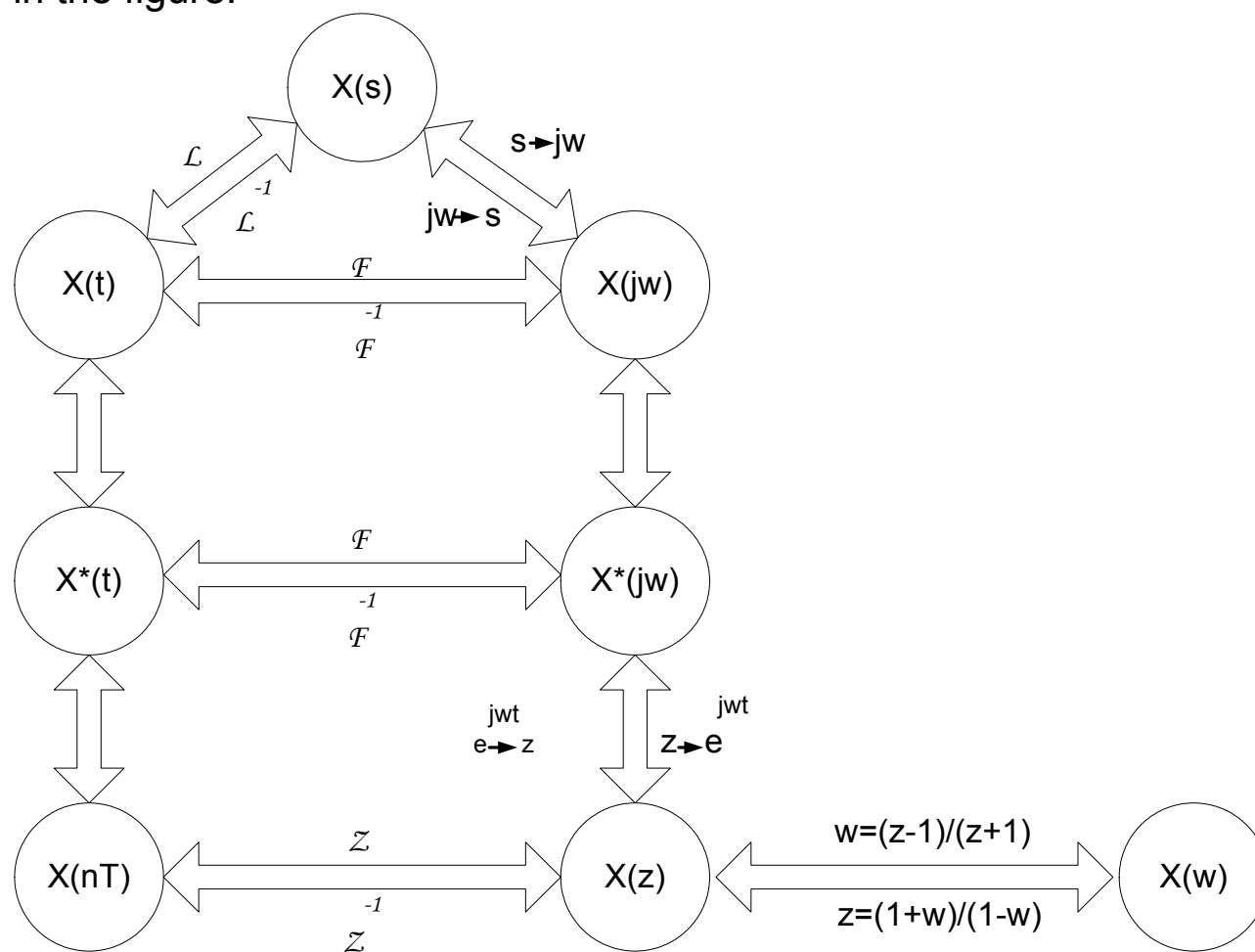


1

Transform methods

Some of the different forms of a signal, obtained by transformations, are shown in the figure.



We will very briefly review the use of Fourier and Laplace transforms, and then have a quick look at the DFT (Discrete Fourier Transform), the FFT (Fast Fourier Transform, a fast computational method for evaluating the DFT) and the z transform.

1.1 The s-plane and the Laplace domain

Introduction

You are familiar with the fundamental circuit laws and how they are applied to both direct current and alternating current circuits. You have also used the Laplace transform to help you with the solution of differential equations that you encounter in solving circuit problems. We will commence this course with a brief review of the s-plane, with the intention of looking at some of the characteristic properties of selected circuits from a slightly different angle. A formal presentation of the Laplace transform will not be attempted, rather, only its applications in the study of circuits (and later) and control systems.

