + Bs(s+2)}{(s+2)(s+4)}$$

$$A+B = 5/8$$

$$4A+2B = 7/4 \Rightarrow 2A+B = 7/8$$

$$A = 1/4, B = 3/8$$

$$Y(s) = \frac{3}{8} + \frac{s/4}{s+2} + \frac{3s/8}{s+4}$$

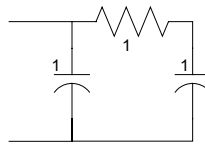


First Cauer form:

Example 1

$$Z(s) = \frac{(s+1)}{s(s+2)} = \frac{1}{s(s+2)} = \frac{1}{s + \frac{s}{s+1}}$$

$$= \frac{1}{s + \frac{1}{\frac{s+1}{s}}} = \frac{1}{s + \frac{1}{1 + \frac{1}{s}}}$$



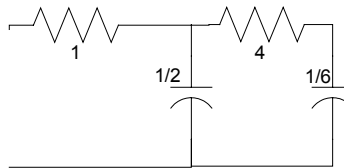
Example 2

$$Z(s) = \frac{(s+1)(s+3)}{s(s+2)} = \frac{s^2 + 4s + 3}{s^2 + 2s}$$

$$= 1 + \frac{2s+3}{s^2+2s} = 1 + \frac{1}{\frac{s^2+2s}{2s+3}} = 1 + \frac{1}{s/2 + \frac{s/2}{2s+3}}$$

$$= 1 + \frac{1}{s/2 + \frac{1}{\frac{2s+3}{s/2}}} = 1 + \frac{1}{s/2 + \frac{1}{4 + \frac{3}{s/2}}}$$

$$= 1 + \frac{1}{s/2 + \frac{1}{4 + \frac{1}{s/6}}}$$

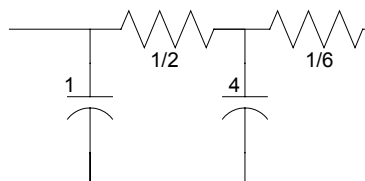


Example 3

$$Z(s) = \frac{(s+2)}{(s+1)(s+3)} = \frac{1}{\frac{s^2+4s+3}{s+2}} = \frac{1}{s + \frac{2s+3}{s+2}}$$

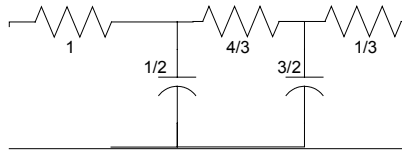
$$= \frac{1}{s + \frac{1}{\frac{1/2 + 1/2}{1/2 + \frac{1}{2s+3}}}} = \frac{1}{s + \frac{1}{1/2 + \frac{1}{4s+6}}}$$

$$= \frac{1}{s + \frac{1}{1/2 + \frac{1}{4s + \frac{1}{1/6}}}}$$



Example 4

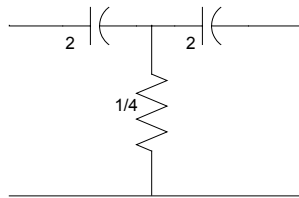
$$\begin{aligned}
 Z(s) &= \frac{(s+2)(s+4)}{(s+1)(s+3)} = \frac{s^2+6s+8}{s^2+4s+3} = 1 + \frac{2s+5}{s^2+4s+3} \\
 &= 1 + \frac{1}{\frac{s^2+4s+3}{2s+5}} = 1 + \frac{1}{s/2 + \frac{3s/2+3}{2s+5}} \\
 &= 1 + \frac{1}{s/2 + \frac{1}{\frac{4/3 + \frac{1}{3s/2+3}}{1}}} \\
 &= 1 + \frac{1}{s/2 + \frac{1}{4/3 + \frac{1}{3s/2 + 1/3}}}
 \end{aligned}$$



Second Cauer form:

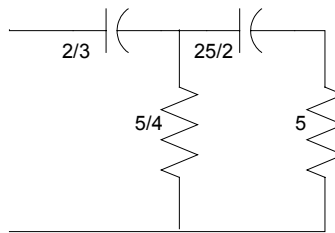
Example 1

$$\begin{aligned}
 Z(s) &= \frac{(s+1)}{s(s+2)} = \frac{1+s}{2s+s^2} = \frac{1}{2s} + \frac{s/2}{2s+s^2} \\
 &= \frac{1}{2s} + \frac{1}{\frac{2s+s^2}{s/2}} = \frac{1}{2s} + \frac{1}{4 + \frac{s^2}{s/2}} \\
 &= \frac{1}{2s} + \frac{1}{4 + \frac{1}{\frac{1}{2s}}}
 \end{aligned}$$



Example 2

$$\begin{aligned}
 Z(s) &= \frac{(s+1)(s+3)}{s(s+2)} = \frac{3+4s+s^2}{2s+s^2} = \frac{1}{2s/3} + \frac{5s/2+s^2}{2s+s^2} \\
 &= \frac{1}{2s/3} + \frac{1}{\frac{2s+s^2}{5s/2+s^2}} = \frac{1}{2s/3} + \frac{1}{4/5 + \frac{s^2/5}{5s/2+s^2}} \\
 &= \frac{1}{2s/3} + \frac{1}{4/5 + \frac{1}{\frac{5s/2+s^2}{s^2/5}}} = \frac{1}{2s/3} + \frac{1}{4/5 + \frac{1}{25/2s + 1/5}}
 \end{aligned}$$



