2

State and state space

We have been studying electrical circuits with reference to the complex frequency s and the s-plane. You have seen this to be a technique that is most convenient, especially when you are interested in the steady-state behaviour of a circuit. There is an alternative way in which we can model dynamic systems, including electrical circuits, which is more suited for the study of their transient behaviour.

The state of a system in this context means exactly the same as the common English meaning of the word. However, we need to be able to define it more precisely. For a known system, a knowledge of its present state and any future inputs from the external world should be sufficient for us to be able to predict its future state. This then is what the state means. The variables, whose values tell us what the present state is, are known as state variables.

A set of state variable is the minimum set of variables, whose values at any instant, along with the inputs from then onwards, would enable the complete determination of the values of the state variables in the future. In effect, the state variables contain all the information about the past and the present, necessary for the prediction of their future values.

2.1   State variables and state space representation

Introduction

We are used to the description of dynamic systems using differential equations. We will examine their relationship through a simple example.

A pendulum, swinging (with small amplitude) in a gravitational field, in a vacuum will continue to swing indefinitely as there is no resistance to slow it down. Similarly, an initially charged capacitor shorted through an inductor (assumed to be without resistance) will give rise to a cyclic charge-discharge activity. In the absence of any circuit resistance or leakage across the capacitor, this will continue indefinitely. Both these phenomena may be described by a simple second order differential equation.

$$d^2 x / dt^2 + \omega_n^2 x = 0$$

[If there were any resistance, the equation would be of the form
:
$$d^2 x / dt^2 + 2\zeta\omega_n \, dx / dt + \omega_n^2 x = 0 \quad]$$

We can break this down into two simultaneous first order differential equations by defining two new variables:

$$x_1 = x$$
$$x_2 = dx / dt = dx_1 /dt$$

Then,

$$dx_1 / dt = x_2$$

$$dx_2 /dt = -\omega_n^2 x_1$$

If we know the values of x_1 and x_2 at any instant of time t_1, then, it is possible to compute their values for all $t > t_1$ (We will also need to know all inputs for $t \geq t_1$., assumed to be zero in the above formulation) Thus, x_1 and x_2 are a valid set of state variables. They are of course not the only valid set, for, we could define (say)

$$z_1 = x_1 + x_2$$
$$z_2 = x_1 - x_2,$$

Then, we can obtain x_1 and x_2 from z_1 and z_2, and hence, (z_1, z_2) is also a valid set of state variables. There are thus an infinite number of such sets of state variables.

The space defined by the state variables is known as the state-space. In the above example, the two-dimensional space x_1-x_2 is the state-space, and any point on it will represent a state of the system. If the state vector in a 3-vector, then its corresponding state-space is also three-dimensional. An n-dimensional state vector will describe a motion in an n-dimensional state-space.

We saw how the state space representation relates to the description of a physical system otherwise described by a set of differential equations. We will now examine the relationship between state space and the system function of a network. The system function is, as we saw earlier, a function of the complex frequency s, representing the ratio of the Laplace transform of a response to the Laplace transform of the excitation causing the response. In this sense, we can look at all such functions as transfer functions.

We have already seen that if the (network) model can be arranged as a set of interconnections among integrators, their outputs constitute a possible set of state variables. With this background, we will attempt to breakdown a function of the complex variable s into a set of relations that can be easily represented by integrators.

We have also noted that the most general form of a system function is given by a *real rational function of s*, expressed as the quotient of two polynomials in s.

$H(s) = q(s) / p(s)$, where

$q(s) = b_m(s-z_1)(s-z_2) \ . \ . \ . \ (s-z_m)$
$p(s) = (s-p_1)(s-p_2) \ . \ . \ . \ . \ (s-p_n)$

Unlike in the study of other more general systems (such as in Control Systems), we have the advantage that network functions of passive networks are subject to certain constraints. The poles and zeros are simple, that is there are no higher order poles or zeros, and there are no poles or zeros on the right hand side of the s-plane.

We can expand $H(s)$ into partial fractions, so that each term corresponds to a single integrator, whose output can then be considered as a state variable. We will, for completeness, examine the general case of how to obtain a set of state variables, given a real rational function of s.

This is of course a very mechanical treatment, giving no physical insight into the problem being studied. It is much more useful to recognise suitable physical quantities as the state variables, and obtain the equations governing their relationships.

We saw that there are many alternate ways of describing the behaviour of a dynamic system, using different state variables. They are all transformations of each other, and most have no physical significance. In the case of the simple pendulum considered earlier, we could have derived the state space description using the angular displacement and the angular velocity as state variables. These obviously have physical significance. But an equally valid, from a representational point of view, pair of state variables would have been the sum of the angular displacement and the angular velocity as one variable and their difference as the other. This combination does not make physical sense.

In engineering, we would always prefer to formulate our equations in terms of variables that have some physical significance. This has a number of advantages, including the ability to make reality checks and providing us with an insight into the problem under study.

We noted earlier on that the state variables contain all the information about the past and the present necessary for the prediction of future behaviour. There are many ways in which these can be visualised. They are connected with "memory", and they are in some manner associated with energy storage. In linear electrical circuits, only inductors and capacitors can store energy (Resistors dissipate energy, but cannot store it.)