We start by recognising the s-plane, a complex plane on which we can place the poles and zeros of a network function. (A network function is a function, which represents some characteristic, such as the input impedance, of the network. We will deal with each of these ideas later on.) The s-plane is a complex plane, whose coordinates are the real part σ and the so-called “imaginary” part $j\omega$

We can represent any function of the complex variable $s (= \sigma + j\omega)$ by its poles and zeros, if we disregard any multiplying factors. The poles are those values of s for which the function goes to infinity, and zeros are those values of s for which the function vanishes.

The general complex exponential excitation function

You have studied the behaviour of circuits under different excitations, by dc, ac and special functions such as impulse and step functions (In practice, we have only approximations to these special functions, as sudden changes are not possible in real life.) We will now look at a more general excitation function that can be used to study the behaviour of circuits in a more convenient and uniform manner.

Consider the general complex exponential excitation function, $x(t) = X e^{s_0 t} u_{-1}(t)$.

$u_{-1}(t)$ represents a unit step function at the origin, and is introduced to ensure that the excitation is applied at time zero. We are only interested in causal systems, that is, where the response occurs only after the excitation. The complex variable $s_0 (= \sigma_0 + j\omega_0)$ is general enough to represent most types of input that we are interested in, and in particular, sinusoidal inputs.

With σ_0 zero, we have a pure, steady, sinusoid; while with ω_0 zero, it is either an exponentially growing or an exponentially decaying quantity, depending on whether σ_0 is positive or negative. With complex s_0 , we would have either growing or decaying sinusoids.

Network functions

We said earlier that a network function is a function, which represents some characteristic, such as the input impedance, of a network. Let us now look at this in a little more detail.

The simplest network that we can think of is a single-element, two-terminal network, such as a resistor. The characteristics of a resistor are described by the Ohm's Law equation:

$$v = iR$$

Where v is the voltage difference *across* the resistor, i is the current *through* it, and R is its resistance, which defines the relationship between v and i . With inductors (inductance L) and capacitors (capacitance C), we have relationships that are dependant on rates of change, giving rise to differential equations:

$$\begin{aligned} v &= L \, di / dt && \text{for an inductor} \\ i &= C \, dv / dt && \text{for a capacitor} \end{aligned}$$

Taking Laplace transforms (assuming zero initial conditions), we get:

$$\begin{aligned} V(s) &= [Ls] I(s) && \text{for an inductor} \\ V(s) &= [1/Cs] I(s) && \text{for a capacitor} \end{aligned}$$

We can now write the relationship between voltage and current for resistors, inductors or capacitors (or any combination of them) as:

$$\begin{aligned} V(s) &= Z(s) I(s) \\ \text{Or } Z(s) &= V(s) / I(s) \end{aligned}$$

$Z(s)$, a function of the complex variable s , is known as the impedance of the two-terminal network. We also have $Y(s)$, the inverse of $Z(s)$ defined by the relationship

$$Y(s) = I(s) / V(s)$$

$Y(s)$ is known as the admittance of the network.

We can visualise a two-terminal network as a single-port network. If it is to be excited by a source, the source has to be connected across the two terminals, thus making it appear as a single port to the source. Most networks, if they are to accomplish any useful purpose, have to have more than one port. A network may be connected between (say) a source and a load to achieve some desired connectivity characteristic, a kind of transformation. We then have, at the simplest, two-port networks with just one pair of ports, which we naturally identify as the input port and the output port. A complex network is made up of interconnections of a large number of such networks, not necessarily two-port

networks. However, multi-port networks can, in most cases, be analysed using two-port network theory, and we confine our attention to them in this course.

We saw earlier how impedance and admittance functions can describe a single-port network. In a two-port network, we would naturally have to extend these concepts to both ports, so that we will have:

- Input impedance
- Input admittance
- Output impedance
- Output admittance
- Transfer impedance and admittance

as functions that characterise the network. There are other functions too which are of significance, such as:

- Transfer function
- Characteristic impedance
- Scattering matrix, which defines the relationship between the incident power and the reflected power.

Some of these ideas will be discussed later.

We arrived at the concept of network function through the Laplace transform. *The system function of a network (a network function) is a function of the complex frequency s , representing the ratio of the Laplace transform of a response to the Laplace transform of the excitation causing the response.* We assume that the network is initially relaxed, that is the stored energy in the network is initially zero.