3.1.3 RL Driving point functions

The easiest way to start is to make use of the duality of RC and RL network functions. We used the following general form of the impedance of an RC function:

$$Z(s) = \frac{H(s-s_2)(s-s_4)\dots(s-s_m)}{(s-s_1)(s-s_3)\dots(s-s_n)}$$

We now consider a function of the same form as the admittance function of an RL network:

$$Y(s) = \frac{H(s-s_2)(s-s_4)\dots(s-s_m)}{(s-s_1)(s-s_3)\dots(s-s_n)}$$

As the smallest critical frequency corresponds to a pole and as poles and zeros alternate,

$$0 \leq s_1 < s_2 < s_3 < \dots$$

There are four possible combinations that we need to study. An example of each is given below:

	$m = n - 1$ (Order of numerator is one less than that of denominator)	$m = n$ (Both numerator and denominator are of the same order)
$s_1 = 0$	Example 1 $Y(s) = \frac{(s+1)}{s(s+2)}$	Example 2 $Y(s) = \frac{(s+1)(s+3)}{s(s+2)}$
$s_1 > 0$	Example 3 $Y(s) = \frac{(s+2)}{(s+1)(s+3)}$	Example 4 $Y(s) = \frac{(s+2)(s+4)}{(s+1)(s+3)}$

Their realisations follow the same pattern as in the case of the RC network functions.

3.1.4 Synthesis of RLC circuits

Out of the many procedures available in classical design for the synthesis of RLC circuits, we will consider the Brune Synthesis, which starts off with what is called the Foster Preamble.

Foster Preamble

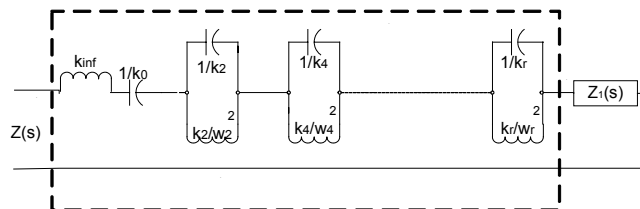
Step 1: Reactance reduction

We have to realise a given positive real function $Z(s)$.

If the function contains poles on the imaginary axis, we can expand this in partial fraction form to yield:

$$Z(s) = k_\infty s + \frac{k_0}{s} + \sum \frac{k_r s}{s^2 + \omega^2} + Z_1(s)$$

The first part of this can be realised as a Foster network, while $Z_1(s)$ is still a positive real function.



The first part of the network has poles only on the $j\omega$ axis, and its real part is zero. The remainder function, that is, $Z_1(s)$ is still positive real, for its real part is still positive and all its poles are on the non-positive (left) half of the s -plane.

$Z_1(s)$ is a minimum reactance function. It is of a lower order than $Z(s)$

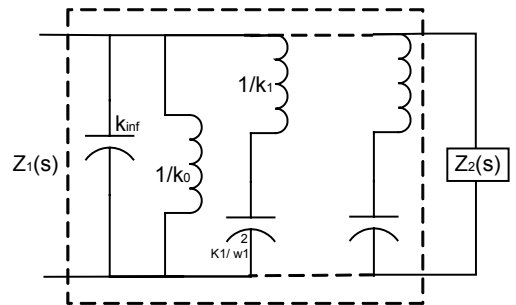
Step 2: Susceptance reduction

$Z_1(s)$ will not have poles on the $j\omega$ axis, but it may have zeros on the $j\omega$ axis. This may be so even if the original function $Z(s)$ did not have zeros on the $j\omega$ axis, for they may be introduced during the removal of poles on the $j\omega$ axis in step 1.

We will now remove them, by partial fraction expansion of the admittance function $Y_1(s) = 1/Z_1(s)$.

$$Y_1(s) = 1/Z_1(s) = k_\infty s + \frac{k_0}{s} + \sum \frac{k_r s}{s^2 + \omega^2} + Y_2(s)$$

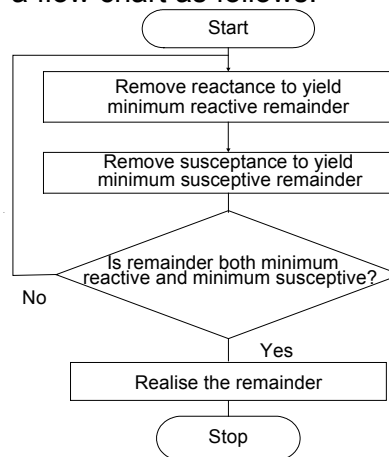
The first part of this may be realised as:



Step 3: Reactance reduction

We may now find that even though $Z_1(s)$ was minimum reactive, $Z_2(s)$ is no longer so, for $j\omega$ axis poles may have been introduced during the removal of $j\omega$ axis zeros. We now repeat Step 1, and then Step 2, repeatedly until the remainder is both minimum reactive and minimum susceptive.

This may be illustrated by a flow chart as follows:



Note: Steps 1 and 2 may be interchanged. We should choose the sequence that gives rise to a simpler realisation.