We have a choice in the selection of state variables with physical significance. We could select either (flux ϕ, charge q) or (current i, voltage v) as our preferred set of variables. Other combination including different combinations of these are of course possible as we saw earlier, but these seem to offer alternatives with real physical significance.
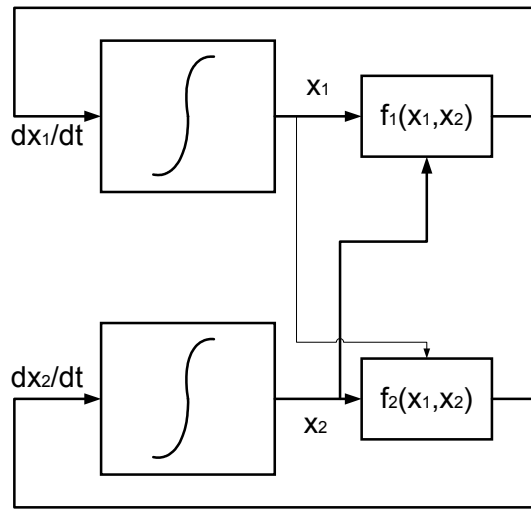
We may use the methods of nodal and mesh analysis, or alternatively, energy function methods to obtain the state equations of a system.

2.1.1 State-space

A second order system (with no inputs) may be represented by a set of two state equations as follows:

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

We can draw a block diagram of these two equations, using two integrators, as follows:

The outputs of the integrators may be taken as a possible set of state variables.

We can then visualise this system as moving in a two-dimensional space as shown below.
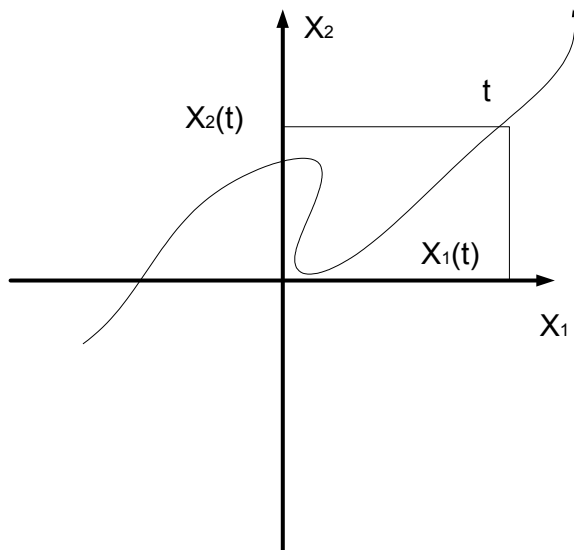
A point $[x_1(t), x_2(t)]$ on the path represents the state of the system at time t. It is obvious that the system shown above would be non-linear, and be a rather complex one.

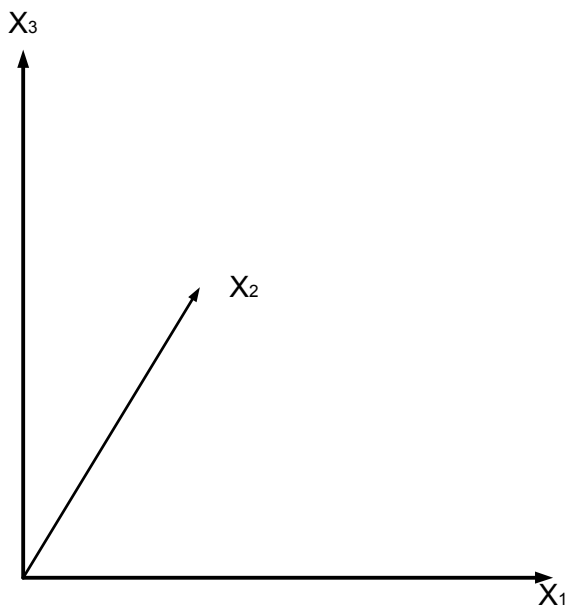The simple harmonic motion represented by the set of linear equations

$$dx_1 / dt = x_2$$

$$dx_2 / dt = -\omega_n^2 x_1$$

considered earlier would be represented on the state-space as follows:
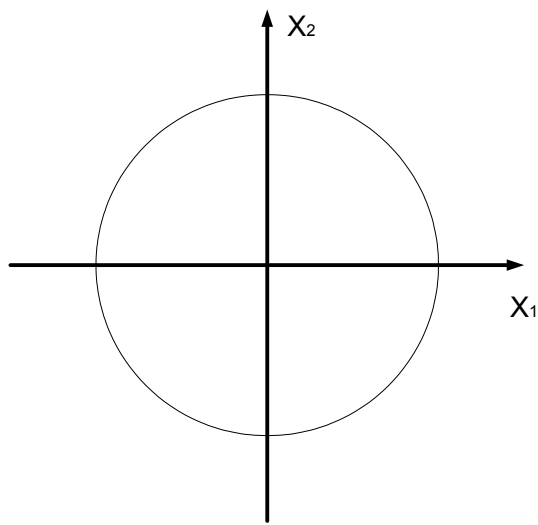
A trajectory on a three-dimensional state-space may be visualised as follows:

Unfortunately, it is not possible to directly represent spaces of higher dimensions on a two-dimensional surface, but you should not have much difficulty in visualising the extension of the concept of state – space to n dimensions.