Pole-zero patterns

The most general form of a system function is given by a *real rational function of s* . It can be expressed as the quotient of two polynomials in s . Such a function is completely described by its poles, zeros and a multiplier factor.

Let us examine such a function, $H(s)$.

$$H(s) = q(s) / p(s), \text{ where}$$

$$q(s) = b_m(s-z_1)(s-z_2) \dots (s-z_m)$$

$$p(s) = (s-p_1)(s-p_2) \dots (s-p_n)$$

The finite zeros of $H(s)$ are z_1, z_2, \dots, z_m while its finite poles are at p_1, p_2, \dots, p_n . In addition to these finite poles and zeros, you can see that there would be additional poles or zeros at infinity, depending whether $m > n$ or $n > m$, of order $(m-n)$ or $(n-m)$.

A study of the pole-zero pattern of a system (or network) function gives us an insight into its behaviour. For example, an examination of the driving point impedance function of a one-port network will define its impedance at all natural frequencies, and enable us to obtain a physical realisation of the impedance using actual components. Similarly, it is possible for us to obtain both the frequency characteristics and a physical realisation of a two-port filter network, given its pole zero pattern.

As a network function is completely defined by its poles and zeros, they are called the critical frequencies of the system function. The (finite) poles of the system function correspond to the natural frequencies of the network, that is the frequencies of natural oscillations. These are also known as the natural modes of the network.

The order of a system function is the highest order (in s) of the denominator or the numerator, and gives an indication of its complexity.

Properties of LC, RC & RLC network functions

Before considering the properties of passive network functions such as those of LC, RC and RLC networks, we need to be familiar with what are known as positive real functions.

LC, RC and RLC network functions all have certain common properties.

All these system functions, whether immittance (impedance or admittance) functions or transfer functions, are quotients of two polynomials in s , with real rational coefficients. They are thus *real rational functions*.

They are all passive functions, with no intrinsic energy sources. Hence, if viewed as response functions to an impulse excitation, they cannot diverge without limit. Therefore, they *cannot have poles (or zeros) on the right-half plane, nor can they have multiple-order poles on the j axes*.

They obey the reciprocity theorem. Thus, any two impedances obtained by interchanging the points of excitation and response are equal. This would mean that *the resulting matrices are symmetric*.

Let us now look at each of these types of networks.

The following properties of an LC network may be derived:

- They are simple, that is there are no higher order poles or zeros.
- They all lie on the j axis.
- Poles and zeros alternate.

The origin and infinity are always critical frequencies, that is, there will be a pole or zero at both the origin and at infinity.
The multiplicative constant is positive.

RC (and RL) network functions have characteristics that are different from those of LC networks. Since an RC network has resistive components, it cannot have a zero-valued impedance at any real frequency. Its zeros (and poles) are on the non-positive σ axis of the s -plane. It should also be noted that the form of the impedance function is different from that of the admittance function, unlike in the case of the LC network. However, the impedance function of an RC network is similar in form to the admittance function of an RL network, and vice versa.

The poles and zeros of an RC driving point function lie on the non-positive real axis.
They are simple.
Poles and zeros alternate.
The slopes of impedance functions are negative, those of admittance functions are positive.

Energy functions

Energy functions or “energy-like-functions” may be used for the derivation of properties of driving point impedance and admittance functions of passive networks. When $s = j\omega$, these functions are directly related to the energy stored in the network, as currents through inductances or voltages across capacitances, that is, as electromagnetic or electrostatic energy.

These functions are positive semi-definite functions.

We define three functions T , F and V , corresponding to the kinetic, dissipation and potential energies as follows:

Consider the loop equations, in matrix form:

$$[Z] [I] = [E]$$

where $[Z] = s[L] + [R] + [S] / s$,

(S is the loop elastance, the reciprocal of loop capacitance)

Pre-multiplying both sides of the equation by the transpose of the complex conjugate of $[I]$ will yield an energy-like expression with three terms (denoted by T , F and V) related to the inductance, resistance and elastance of the loop circuits.

We may define another set of parallel functions V^* , F^* and T^* starting with the node-pair equations:

$$[Y][E] = [I]$$

Again, pre-multiplying both sides by the transpose of the complex conjugate of $[E]$, we obtain the expressions for V^* , F^* and T^* .