Example
$$Z(s) = \frac{12s^5 + 8s^4 + 32s^3 + 14s^2 + 8s + 2}{4s^5 + 4s^4 + 10s^3 + 9s^2 + 4s + 1}$$

We will first check whether either the numerator or the denominator contains a quadratic factor that can be removed:

$$\begin{array}{r} 3s/2 \\ \hline 8s^4 + 14s^2 + 2 \left| \begin{array}{l} 12s^5 + 32s^3 + 8s \\ 12s^5 + 21s^3 + 3s \\ \hline 11s^3 + 5s \end{array} \right. \end{array} \quad \begin{array}{r} 8s/11 \\ \hline 11s^3 + 5s \left| \begin{array}{l} 8s^4 + 14s^2 + 2 \\ 5s^4 + 40s^2/11 \\ \hline 114s^2/11 + 2 \end{array} \right. \end{array} \quad \begin{array}{r} 121s/114 \\ \hline 114s^2/11 + 2 \left| \begin{array}{l} 11s^3 + 5s \\ 11s^2 + 121s/57 \\ \hline 164s/57 \end{array} \right. \end{array}$$

$$\begin{array}{r} 114 \times 57 s \\ 11 \times 164 \\ \hline 164s/57 \left| \begin{array}{l} 114s^2/11 + 2 \\ 114s^2/11 \\ \hline 2 \end{array} \right. \end{array} \quad \begin{array}{r} 82s/57 \\ \hline 2 \left| \begin{array}{l} 164s/57 \\ 164s/57 \\ \hline 0 \end{array} \right. \end{array}$$

We will now repeat this with the denominator polynomial:

$$\begin{array}{r} 2s/7 \\ \hline 14s^4 + 9s^2 + 1 \left| \begin{array}{l} 4s^5 + 10s^3 + 4s \\ 4s^5 + 18s^3/7 + 2s/7 \\ \hline 52s^3/7 + 26s/7 \end{array} \right. \end{array} \quad \begin{array}{r} 49s/28 \\ \hline 52s^3/7 + 26s/7 \left| \begin{array}{l} 14s^4 + 9s^2 + 1 \\ 14s^4 + 7s^2 \\ \hline 2s^2 + 1 \end{array} \right. \end{array} \quad \begin{array}{r} 26s/7 \\ \hline 2s^2 + 1 \left| \begin{array}{l} 52s^3/7 + 26s/7 \\ 52s^3/7 + 26s/7 \\ \hline 0 \end{array} \right. \end{array}$$

The remainder is zero, and the last non-zero remainder is $(2s^2+1)$, indication a pair of $j\omega$ axis poles at $s = +j/\sqrt{2}$ and $-j/\sqrt{2}$

This means that we can remove a partial fraction of the form:
$$\frac{s}{s^2 + \frac{1}{2}}$$

$$\begin{array}{r} 4s^3 + 14s^4 + 8s + 2 \\ \hline s^2 + 1/2 \left| \begin{array}{l} 4s^5 + 14s^4 + 10s^3 + 9s^2 + 4s + 1 \\ 4s^5 + \quad \quad 2s^3 \\ \hline 14s^4 + 8s^3 + 9s^2 + 4s + 1 \\ 14s^4 \quad \quad + 7s^2 \\ \hline 8s^3 + 2s^2 + 4s + 1 \\ 8s^3 \quad \quad + 4s \\ \hline 2s^2 + 1 \\ 2s^2 + 1 \\ \hline \end{array} \right. \end{array}$$

$$(s^2 + 1/2)(4s^3 + 14s^4 + 8s + 2) = 4s^5 + 14s^4 + 10s^3 + 9s^2 + 4s + 1$$

$$\begin{aligned}
 \therefore Z(s) &= \frac{12s^5 + 8s^4 + 32s^3 + 14s^2 + 8s + 2}{4s^5 + 4s^4 + 10s^3 + 9s^2 + 4s + 1} \\
 &= \frac{12s^5 + 8s^4 + 32s^3 + 14s^2 + 8s + 2}{(s^2 + \frac{1}{2})(4s^3 + 14s^2 + 8s + 2)} \\
 &= \frac{6s^5 + 4s^4 + 16s^3 + 7s^2 + 4s + 1}{(s^2 + \frac{1}{2})(2s^3 + 7s^2 + 4s + 1)} \\
 &= \frac{As}{s^2 + \frac{1}{2}} + \frac{Bs^3 + Cs^2 + Ds + E}{(2s^3 + 7s^2 + 4s + 1)}
 \end{aligned}$$

We can compute the values of A, B, C, D and E by equating coefficients:

$$B = 6, \quad 2A + C = 4, \quad 7A + D + B/2 = 16, \quad 4A + E + C/2 = 7, \quad a + D/2 = 4, \quad E/2 = 1$$

This gives: $A = 1, B = 6, C = 2, D = 6, E = 2$

$$\therefore Z_1(s) = Z(s) - \frac{s}{s^2 + \frac{1}{2}} = \frac{6s^3 + 2s^2 + 6s + 2}{2s^3 + 7s^2 + 4s + 1}$$