2.1.2 Obtaining state variables and state equations from a transfer function

Consider the general form of a transfer function:

$$H(s) = \frac{b_n s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

Let the function H(s) denote a relationship between two functions U(s) and Y(s) such that:

$$\frac{Y(s)}{U(s)} = H(s)$$

$$Y(s)[s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0] = U(s)[b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0]$$
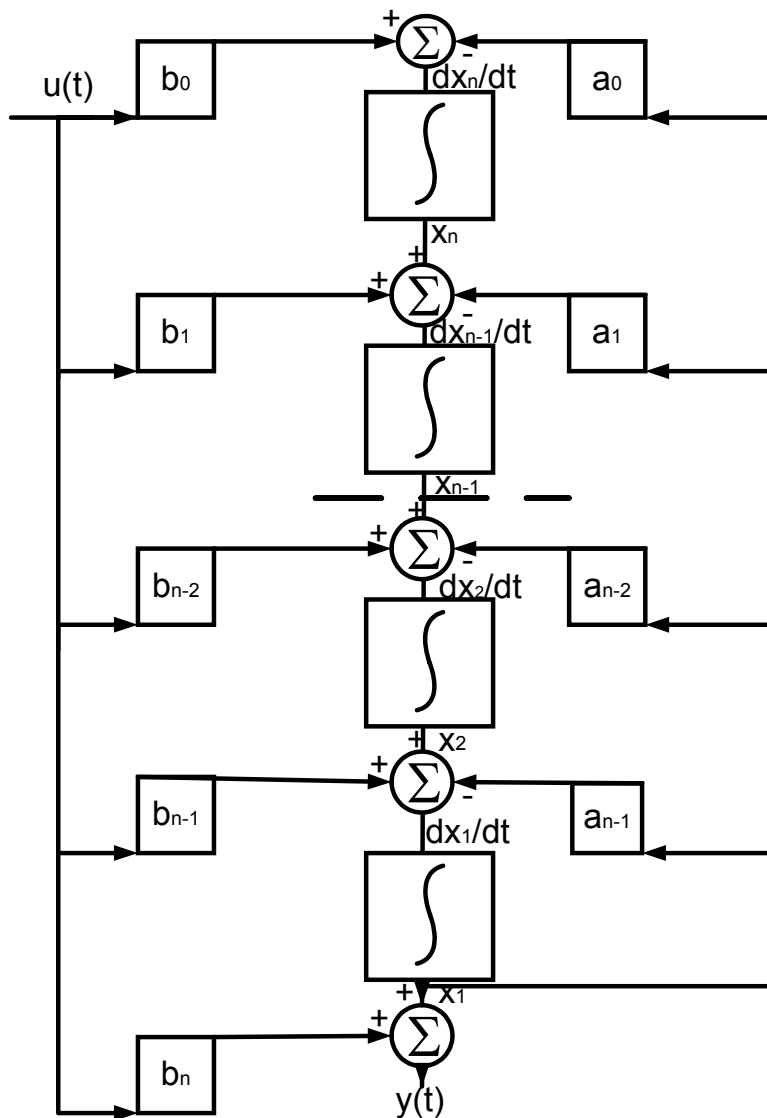
Dividing by s^n and rearranging,

$$Y(s) = b_n U(s) + \frac{1}{s}[b_{n-1}U(s) - a_{n-1}Y(s)] + \frac{1}{s^2}[b_{n-2}U(s) - a_{n-2}Y(s)] +$$

$$+ \ldots +$$

$$\frac{1}{s^{n-1}}[b_1 U(s) - a_1 Y(s)] + \frac{1}{s^n}[b_0 U(s) - a_0 Y(s)]$$

The implementation of these using integrators is shown in the figure.

From this, we can write down the state space description by inspection, if we chose the outputs of the integrators as the state variables.

$$\dot{x}_1 = -a_{n-1}x_1 + x_2 + b_{n-1}u$$

$$\dot{x}_2 = -a_{n-2}x_1 + x_3 + b_{n-2}u$$

. . . . .

$$\dot{x}_{n-1} = -a_1 x_1 + x_n + b_1 u$$

$$\dot{x}_n = -a_0 x_1 + b_0 u$$

The output y is given by:

$$y = x_1 + b_0 u$$

In matrix form, we can write:

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ . \\ . \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix}
=
\begin{bmatrix}
-a_{n-1} & 1 & 0 & . & . & . & 0 \\
-a_{n-2} & 0 & 1 & 0 & . & . & 0 \\
. \\
. \\
-a_1 & 0 & 0 & 0. & . & . & 1 \\
-a_0 & 0 & 0 & 0 & . & . & 0.
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_{n-1} \\ x_n \end{bmatrix}
+
\begin{bmatrix} b_{n-1} \\ b_{n-2} \\ . \\ . \\ b_1 \\ b_0 \end{bmatrix} u
$$

$$
\begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & . & . & . & 0 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ .x_{n-1} \\ x_n \end{bmatrix}
+ b_n u
$$

In concise form:

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

As has been repeatedly emphasised, there is no unique set of state variables, and what we defined in the above analysis is just one possible selection of state variables, given the function H(s). We will now examine one other possible form, obtained by the partial fraction expansion of the given function.