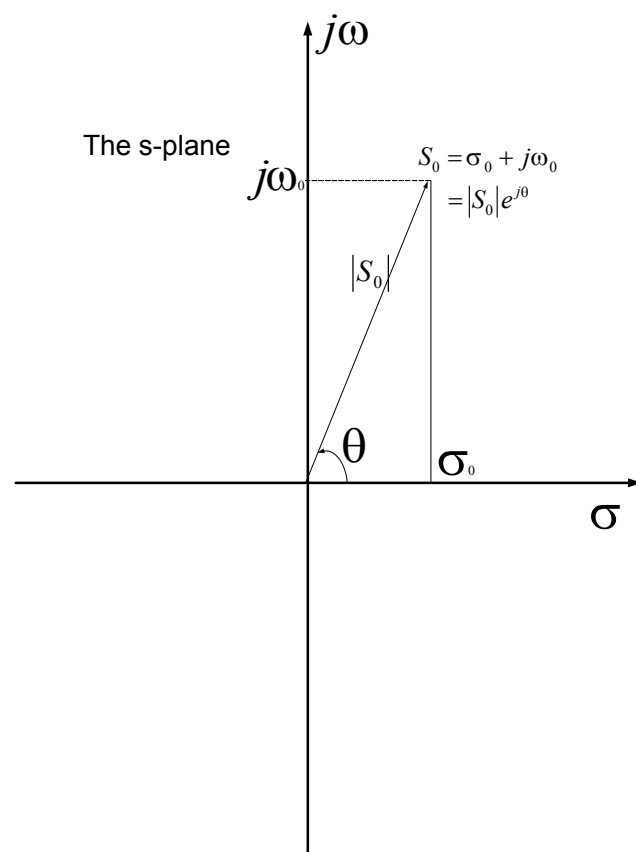
The two derivations give us two forms of the energy function as:

$$\begin{aligned} sT + F + V/s \\ sV^* + F^* + T^*/s \end{aligned}$$

These may be used to evaluate the driving point impedances and admittances by imposing suitable conditions.

1.1.1 The s-plane

The s-plane is a complex plane (axes σ and $j\omega$) as shown in the figure.



Any point s_0 on the plane will have two coordinates σ_0 and $j\omega_0$ as shown. They are known as the “real” part and the “imaginary” part of the complex number s_0 ($= \sigma_0 + j\omega_0$). However, there is nothing more real in the real part than in the imaginary, both are real enough. This terminology has arisen due to historical reasons, and should be treated as mere names, with no significance in the meaning.

s_0 can also be represented in polar form, as having a magnitude $|s_0|$, and angle θ . This of course further illustrates that all quantities are real in the normal meaning of the word.

The relationships among these various quantities are obvious from the diagram:

$$s_0 = \sigma_0 + j\omega_0 = |s_0|e^{j\theta};$$

$$|s_0| = \sqrt{(\sigma_0^2 + \omega_0^2)}$$

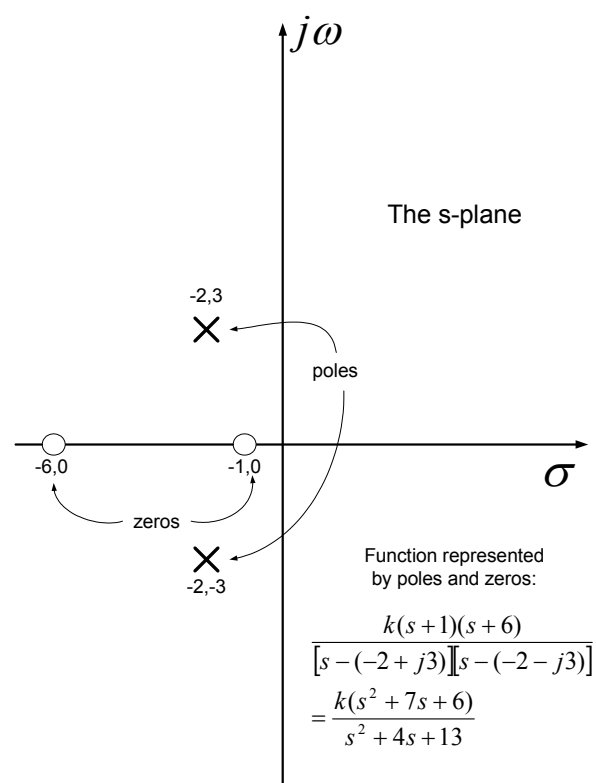
$$\theta = \tan^{-1} \frac{\omega_0}{\sigma_0}$$

$$\sigma_0 = |s_0| \cos \theta$$

$$\omega_0 = |s_0| \sin \theta$$

$$s_0 = |s_0| (\cos \theta + j \sin \theta)$$

We can represent a function of a complex variable $s (= \sigma + j\omega)$. by its poles and zeros. By poles we mean those values of s for which the function becomes infinite, that is, its denominator becomes zero; and by zeros we mean those values of s for which the function vanishes, that is, the numerator becomes zero.