We will now consider $Y_1(s)$:

$$Y_1(s) = \frac{2s^3 + 7s^2 + 4s + 1}{6s^3 + 2s^2 + 6s + 2} = \frac{2s^3 + 7s^2 + 4s + 1}{2(s^2 + 1)(3s + 1)} = \frac{s}{s^2 + 1} + \frac{s + \frac{1}{2}}{3(s + \frac{1}{3})} = \frac{s}{s^2 + 1} + Y_2$$

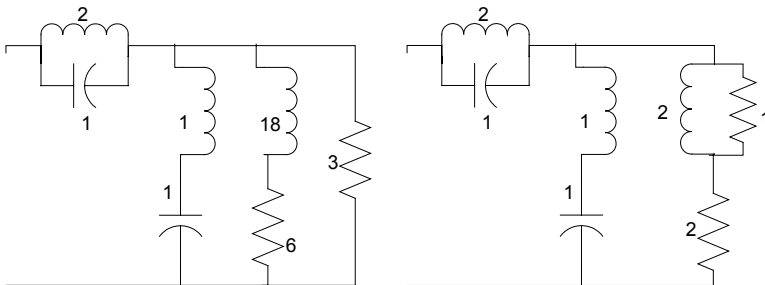
We can now write either

$$Y_2 = \frac{s + \frac{1}{2}}{3(s + \frac{1}{3})} = \frac{1}{3} + \frac{\frac{1}{18}}{s + \frac{1}{3}}$$

or,

$$Z_2 = \frac{1}{Y_2} = \frac{3s + 1}{s + \frac{1}{2}} = 2 + \frac{s}{s + \frac{1}{2}}$$

These two alternatives lead to the following realisations:



Brune synthesis

Brune synthesis starts with the Foster preamble, to remove reactive and susceptive components to yield a minimum reactive and minimum susceptive function. This would be of the form:

$$Z(s) = \frac{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_0}$$

where $a_n, b_n, a_0, b_0 \neq 0$

We then proceed to remove a “minimum resistance” from the remainder to obtain a minimum resistive function. As the real part of $Z(j\omega)$ is a function of ω , care has to be exercised to remove only a minimum resistance, for otherwise, we will be left with a non-positive real function that will not be realisable.

$$Z_1(s) = Z(s) - R_1$$

At the frequency at which the resistance is a minimum, we now have a pure reactive network.

$$Z_1(j\omega_1) = X_1(j\omega_1) = jX_1$$

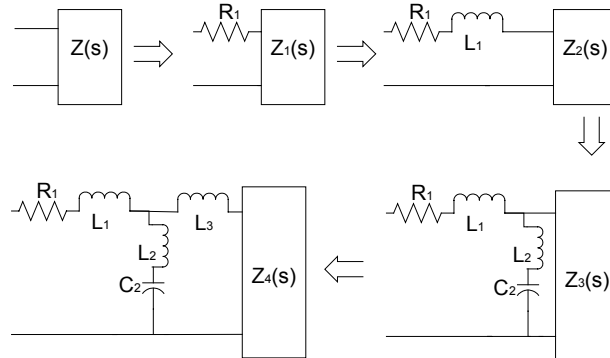
We now remove this inductance from the network.

$$L_1 = \frac{X_1}{\omega_1}, \quad Z_2(s) = Z_1(s) - L_1 s$$

Since $Z(s)$ had no poles or zeros on the $j\omega$ axis, the removal of $L_1 s$ from $Z_1(s)$ to yield $Z_2(s)$ has introduced a zero at $s_1 = \pm j\omega_1$ to $Z_2(s)$. We now proceed to remove this:

$$Y_3(s) = \frac{1}{Z_3(s)} = \frac{1}{Z_2(s)} - \frac{\frac{s}{L_2}}{s^2 - s^2_1}$$

As we saw in the Foster realisation, the removed branch is a shunt element consisting of an inductance and a capacitance in series.



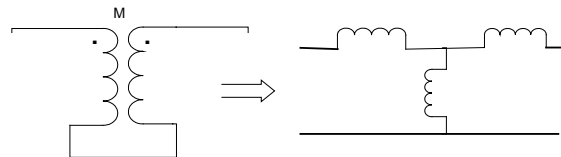
$Z_3(s)$ now has a pole at infinity, which is removed to yield $Z_4(s)$.

$Z_4(s)$ is of the same form as $Z(s)$, except that it is of order $(n-2)$ instead of n . We have completed one cycle of the Brune cycle.

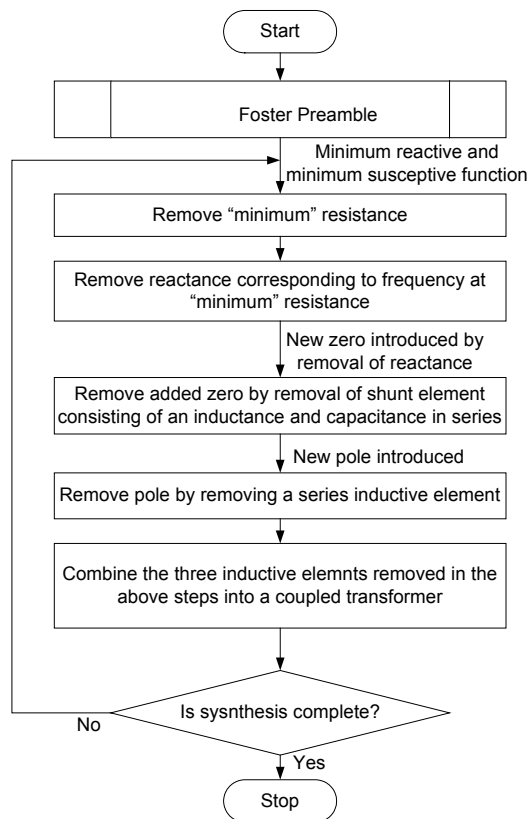
There are two possible cases to be considered in going through this cycle. One is the possibility that L_1 is negative, leaving $Z_2(s)$ positive real, and the other is when L_1 is positive, leaving $Z_2(s)$ non-positive real.