Let us first assume that all the poles are distinct and real. Then, we can write:

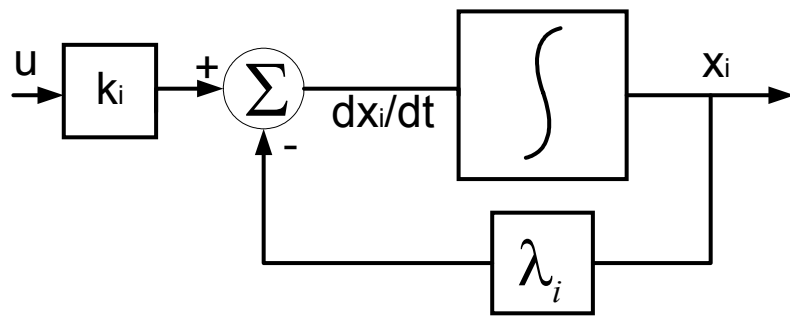$$H(s) = \frac{Y(s)}{U(s)} = \sum_{i=1}^{n} \frac{k_i}{s + \lambda_i}$$

Considering each term,

$$\frac{X_i(s)}{U(s)} = \frac{k_i}{s + \lambda_i}$$

$$sX_i(s) = k_i U(s) - \lambda_i X(s)$$

$$X_i(s) = \frac{1}{s}\left[k_i U(s) - \lambda_i X(s)\right]$$

This leads to:

If we again select the outputs of the integrators as the state variables:

$$\dot{x}_i = -\lambda_i x_i + k_i u$$

$$y = \sum_{i=1}^{n} x_i$$

In matrix form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ . \\ . \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -\lambda_1 & 0 & 0 & . & . & . & 0 \\ 0 & -\lambda_2 & 0 & 0 & . & . & 0 \\ . & & & & & & \\ . & & & & & & \\ 0 & 0 & 0 & . & . & -\lambda_{n-1} & 0 \\ 0 & 0 & 0 & . & . & . & -\lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \\ . \\ . \\ k_{n-1} \\ k_n \end{bmatrix} u$$

$$[y] = \begin{bmatrix} 1 & 1 & 1 & . & . & . & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ .x_{n-1} \\ x_n \end{bmatrix}$$

This is again of the form

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

even though the structure of the matrices are different.

2.1.3 Resistors, inductors and capacitors

Consider the following circuit elements:

> Resistance R (ohms)
> Conductance G (mhos) (=1/R)
>
> Capacitance C (farads)
> Elastance S (darafs) (=1/C)
>
> Inductance L (henrys)
> Inverse inductance Γ
>   (inverse henrys) (=1/L)

For the class of linear elements, we have the following relationships:

Defining relationships:

$$v = Ri, \qquad i = Gv$$

$$q = Cv, \qquad v = Sq$$

$$\phi = Li, \qquad i = \Gamma\phi$$

i-v relationships:

$$v = Ri, \qquad\qquad i = Gv$$

$$i = C\frac{dv}{dt}, \qquad\qquad \frac{dv}{dt} = Si$$

$$v = L\frac{di}{dt}, \qquad\qquad \frac{di}{dt} = \Gamma v$$

The relationships for capacitors and inductors may also be written in integral form as:

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau + v_0$$

$$i(t) = \frac{1}{L}\int_{t_0}^{t} v(\tau)d\tau + i_0$$

Relationships derived from resistances do not yield state equations, as they can only be algebraic equations and not differential equations. If we select v-i relationships as the relationships of choice (this is not necessary, we could equally well select others such as flux – charge relationships), we are led naturally to select:

> voltages across capacitors

and

> currents through inductors

as the state variables of choice.

[In the generalised theory of dynamic systems, there are analogous choices to be made for the representation of physical phenomena in mechanical systems, thermodynamic systems and fluidic systems]

2.1.4 Formulation of state equations by nodal and mesh analysis

We will demonstrate the derivation of state equations through nodal and mesh analysis through a simple example. Consider the network shown below:
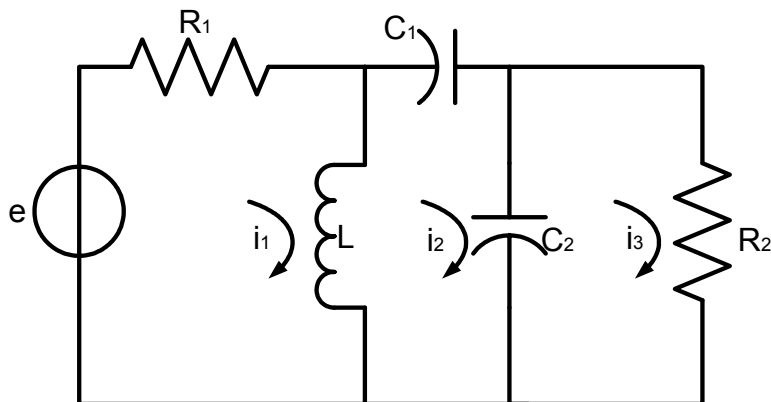
Writing the mesh equations:

$$R_1 i_1 + L\dot{i}_1 - L\dot{i}_2 = e$$

$$-L\dot{i}_1 + L\dot{i}_2 + \frac{1}{C_1}\int i_2 dt + \frac{1}{C_2}\int i_2 dt - \frac{1}{C_2}\int i_3 dt = 0$$

$$-\frac{1}{C_2}\int i_2 dt + \frac{1}{C_2}\int i_3 dt + R_2 i_3 = 0$$

Differentiating the second and third equations to transform from integral to differential form, these may be rewritten as:

$$R_1 i_1 + L\dot{i}_1 - L\dot{i}_2 = e$$

$$-LC_1 C_2 \ddot{i}_1 + LC_1 C_2 \ddot{i}_2 + (C_1 + C_2)i_2 - C_1 i_3 = 0$$

$$-i_2 + i_3 + C_2 R_2 \dot{i}_3 = 0$$

From the above, it appears that there are two variables with second derivatives. In fact, there is only one, $(i_1 - i_2)$. We need to recognise this fact, that there is only one variable with a second derivative.