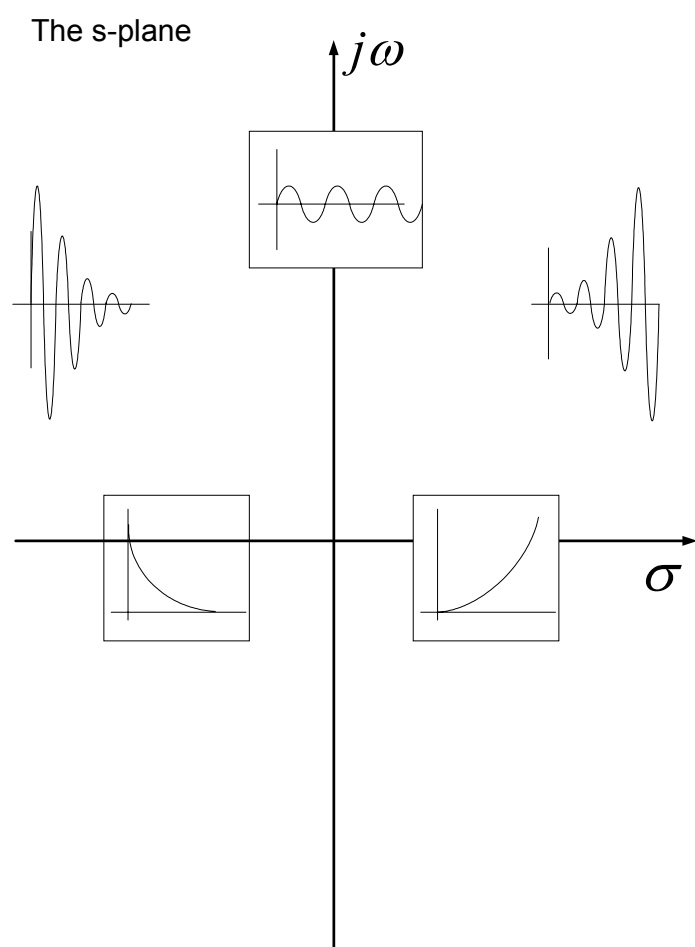
The poles are marked by crosses (X) and zeros by noughts (O) as shown. Note that any multiplying factor (k , in the illustration) is not represented on the s-plane plot

1.1.2 The general complex exponential excitation function

The behaviour of the general exponential excitation function

$$x(t) = X e^{s_0 t} u_{-1}(t).$$

for s_0 lying in different parts of the s-plane is shown in the figure.

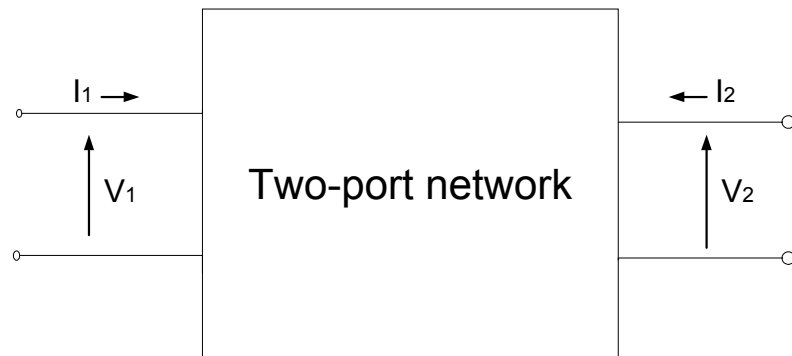


With σ_0 zero, we have a pure, steady, sinusoid; while with ω_0 zero, it is either an exponentially growing or an exponentially decaying function, depending on whether σ_0 is positive or negative. With complex s_0 , we would have either growing or decaying sinusoids.

Fourier analysis enables us to express any repetitive waveform as the summation of a series of sinusoids, with arbitrary accuracy. Thus, any required repetitive excitation pattern might be implemented using the general complex exponential excitation. This also applies to non-repetitive waveforms, as they can be treated as repetitive waveforms with an infinite period. The use of the complex excitation function further enable us to decay any component of the excitation, using the real part of the complex exponential.

With s_0 equal to zero and X equal to one, we can obtain the unit step function.

1.1.3 Two-port networks



Consider a two-port network as shown.