In either case, we will end up with the structure shown at the end of the Brune cycle, with either L_1 or L_3 negative.

The combination of L_1 , L_2 and L_3 may be realised with a coupled transformer as shown:

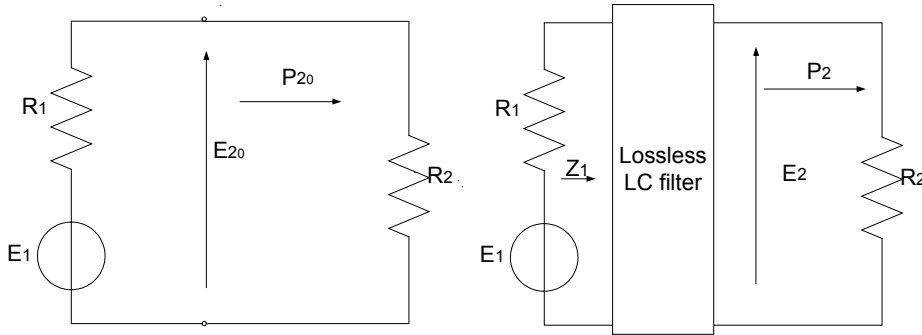


The complete synthesis procedure is illustrated in the following flow chart:



3.1.5 Insertion power function and the reflection coefficient

Let us consider an LC filter. As there are no resistance elements, there will be no filter loss in such a filter. Let us see what happens when we connect such a lossless LC filter between a source and a load.



The figure shows the voltage and power distribution before and after the insertion of a lossless LC filter.

$$\text{Voltage insertion function } \left| \frac{E_{20}}{E_2} \right| = e^\alpha$$

$$\text{Power insertion function } \left| \frac{P_{20}}{P_2} \right| = \left| \frac{E_{20}}{E_2} \right|^2 = e^{2\alpha}$$

$$\text{We also have: } [P_{20}] = \frac{E_1^2}{(R_1 + R_2)^2} R_2$$

For $[P_{20}]$ to be a maximum,

$$\frac{\partial P_{20}}{\partial R_2} = 0$$

$$-2R_2(R_1 + R_2)^{-3} + (R_1 + R_2)^{-2} = 0$$

$$2R_2 = (R_1 + R_2)$$

$$R_2 = R_1$$

$$[P_{20}]_{\max} = \frac{E_1^2}{4R_1}$$

The maximum value that P_2 can take is only the maximum value of P_{20}

$$\therefore [e^{2\alpha}]_{\min} = \frac{P_{20}}{[P_2]_{\max}} = \frac{E_1^2 R_2}{(R_1 + R_2)^2} \frac{4R_1}{E_1^2} = \frac{4R_1 R_2}{(R_1 + R_2)^2} \leq 1$$

It is of course obvious that this has to be less than or equal to unity, for the transmitted power cannot be more than the available power. This is also known as the transmission ratio.

The fractional power rejected by the load is used to define the reflection coefficient $\Gamma(s)$.

$$e^{2\alpha} = \frac{P_{2_0}}{P_2} = \frac{R_2}{(R_1 + R_2)^2} \frac{|(Z_1 + R_1)^2|}{R_1}$$

$$\therefore \frac{(R_1 + R_2)^2}{R_2} e^{2\alpha} = \frac{|(Z_1 + R_1)^2|}{R_1}$$

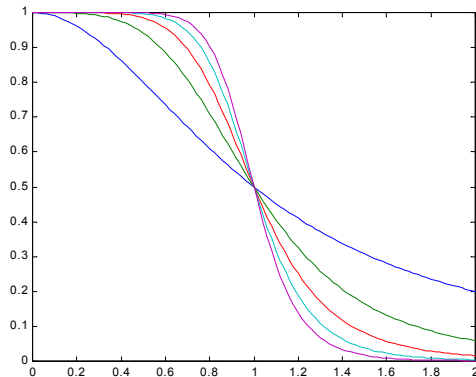
The reflection coefficient is defined as

$$\Gamma(s) = \frac{Z_1(s) - R_1}{Z_1(s) + R_1}$$

From the above, we get

$$\Gamma(s)\Gamma(-s) = 1 - e^{-2\alpha} \frac{4R_1 R_2}{(R_1 + R_2)^2}$$

3.1.6 Butterworth filters



The figure shows a plot of the function $\frac{1}{1 + \Omega^{2n}}$ for different values of n from 1 to 5.