If we rewrite the equations using $(i_1 - i_2)$ as a single variable (and also abandoning (say) i_1 as an independent variable):

$$R_1(i_1 - i_2) + R_1 i_2 + L\overbrace{(\dot{i}_1 - \dot{i}_2)} = e$$

$$-LC_1 C_2 \overbrace{(\ddot{i}_1 - \ddot{i}_2)} + (C_1 + C_2)i_2 - C_1 i_3 = 0$$

$$-i_2 + i_3 + C_2 R_2 \dot{i}_3 = 0$$

As there are no derivatives of i_2, it can be eliminated (by substituting for i_2 from the third equation, into the other two) to yield:

$$R_1(i_1 - i_2) + L\overbrace{(\dot{i}_1 - \dot{i}_2)} + R_1 i_3 + C_2 R_1 R_2 \dot{i}_3 = e$$

$$-LC_1 C_2 \overbrace{(\ddot{i}_1 - \ddot{i}_2)} + (C_1 + C_2)(i_3 + C_2 R_2 \dot{i}_3) - C_1 i_3 = 0$$

If we now define:

$$x_1 = i_1 - i_2$$

$$x_2 = \overbrace{(\dot{i}_1 - \dot{i}_2)}$$

$$x_3 = i_3$$

We get the state equations:

$$\dot{x}_1 = x_2$$
$$R_1 x_1 + L x_2 + R_1 x_3 \_ C_2 R_1 R_2 \dot{x}_3 = e$$
$$- L C_1 C_2 \dot{x}_2 + (C_1 + C_2) C_2 R_2 \dot{x}_3 + C_2 x_3 = 0$$

Rearranging, we get:

$$\dot{x}_1 = x_2$$

$$\dot{x}_3 = \frac{1}{C_2 R_2} x_1 + \frac{L}{C_2 R_1 R_2} x_2 + \frac{1}{C_2 R_2} x_3 - \frac{1}{C_2 R_1 R_2} e$$

$$\dot{x}_2 = \frac{1}{L C_1 C_2} \left[(C_1 + C_2) C_2 R_2 \dot{x}_3 + C_2 x_3 \right]$$

$$= \frac{(C_1 + C_2) R_2}{L C_1} \left[\frac{1}{C_2 R_2} x_1 + \frac{L}{C_2 R_1 R_2} x_2 + \frac{1}{C_2 R_2} x_3 - \frac{1}{C_2 R_1 R_2} e \right] + \frac{1}{L C_1} x_3$$

$$= \frac{C_1 + C_2}{L C_1 C_2} x_1 + \frac{C_1 + C_2}{C_1 C_2 R_1} x_2 + \frac{C_1 + 2 C_2}{L C_1 C_2} x_3 - \frac{C_1 + C_2}{L C_1 C_2 R_1} e$$

In matrix form,

$$\dot{X} = AX + BU$$

where:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ \dfrac{C_1 + C_2}{L C_1 C2} & \dfrac{C_1 + C}{C_1 C_2 R_1} & \dfrac{C_1 + 2 C_2}{L C_1 C_2} \\ \dfrac{1}{C_2 R_2} & \dfrac{L}{C_2 R_1 R_2} & \dfrac{1}{C_2 R_2} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ -\dfrac{C_1 + C_2}{L C_1 C_2 R_1} \\ -\dfrac{1}{C_2 R_1 R_2} \end{bmatrix}$$

We could, alternatively, have started by writing the nodal equations, and obtained the state equations in terms of the node-pair voltages and their derivatives.

Another approach (to both nodal and mesh analysis) is to define the state variables in terms of physical variables that correspond directly to the state – the currents through inductors and voltages across capacitors. This would be the most direct approach, and we will repeat the analysis of this circuit using his approach. (This is a method that can be carried across to the study of other systems such as mechanical, thermal and fluid systems, which all have identifiable "through" variables and "across" variables)
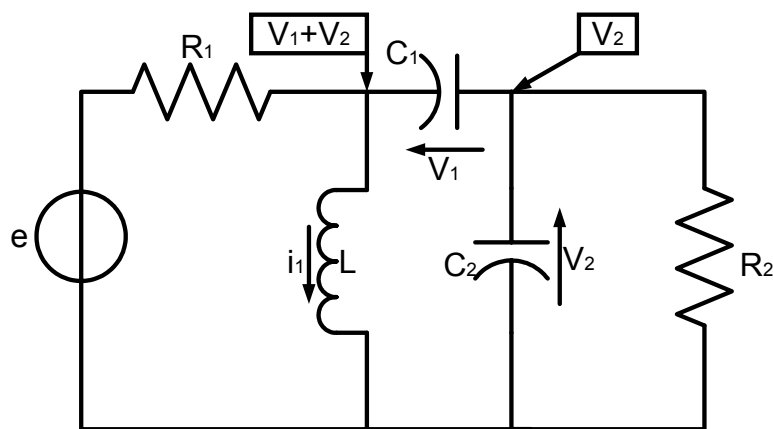
Writing the nodal equations in terms of i_1, v_1 and v_2, we have:

$$\frac{v_1 + v_2 - e}{R_1} + i_1 + C_1 \dot{v}_1 = 0$$

$$-C_1 \dot{v}_1 + C_2 \dot{v}_2 + \frac{v_2}{R_2} = 0$$

We also have the additional relationship:

$$v_1 + v_2 = L \dot{i}_1$$

Rearranging, we have:

$$\dot{i}_1 = \frac{1}{L} v_1 + \frac{1}{L} v_2$$

$$\dot{v}_1 = -\frac{1}{C_1} i_1 - \frac{1}{R_1 C_1} v_1 - \frac{1}{R_1 C_1} v_2 + \frac{1}{R_1 C_1} e$$

$$\dot{v}_2 = -\frac{1}{C_2} i_1 - \frac{1}{R_1 C_2} v_1 - \frac{R_1 + R_2}{C_2 R_1 R_2} v_2 + \frac{1}{R_1 C_2} e$$

This is in the standard form. Note how easily the state equations could be obtained, if we make a proper choice of state variables.

We could equally well have obtained the state equations by writing the mesh equations.

We can now formally state the procedure for writing down the state equations as follows:

1. Select the currents through inductors and voltages across capacitors as the state variables.
2. Write the loop *(node-pair)* equations for all loops *(node-pairs)* that contain *(are connected to)* at least one storage element (that is, an inductor or capacitor)
3. If there are n storage elements and only m (m<n) loop (node-pair) equations, then there will be an additional (n-m) relationships between the variables we have chosen. Altogether, there will be n equations.

2.1.5 Energy functions

We have defined the loop-based energy functions as:

$$T = \bar{I}^T L I = \sum_{i,k=1}^{l} L_{ik} I_i \bar{I}_k$$

$$F = \bar{I}^T R I = \sum_{i,k=1}^{l} R_{ik} I_i \bar{I}_k$$

$$V = \bar{I}^T S I = \sum_{i,k=1}^{l} S_{ik} I_i \bar{I}_k$$

where l is the number of independent loops and the node-pair-based energy functions as:

$$V^* = \bar{E}^T C E = \sum_{i,k=1}^{n} C_{ik} E_i \bar{E}_k$$

$$F^* = \bar{E}^T G E = \sum_{i,k=1}^{n} G_{ik} E_i \bar{E}_k$$

$$T^* = \bar{E}^T \Gamma E = \sum_{i,k=1}^{n} \Gamma_{ik} E_i \bar{E}_k$$

where n is the number of independent node-pairs.

L, R and S represent the loop inductance, resistance and elastance (reciprocal capacitance) while C,G and Γ represent the node capacitance, conductance and reciprocal inductance matrices.

Each of these energy functions is a positive semi-definite quadratic form. Their positive-semi-definiteness may be established qualitatively by considering that the loop and node-pair matrices in a passive network have to be positive-semi-definite because in such a network, the energy stored or dissipated cannot be negative.

Considering the energy functions derived from the loop equations, if we set

$$I_i = 0, \ i \neq 1$$
$$I_1 = 1$$

then we get:

$$Z_{11} = sT + F + V / s$$

Similarly, starting with the energy functions defined using node-pair equations and setting

$$E_i = 0, \qquad i \neq 1$$
$$E_1 = 1$$

we get

$$Y_{11} = sV^* + F^* + T^* / s$$

We may derive special cases from these results. For example, for an LC network, F is zero and for an RC network, T is zero. Therefore:

$$Z_{11} = sT + V / s \qquad \textit{for an LC network}$$
$$Z_{11} = F + V / s \qquad \textit{for a RC network}$$

Similar results may be obtained for the driving point admittances using the node-pair-based energy functions. Obviously, these are not the methods used for evaluating driving point impedances and admittances, as they lead to complex expressions. However, the insight provided by this analysis is very useful.

## 2.1.6 Formulation of state equations using energy functions

We have already briefly come across the use of energy functions in the previous lecture, where we referred to the modelling of the capacitor microphone. Let us now look at it in a little more detail.

Conservative systems

We will first consider conservative systems, that is, systems without energy sources or sinks, and later go on to consider non-conservative systems.

Example 1

Consider a very simple example, with only one inductor and capacitor as shown.
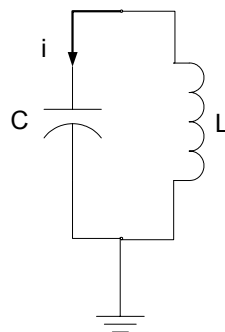
Let us define T as the total system kinetic energy and V as the total system potential energy. We will call the total energy E