The driving point impedance (or the input impedance) looking in at port 1 would obviously depend on the type of termination at port 2. If we recognise port 1 as the input and port 2 as the output, it is usual to compute the driving point impedance at port 1 with port 2 open circuited, that is, with I_2 equal to zero.

We can describe the behaviour of the two-port network by the following set of two equations:

$$\begin{aligned} V_1 &= z_{11}I_1 + z_{12}I_2 \\ V_2 &= z_{21}I_1 + z_{22}I_2 \end{aligned}$$

The z 's are called the open circuit impedances. As noted earlier, when I_2 is zero, we have, from the first equation:

$$z_{11} = \frac{V_1}{I_1}, \quad z_{21} = \frac{V_2}{I_1}$$

z_{11} is the driving point impedance at port 1, or the input impedance of the network, looking at port 1. z_{21} is a transfer impedance.

Similarly, with I_1 zero, we obtain:

$$z_{22} = \frac{V_2}{I_2}, \quad z_{12} = \frac{V_1}{I_2}$$

We could have described the network in terms of admittances, instead of impedances, as given below:

$$I_1 = y_{11}V_1 + y_{12}V_2$$

$$I_2 = y_{21}V_1 + y_{22}V_2$$

We saw earlier why the impedances z_{11} , z_{12} , z_{21} and z_{22} are known as open circuit impedances, for we obtained them by setting I_1 and I_2 equal to zero. Similarly, the admittance functions turn out to be short circuit admittances.

With V_2 equal to zero, we have:

$$y_{11} = \frac{I_1}{V_1}, \quad y_{21} = \frac{I_2}{V_1}$$

and setting V_1 to zero gives:

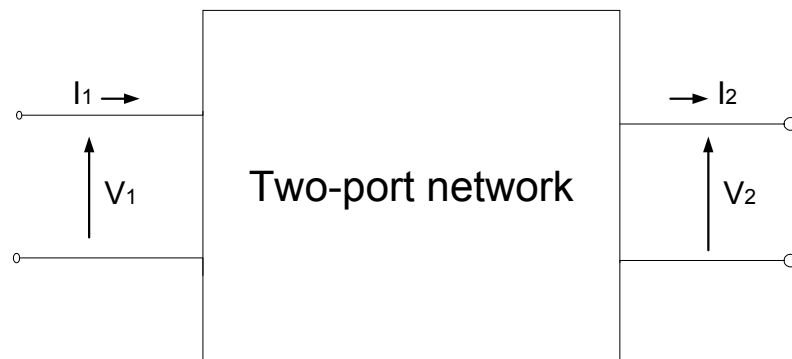
$$y_{22} = \frac{I_2}{V_2}, \quad y_{12} = \frac{I_1}{V_2}$$

Other representations of two-port networks are also in use. One such is the hybrid parameter representation used in electronic circuit analysis.

$$V_1 = h_{11}I_1 + h_{12}V_2$$

$$I_2 = h_{21}I_1 + h_{22}V_2$$

ABCD representation is commonly used in the study of transmission lines. Here, the current I_2 is, by convention, shown as leaving the network rather than as entering the network.



$$V_1 = AV_2 + BI_2$$

$$I_1 = CV_2 + DI_2$$

1.1.4 Positive real functions

A complex function $G(s)$ is said to be positive real when:

$G(s)$ is real for all real s
 $\text{Re}(G(s)) \geq 0$, for $\text{Re}(s) \geq 0$

The driving point function of any physical network is positive real, and every rational function that is positive real can be realised as the driving point function of a network. Hence, the study of such functions is important in the study of networks.

If G is positive real, then $1/G$ is also positive real.

It can be shown that $G(s)$ is positive real if it satisfies the following conditions:

1. $G(s)$ is real for all real s
2. $G(s)$ is analytic in the right-half plane.
3. Any poles on the j axis are simple, with real positive residues.
4. $\text{Re}(G(j\omega)) \geq 0$ for all ω

All conditions relating to poles also apply to zeros, as $1/G$ also has to be positive real.

The following properties of positive real functions may be deduced from these conditions:

No negative coefficients occur in either the numerator or denominator.

The highest (and lowest) powers of the numerator and denominator cannot differ by more than unity.

Poles (and zeros) on the j axes occur in conjugate pairs.

1.1.5 Positive semi-definite functions

We define a positive semi-definite function as one which is always either positive or zero. We will consider only quadratic forms in the present situation.

All the terms of a quadratic form are of the second order. In matrix form, we can represent a quadratic form as:

$$X^T A X$$

where X is a column vector and A is a symmetric matrix.