This general form is used in the specification of the Butterworth filter to denote the attenuation of power over the frequency range of interest. In general, we consider the y-axis as representing power or the square of the voltage, and the x-axis as representing the normalised frequency. We also usually consider the range of frequencies up to the half-power point ($\Omega=1$) as the pass band, and the region $\Omega > 1$ as the stop band.

Example:

Assume that we want to design a filter with a pass band of 0 – 5 MHz. and an attenuation of at least 20 dB at 7.5 MHz.

We define the normalised frequency as: $\Omega = \frac{\omega}{\omega_0}$

At 7.5 MHz., $\Omega = 1.5$.

$$-20 = 10 \log_{10} \frac{1}{1 + (1.5)^{2n}}$$

$$\frac{1}{1 + (1.5)^{2n}} = 10^{-2}$$

$$(1.5)^{2n} = 99$$

$$2n \ln 1.5 = \ln 99$$

$$2n (0.405) = 4.595$$

$$2n = \frac{4.595}{0.405} = 11.35$$

$$n = 5.675 \Rightarrow 6$$

$$2n = 12$$

Now let us see what would happen if we had tighter specifications for the pass band. Instead of 3dB attenuation (corresponding to half-power), let us assume that we wanted the attenuation in the pass band to be limited to 0.25 dB. To accommodate this, we use a slightly different form of the general function defining the filter characteristics:

$$\frac{1}{1 + \varepsilon^2 \Omega^{2n}}$$

At $\Omega=1$,

$$-0.25 = 10 \log_{10} \frac{1}{1 + \varepsilon^2}$$

$$\log_{10} \frac{1}{1 + \varepsilon^2} = -0.025$$

$$(1 + \varepsilon^2) = 10^{0.025}$$

$$\varepsilon^2 = 1.05925 - 1 = 0.05925$$

$$-20 = 10 \log_{10} \frac{1}{1 + 0.0(1.5)^{2n}}$$

$$\frac{1}{1 + 0.05925(1.5)^{2n}} = 10^{-2}$$

$$0.05925(1.5)^{2n} = 99$$

$$2n \ln 1.5 = \ln \frac{99}{0.05925} = \ln 1670$$

$$2n (0.405) = 7.421$$

$$2n = \frac{7.421}{0.405} = 18.3$$

$$n = 9.15 \Rightarrow 10$$

$$2n = 20$$

We see that with the tighter specifications, we need a much more complex filter with more components.

Further example: Low pass filter, with the following specifications:

Pass band	0 – 20 kHz.
Pass band tolerance	3 dB
Load and source impedance	600 Ω
High frequency attenuation	≥ 40 dB at 100 kHz.

$$|M(\omega)|^2 = \frac{1}{1 + \varepsilon^2 \Omega^{2n}}$$

$$\text{At } \Omega = 1:$$

$$3 = 10 \log_{10}(1 + \varepsilon^2)$$

$$\therefore \varepsilon = 1$$

For an attenuation of 40 dB at 100 kHz.,

$$\text{At } \Omega = \frac{100}{20} = 5:$$

$$40 = 10 \log_{10}(1 + \Omega^{2n}) = 10 \log_{10}(1 + 5^{2n})$$

$$1 + 5^{2n} = 10000$$

$$5^{2n} = 9999$$

$$2n \ln 5 = \ln 9999$$

$$2n * 1.609 = 9.210$$

$$n = 2.86 \Rightarrow 3$$

$$|M(\omega)|^2 = \frac{1}{1 + \varepsilon^2 \Omega^{2n}} = \frac{1}{1 + \Omega^6},$$

Substituting $s = j\Omega$:

$$\Gamma(s)\Gamma(-s) = 1 - \frac{4R_1R_2}{(R_1 + R_2)^2} \frac{1}{1 - s^6}$$

As R_1 and R_2 are given as 600Ω ,

$$\Gamma(s)\Gamma(-s) = 1 - \frac{4R_1R_2}{(R_1 + R_2)^2} \frac{1}{1 - s^6} = 1 - \frac{1}{1 - s^6} = \frac{-s^6}{1 - s^6} = \frac{-s^6}{(1 - s^3)(1 + s^3)}$$

$$= \frac{-s^6}{(1 - s)(1 + s + s^2)(1 + s)(1 - s + s^2)} = \frac{(s^3)}{(1 + s)(1 + s + s^2)} \frac{(-s^3)}{(1 - s)(1 - s + s^2)}$$

$$\text{Let us chose } \Gamma(s) = \frac{s^3}{(1 + s)(1 + s + s^2)}$$

$$\text{But } \Gamma(s) = \frac{Z(s) - 1}{Z(s) + 1},$$

where $Z(s)$ is the normalised impedance

$$\Gamma(s)Z(s) + \Gamma(s) = Z(s) - 1$$

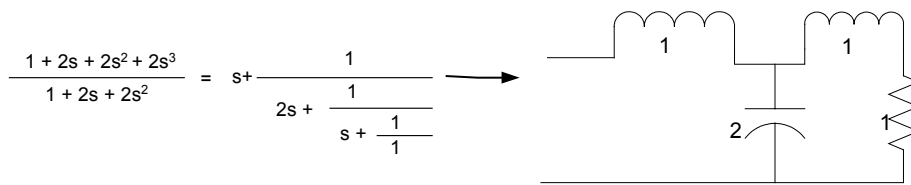
$$Z(s)[\Gamma(s) - 1] = -[\Gamma(s) + 1]$$

$$Z(s) = \frac{1 + \Gamma(s)}{1 - \Gamma(s)} = \frac{1 + \frac{s^3}{1 + 2s + 2s^2 + s^3}}{1 - \frac{s^3}{1 + 2s + 2s^2 + s^3}} = \frac{1 + 2s + 2s^2 + 2s^3}{1 + 2s + 2s^2}$$