In terms of the charge q and its derivative, we can write:

$$T = \frac{1}{2} L \dot{q}^2$$

$$V = \frac{1}{2C} q^2$$

$$E = T + V = \frac{1}{2} L \dot{q}^2 + \frac{1}{2C} q^2$$

Since the system is conservative,

$$\frac{dE}{dt} = 0$$

$$L \dot{q} \ddot{q} + \frac{1}{C} q \dot{q} = 0$$

$$LC \ddot{q} + q = 0$$

In terms of the flux linkage λ and its derivatives:

$$E = \frac{1}{2L}\lambda^2 + \frac{1}{2}C\dot{\lambda}^2$$

$$\frac{dE}{dt} = \frac{1}{L}\lambda\dot{\lambda} + C\dot{\lambda}\ddot{\lambda} = 0$$

$$LC\ddot{\lambda} + \lambda = 0$$

We could use Lagrange's energy balance equation for conservative systems, which states that:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_n}\right) - \frac{\partial T}{\partial q_n} + \frac{\partial V}{\partial q_n} = 0$$

where

        T = total system kinetic energy
        V = total system potential energy
        n = 1, 2, … refers to the independent   coordinates in the system
        q_n = generalised coordinate
        $\dot{q}_n$ = generalised velocity

$$T = \frac{1}{2}L\dot{q}^2$$

$$V = \frac{1}{2C}q^2$$

$$\frac{\partial T}{\partial \dot{q}} = L\dot{q}$$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}}\right] = L\ddot{q}$$
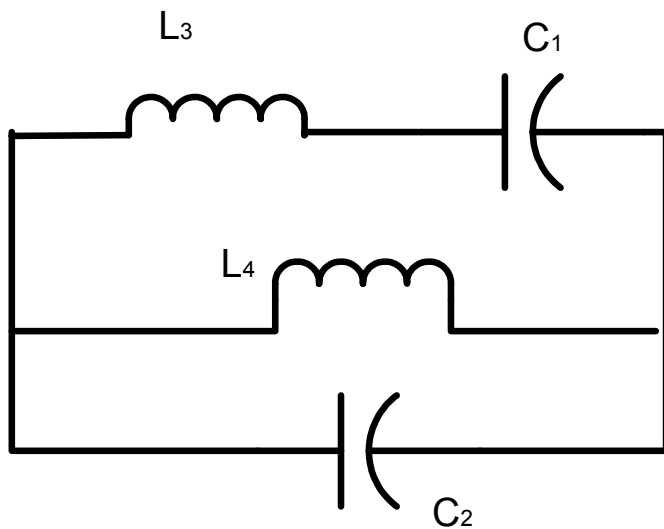
$$\frac{\partial T}{\partial q} = 0$$

$$\frac{\partial V}{\partial q} = \frac{1}{C}q$$

$$\therefore L\ddot{q} + \frac{1}{C}q = 0$$

$$LC\ddot{q} + q = 0$$

Example 2

Let us consider one more example:

We will consider the flux linkages λ associated with each of the elements as candidates for the selection of independent coordinates.

[The meaning of a flux linkage associated with a capacitor is not quite clear. We will assume that a capacitor is in parallel with an infinite inductor, carrying zero current, but with a flux linkage λ such that its time derivative is equal to the voltage across the capacitor.]

We could select (say) λ_1 and λ_2 as our coordinates (but not λ_2 and λ_4, as they are not independent)

$$T = \frac{1}{2} C_1 \dot{\lambda}_1^2 + \frac{1}{2} C_2 \dot{\lambda}_2^2$$

$$V = \frac{1}{2L_3} \lambda_3^2 + \frac{1}{2L_4} \lambda_4^2$$

$$= \frac{1}{2L_3} (\lambda_2 - \lambda_1)^2 + \frac{1}{2L_4} \lambda_2^2$$

Evaluating with respect to λ_1 we have:

$$\frac{\partial T}{\partial \dot{\lambda}_1} = C_1 \dot{\lambda}_1$$

$$\frac{d}{dt} \left[\frac{\partial T}{\partial \dot{\lambda}_1} \right] = C_1 \ddot{\lambda}_1$$

$$\frac{\partial T}{\partial \lambda_1} = 0$$

$$\frac{\partial V}{\partial \lambda_1} = -\frac{1}{L_3} (\lambda_2 - \lambda_1)$$

This yields the state equation:

$$C_1 \ddot{\lambda}_1 - \frac{1}{L_3}(\lambda_2 - \lambda_1) = 0$$

Similarly, evaluating with respect to λ_2, we get:

$$\frac{\partial T}{\partial \dot{\lambda}_2} = C_2 \dot{\lambda}_2$$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\lambda}_1}\right] = C_2 \ddot{\lambda}_2$$

$$\frac{\partial T}{\partial \lambda_2} = 0$$

$$\frac{\partial V}{\partial \lambda_2} = \frac{1}{L_3}(\lambda_2 - \lambda_1) + \frac{1}{L_4}\lambda_2$$

leading to the state equation:

$$C_2 \ddot{\lambda}_2 + \frac{1}{L_3}(\lambda_2 - \lambda_1) + \frac{1}{L_4}\lambda_2 = 0$$

These two equations may now be expressed in standard form by defining a new set of state variables as:

$$x_1 = \lambda_1, \qquad x_2 = \lambda_2,$$
$$x_3 = \dot{\lambda}_1, \qquad x_4 = \dot{\lambda}_2$$

Non-conservative systems

Let us now consider non-conservative systems. The complete Lagrange's equations for such systems are as follows:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_n}\right) - \frac{\partial T}{\partial q_n} + \frac{\partial D}{\partial \dot{q}_n} + \frac{\partial V}{\partial q_n} = Q_n$$

where

T = total system kinetic energy
D = total system dissipation factor, and is defined as one half the rate at which energy is dissipated in the system as heat.
V = total system potential energy
n = 1, 2, … refers to the independent   coordinates in the system
Q_n = generalised forcing function relative to coordinate n
q_n = generalised coordinate
$\dot{q}_n$ = generalised velocity

To obtain the forcing function relative to the nth coordinate:

Suppose the system has all its coordinates q_1, q_2, ..., q_n frozen when the system is in an arbitrary configuration. Now, let one coordinate q_i increase by δq_i. Let δw_i be the work done by all external forces in the system during the displacement δq_i. Then,

$$Q_i \triangleq \underset{\delta q_i \to 0}{Lim} \frac{\delta w_i}{\delta q_i}$$

Example 3

Let us consider the following example:
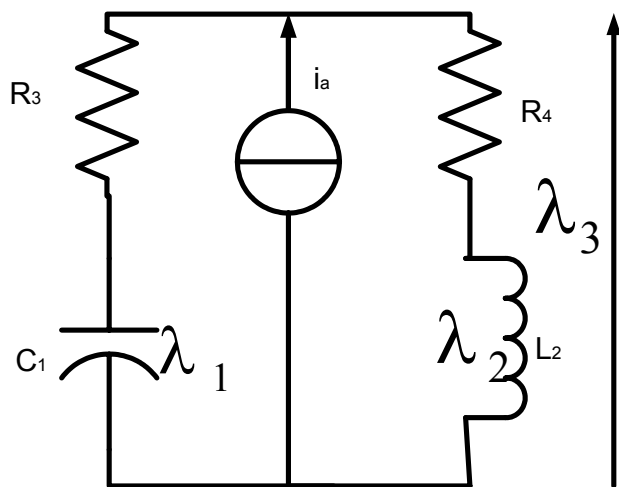
We will choose λ_1, λ_2 and λ_3 as the coordinates.

$$T = \frac{1}{2} C_1 \dot{\lambda}_1^2$$

$$V = \frac{1}{2L_2} \lambda_2^2$$

$$D = \frac{1}{2} \left[\frac{(\dot{\lambda}_3 - \dot{\lambda}_1)^2}{R_3} + \frac{(\dot{\lambda} - \dot{\lambda}_2)^2}{R_4} \right]$$

$$Work\ done\ by\ external\ forces =$$

$$i_a v\, dt = i_a \frac{d\lambda_3}{dt} dt = i_a \delta\lambda_3$$

Therefore,

$$\delta w_1 = 0,\ as\ \lambda_1\ is\ held\ const.,\quad Q_1 = 0$$
$$\delta w_2 = 0,\ as\ \lambda_2\ is\ held\ const.,\quad Q_2 = 0$$

$$\delta w_3 = i_a \delta\lambda_3 \quad Q_3 = \frac{i_a \delta\lambda_3}{\delta\lambda_3} = i_a$$

Now let us write down the Lagrange's equations:

$$T = \frac{1}{2} C_1 \dot{\lambda}_1^2$$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\lambda}_1}\right] = C_1 \ddot{\lambda}_1,$$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\lambda}_2}\right] = 0,$$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\lambda}_3}\right] = 0$$

$$\frac{\partial T}{\partial \lambda_1} = 0, \qquad \frac{\partial T}{\partial \lambda_2} = 0, \qquad \frac{\partial T}{\partial \lambda_3} = 0,$$

$$V = \frac{1}{2 L_2} \lambda_2^2$$

$$\frac{\partial V}{\partial \lambda_1} = 0, \qquad \frac{\partial V}{\partial \lambda_2} = \frac{\lambda_2}{L_2}, \qquad \frac{\partial V}{\partial \lambda_3} = 0,$$

$$D = \frac{1}{2}\left[\frac{(\dot{\lambda}_3 - \dot{\lambda}_1)^2}{R_3} + \frac{(\dot{\lambda} - \dot{\lambda}_2)^2}{R_4}\right]$$

$$\frac{\partial D}{\partial \dot{\lambda}_1} = -\frac{(\dot{\lambda}_3 - \dot{\lambda}_1)}{R_3},$$

$$\frac{\partial D}{\partial \dot{\lambda}_2} = -\frac{(\dot{\lambda} - \dot{\lambda}_2)}{R_4},$$

$$\frac{\partial D}{\partial \dot{\lambda}_3} = \frac{(\dot{\lambda}_3 - \dot{\lambda}_1)}{R_3} + \frac{(\dot{\lambda}_3 - \dot{\lambda}_2)}{R_4}$$

Substituting in the Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_n}\right) - \frac{\partial T}{\partial q_n} + \frac{\partial D}{\partial \dot{q}_n} + \frac{\partial V}{\partial q_n} = Q_n$$

$$C_1\ddot{\lambda}_1 - \frac{(\dot{\lambda}_3 - \dot{\lambda}_1)}{R_3} = 0 \qquad (1)$$

$$\frac{\lambda_2}{L_2} - \frac{(\dot{\lambda}_3 - \dot{\lambda}_2)}{R_4} = 0 \qquad (2)$$

$$\frac{(\dot{\lambda}_3 - \dot{\lambda}_1)}{R_3} + \frac{(\dot{\lambda}_3 - \dot{\lambda}_2)}{R_4} = i_a \qquad (3)$$

From the above, it appears that there are many state variables, as the first derivatives of λ_1, λ_2 and λ_3 and the second derivative of λ_1 appear in the above equations. However, a careful study reveals that most of them can be eliminated:

Substituting from equations (1) and (2) into equation (3) yields:

$$C_1\ddot{\lambda}_1 + \frac{\lambda_2}{L_2} = i_a$$

which correspond to only two state equations, as should have been expected for a system with only two energy storage elements.