(Note: We will confine our attention to symmetric matrices, and both X and A are real)

The following definitions and properties apply:

1. A real matrix A is said to be positive semi-definite if and only if

$$X^T A X \geq 0$$

for all real and finite X

2. A is said to be positive definite if and only if it is positive semi-definite, and in addition

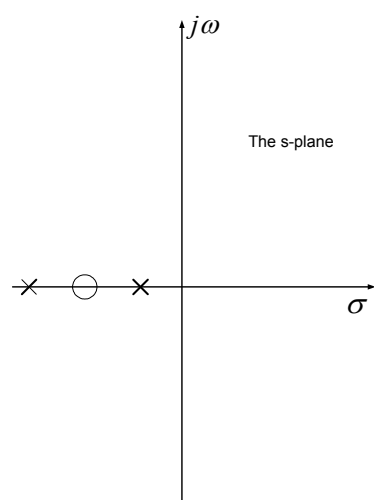
$$X^T A X = 0$$

only if $X=0$

3. For A to be positive definite, each of its principal minors should be positive. (This may be tested by testing a set of determinants obtained by deleting successive rows and columns from A for positiveness.)

4. For A to be positive semi-definite, each of its principal minors should be non-negative.

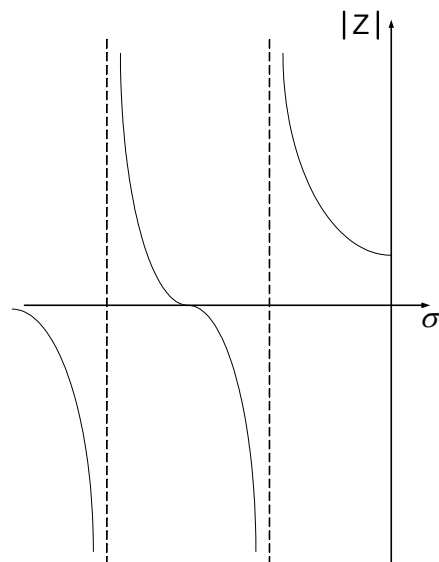
1.1.6 Properties of RC networks



The figure shows a typical pole-zero pattern of the impedance function of an RC network. Note that:

- The poles and zeros are simple.
- They all lie on the non-positive σ - axis.
- Poles and zeros alternate.

The plot of impedance vs' σ , along the non-positive real axis is shown below.

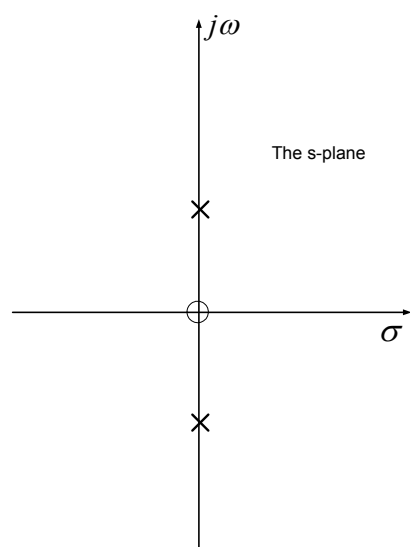


Note that there is a critical frequency at infinity, but not at the origin, in this particular case

In general, the critical frequency of the smallest magnitude is a pole, lying on the non-positive (that is, at the origin or on the negative) real axis.

When the number of finite poles is greater than the number of finite zeros, there will be a zero at infinity, as is the case in the example (two finite poles and only one finite zero). When the degrees of the numerator and denominator polynomials are equal, there will be no critical frequency at infinity. Driving point impedance functions of RC networks have negative gradients along the σ axis.

1.1.7 Properties of LC networks.

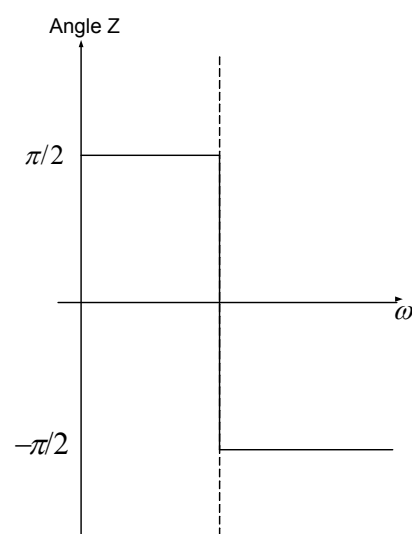
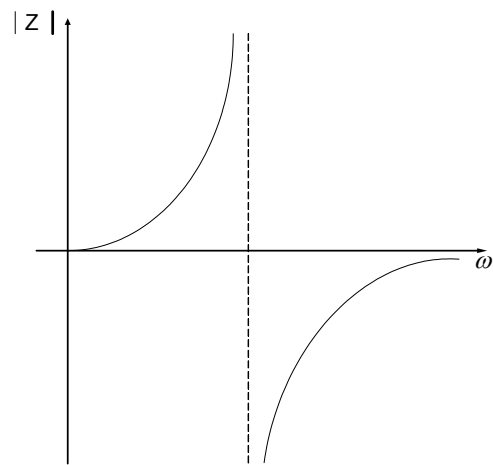


The figure shows a typical pole-zero pattern of the impedance function of an LC network

Note that

- The poles and zeros are simple.
- They all lie on the $j\omega$ axis.
- Poles and zeros alternate.

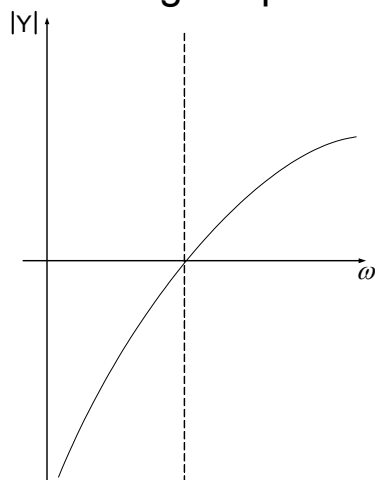
The variation of impedance with frequency corresponding to this pole-zero pattern may be obtained by substituting $j\omega$ for s , and is as shown below.



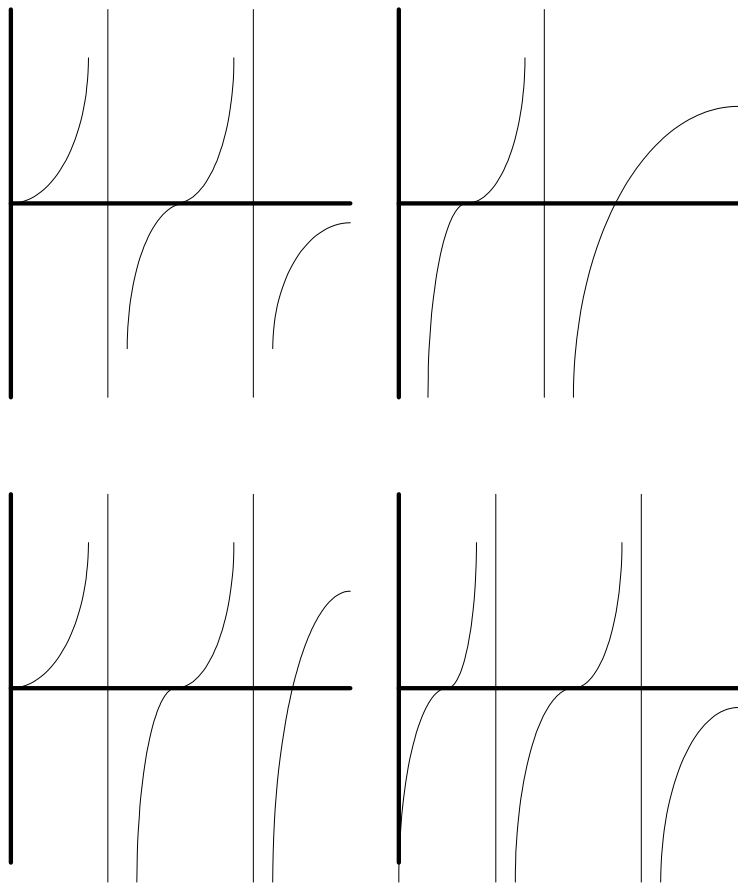
Note that there are critical frequencies at the origin and at infinity.

The characteristics of driving point admittance functions of LC networks are similar to those of impedance functions, and may be obtained by substituting $1/s$ for s .

For the impedance function considered earlier, the admittance function would have the following shape:



Plots of other typical immittance functions of LC networks are shown below.



In addition to the properties mentioned earlier, you may have noticed another property of LC functions.

They all have positive gradients, that is, the value of the function always increases with increase of frequency.

This arises from the fact that all the residues at the poles and zeros are positive, that is, the partial fraction expansion of the functions all have positive coefficients. Substitution of $s = j\omega$ and differentiation leads to this result.