This may now be synthesised. Using continued fraction expansion:

$$2s^2 + 2s + 1 \overline{\left| \begin{array}{l} s \\ 2s^3 + 2s^2 + 2s + 1 \\ 2s^3 + 2s^2 + s \end{array} \right.} \quad s + 1 \overline{\left| \begin{array}{l} 2s \\ 2s^2 + 2s + 1 \\ 2s^2 + 2s \end{array} \right.}$$

$$\frac{\hspace{10em}}{s + 1} \qquad \frac{\hspace{10em}}{1}$$

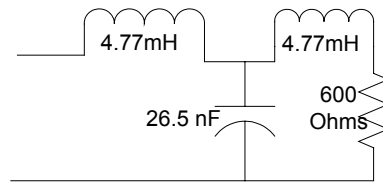


Now we need to de-normalise this, to obtain the final implementation:

$$R_0 = 600 \Omega$$

$$\omega_0 = 20\,000 \times 2\pi$$

The corresponding transformations are: $R \Rightarrow R \times R_0$
 $L \Rightarrow L \times R_0 / \omega_0$
 $C \Rightarrow C / (R_0 \times \omega_0)$



3.1.7 Chebyshev filters

In principle, this (and other mathematically defined filters such as the elliptic filter) is similar to the Butterworth filter in that we define a mathematical function approximating the characteristics that we desire.

$$\text{Here, we define } |M(\omega)|^2 = \frac{1}{1 + \varepsilon^2 C_n^2(\Omega)}$$

$$\text{Where } C_n(\Omega) = \begin{cases} \cos(n \cos^{-1} \Omega), & \text{for } |\Omega| \leq 1 \\ \cosh(n \cosh^{-1} \Omega), & \text{for } |\Omega| > 1 \end{cases}$$

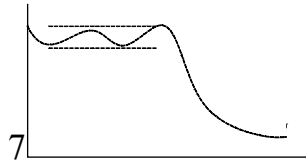
$$\text{We have } C_0(\Omega) = 1$$

$$\text{and } C_1(\Omega) = \Omega$$

We can also obtain a recursive expression for higher order values of C_n as:

$$C_n(\Omega) = 2\Omega C_{n-1}(\Omega) - C_{n-2}(\Omega)$$

The shape of the function generated by this approximation is as shown:



We will consider the same example as before: As the pass band tolerance is 3 dB, we again have $\varepsilon = 1$.

$$40 = 10 \log (1 + C_n^2(5))$$

$$C_n^2(5) = 9999$$

$$C_n(5) = 99.99 = \cosh(n \cosh^{-1} 5)$$

$$n = \frac{\cosh^{-1} 99.99}{\cosh^{-1} 5} = \frac{5.298}{2.312} = 2.29 \Rightarrow 3$$

We can find $C_n(\Omega)$ using the recursive algorithm given earlier, and we get

$$C_2(\Omega) = 2\Omega C_1(\Omega) - C_0(\Omega) = 2\Omega^2 - 1$$

$$C_3(\Omega) = 2\Omega C_2(\Omega) - C_1(\Omega) = 2\Omega(2\Omega^2 - 1) - \Omega = 4\Omega^3 - 3\Omega$$

$$|M(\omega)|^2 = \frac{1}{1 + \varepsilon^2 C_n^2(\Omega)}$$

$$= \frac{1}{1 + (4\Omega^3 - 3\Omega)^2}$$

Substituting $s = j\Omega$;

$$\Gamma(s)\Gamma(-s) = 1 - \frac{1}{1 - 9s^2 - 24s^4 - 16s^6}$$

This has to be separated into two parts, and then we can proceed as before. In this case, we will have to resort to numerical methods to factorise the expression into two parts. As before, we will end up with a filter with three components. However, for filters with tighter specifications, the Chebyshev approximation yields lower degree implementations than the Butterworth approximation.

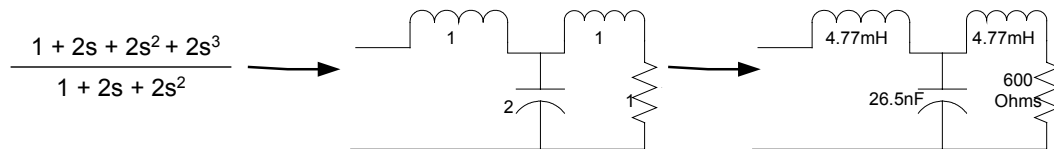
Pole-zero patterns

In the Butterworth filter, the poles lie on a semi-circle centered on the origin, on the left half of the s-plane. All the zeros are at infinity. In the Chebyshev filter, we bring the poles closer to the $j\omega$ axis in an attempt to increase the cut-off slope. However, this introduces a ripple in the pass band. The reverse is attempted in the Bessel filter, where the poles are moved further away from the circular locus. The elliptic filter has its poles closer to the axis as in the Chebyshev filter, but also has the zeros moved closer to the real axis, from infinity. This filter has ripples in the cut-off band as well.

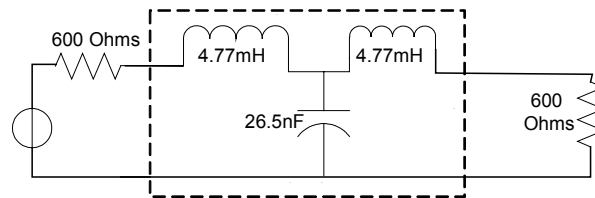
3.1.8 Transformation to obtain high-pass characteristics

Once you have obtained a low-pass filter, it is quite simple to get a high-pass filter by transformation. Consider the first low-pass Butterworth filter we designed:

Pass band	0 – 20 kHz.
Pass band tolerance	3 dB
Load and source impedance	600 Ω
High frequency attenuation	≥ 40 dB at 100 kHz.



The filter itself is only the LC section, the terminal resistance being the load resistance. This is illustrated below, where both the source and load resistances are shown:



Starting with the normalised function, we can obtain a high pass filter merely by the transformation $s \Rightarrow 1/s$. The transformation is invariant at $\Omega = 1$, while all the other frequencies are inverted. In the case under study, we had:

$$|M(\omega)|^2 = \frac{1}{1 + \varepsilon^2 \Omega^{2n}} = \frac{1}{1 + \Omega^6},$$

Substituting $s = j\Omega$:

$$\Gamma(s)\Gamma(-s) = 1 - \frac{1}{1 - s^6}$$

This would transform to:

$$\Gamma(s)\Gamma(-s) = 1 - \frac{1}{1 - s^6} = \frac{1}{1 - s^6} = \frac{1}{(1-s)(1+s+s^2)(1+s)(1-s+s^2)} = \frac{1}{(1+s)(1+s+s^2)} \frac{1}{(1-s)(1-s+s^2)}$$

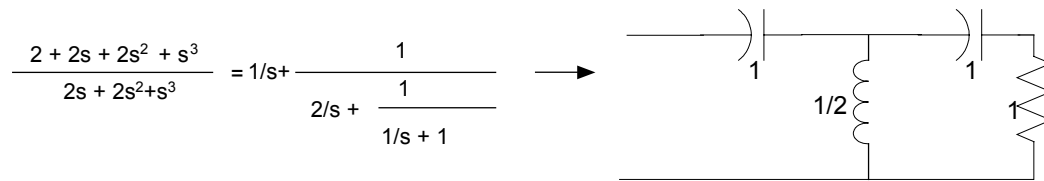
$$\text{Chose } \Gamma(s) = \frac{1}{(1+s)(1+s+s^2)}$$

$$\text{But } \Gamma(s) = \frac{Z(s)-1}{Z(s)+1},$$

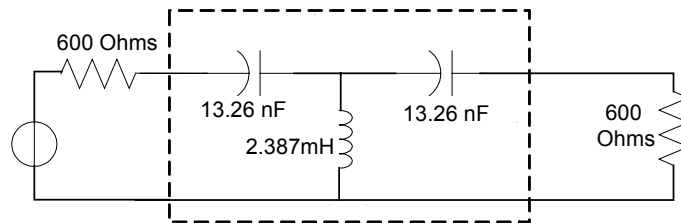
$$\therefore Z(s) = \frac{1 + \Gamma(s)}{1 - \Gamma(s)} = \frac{1 + \frac{1}{1 + 2s + 2s^2 + s^3}}{1 - \frac{1}{1 + 2s + 2s^2 + s^3}} = \frac{2 + 2s + 2s^2 + s^3}{2s + 2s^2 + s^3}$$

By continued fraction expansion, we have:

$$2s + 2s^2 + s^3 \overline{\left| \begin{array}{l} 1/s \\ 2 + 2s + 2s^2 + s^3 \\ 2 + 2s + s^2 \\ \hline s^2 + s^3 \end{array} \right.} \qquad s + s^2 \overline{\left| \begin{array}{l} 2/s \\ 2 + 2s + s^3 \\ 2 + 2s \\ \hline s^2 \end{array} \right.}$$



De-normalising will then yield:



This filter will have the following specifications, they being the inverse of those of the original low pass filter:

Pass band	20 kHz and above.
Pass band tolerance	3 dB
Load and source impedance	600 Ω
Low frequency attenuation	≥ 40 dB at 4 kHz.

The transformation could have been obtained by operating directly on $Z(s)$ (instead of on the insertion power characteristic) or even by transforming each of the filter elements. The filter elements transform as follows:

Low-pass prototype (normalised)	Low-pass	High-pass	Band-pass	Band-stop

Note that the above table allows for de-normalisation for frequency only, and does not take impedance normalisation into account.

3.1.9 Band pass and band stop filters

It is possible to obtain band pass and band stop characteristics by frequency transformation of low pass filters, as we found with high pass filters. However, it is much more convenient to construct these as cascade and parallel combinations of a low pass and a high pass filter as this would give us independent control of both the cut-off frequency and the roll-off characteristics on either side. We illustrate this, using ideal filter characteristics in the following diagram:

