Analysis of Non-Sinusoidal Waveforms

Waveforms

Up to the present, we have been considering direct waveforms and sinusoidal alternating waveforms as shown in figure 1(a) and 1(b) respectively.

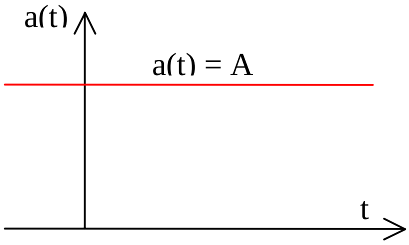

Figure 1(a) – direct waveform

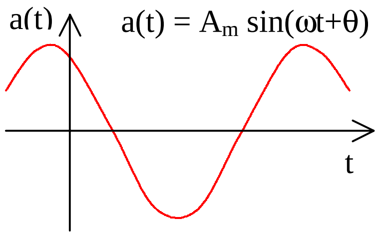

Figure 1(a) – sinusoidal waveform

However, many waveforms are neither direct nor sinusoidal as seen in figure 2.

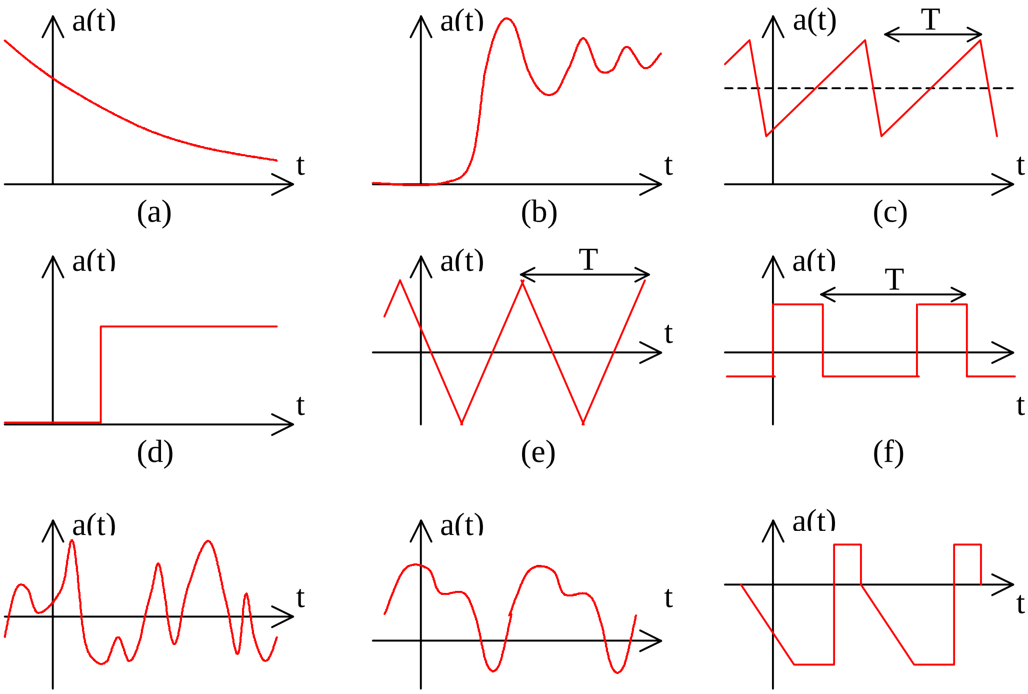

Figure 2

It can be seen that the waveforms of Figure 2 (a), (b), (c) and (d) are uni-directional, although not purely direct. Waveforms of Figure 2 (e) and (f) are repetitive waveforms with zero mean value, while figure2 (c), (h) and (i) are repetitive waveforms with finite mean values. Figure 2 (g) is alternating but non-repetitive and mean value is also non-zero.

Thus we see that there are basically two groups of waveforms, those that are repetitive and those which are non-repetitive. These will be analysed separately in the coming sections.

In a repetitive waveform, only one period "T" needs to be defined and can be broken up to a fundamental component (corresponding to the period T) and its harmonics. A uni-directional term (direct component) may also be present. This series of terms is known as Fourier Series named after the French mathematician who first presented the series in 1822.

Fourier Series

The Fourier series states that any practical periodic function (period T or frequency $\omega_o = 2\pi/T$) can be represented as an infinite sum of sinusoidal waveforms (or sinusoids) that have frequencies which are an integral multiple of ω_o.

$$f(t) = F_o + F_1 \cos(\omega_o t+\theta_1) + F_2 \cos(2\omega_o t+\theta_2) + F_3 \cos(3\omega_o t+\theta_3) + F_4 \cos(4\omega_o t+\theta_4) + F_5 \cos(5\omega_o t+\theta_5) + \ldots\ldots\ldots$$

Usually the series is expressed as a direct term ($A_o/2$) and a series of cosine terms and sine terms.

$$f(t) = A_o/2 + A_1 \cos\omega_o t + A_2 \cos 2\omega_o t + A_3 \cos 3\omega_o t + A_4 \cos 4\omega_o t + \ldots\ldots\ldots + B_1 \sin\omega_o t + B_2 \sin 2\omega_o t + B_3 \sin 3\omega_o t + B_4 \sin 4\omega_o t + \ldots\ldots\ldots$$

$$f(t) = \frac{A_o}{2} + \sum_{n=1}^{\infty}(A_n \cos n\omega_o t + B_n \sin n\omega_o t)$$

This, along with the Superposition theorem, allows us to find the behaviour of circuits to arbitrary periodic inputs.

Before going on to the analysis of the Fourier series, let us consider some of the general properties of waveforms which will come in useful in the analysis.

Symmetry in Waveforms

Many periodic waveforms exhibit symmetry. The following three types of symmetry help to reduce tedious calculations in the analysis.

(i) Even symmetry
(ii) Odd symmetry
(iii) Half-wave symmetry

Even Symmetry

A function *f(t)* exhibits even symmetry, when the region before the *y-axis* is the mirror image of the region after the *y-axis.*

i.e. $f(t) = f(-t)$

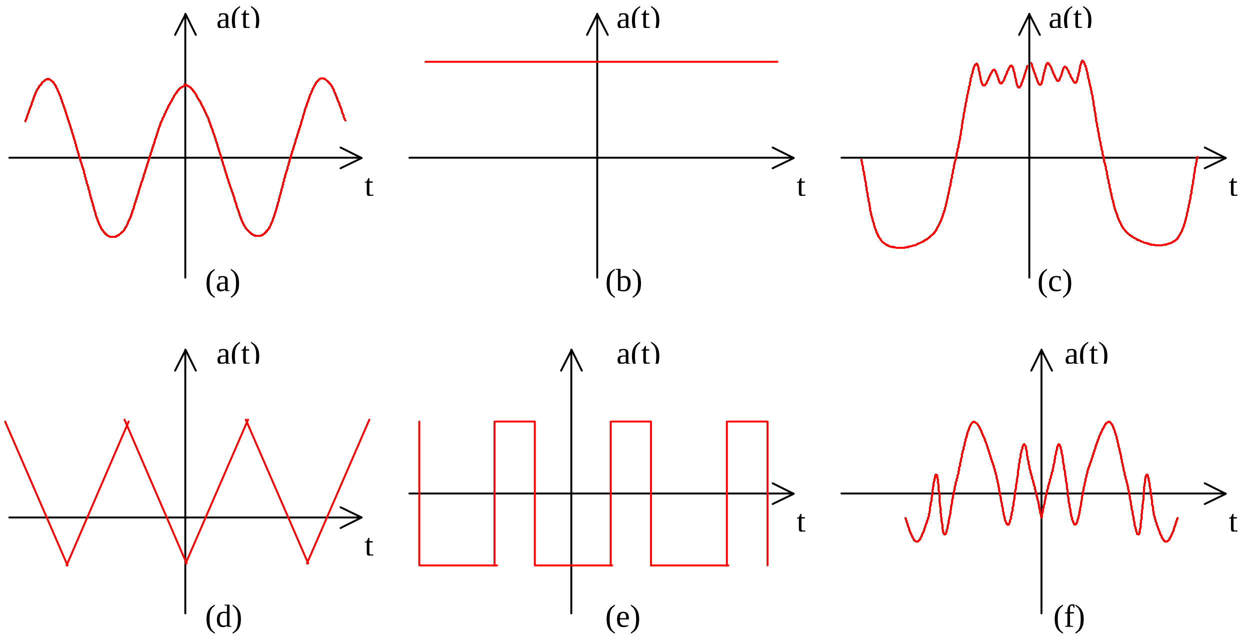

Figure 3 –Waveforms with Even symmetry

The two simplest forms of the *Even Function* or waveform with even symmetry are the *cosine* waveform and the *direct* waveform as shown in figure 3 (a) and (b).

It can also be seen from the waveforms seen in the figure 3 that even symmetry can exist in both periodic and non-periodic waveforms, and that both direct terms as well as varying terms can exist in such waveforms.

It is also evident, that if the waveform is defined for only $t \geq 0$, the remaining part of the waveform is automatically known by symmetry.

Odd Symmetry

A function *f(t)* exhibits even symmetry, when the region before the *y-axis* is the negative of the mirror image of the region after the *y-axis.*

i.e. $f(t) = (-)\, f(-t)$

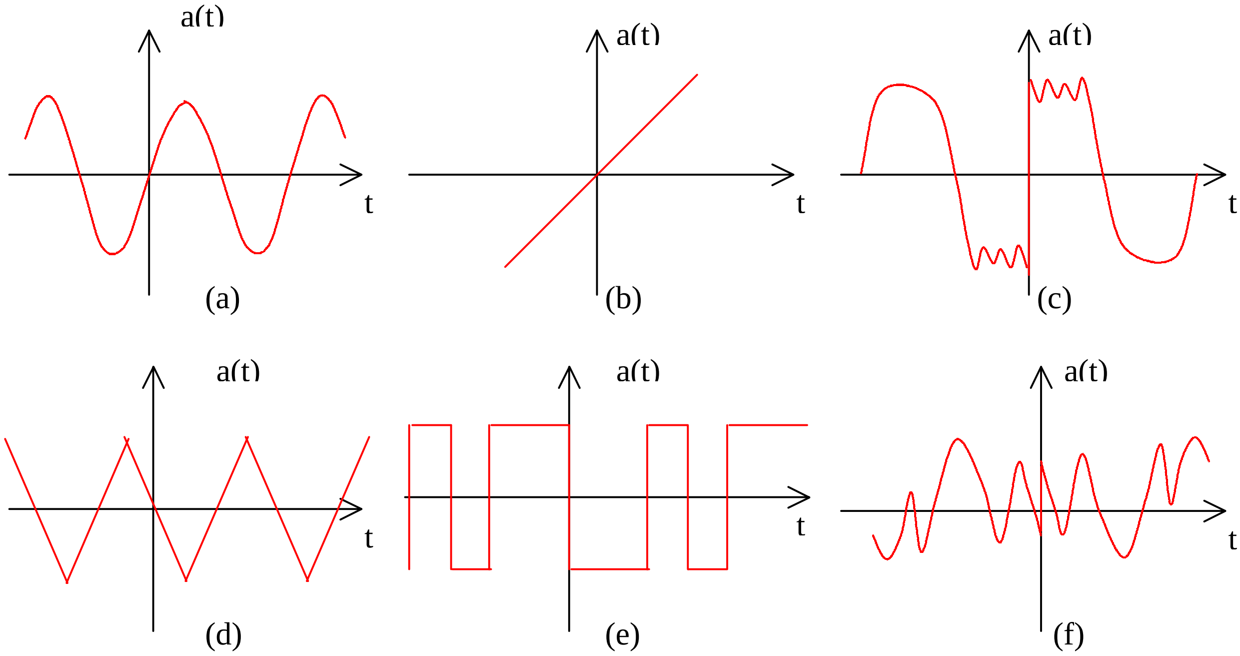

Figure 4 – Waveforms with Odd symmetry

The two simplest forms of the *Odd Function* or waveform with odd symmetry are the *sine* waveform and the *ramp* waveform as shown in figure 4 (a) and (b).

It can also be seen from the waveforms seen in the figure 4 that odd symmetry can exist in both periodic and non-periodic waveforms, and that only varying terms can exist in such waveforms. Note that direct terms cannot exist in odd waveforms.

It is also evident, that if the waveform is defined for only $t \geq 0$, the remaining part of the waveform is automatically known by the properties of symmetry.

Half-wave Symmetry

A function *f(t)* exhibits half-wave symmetry, when one half of the waveform is exactly equal to the negative of the previous or the next half of the waveform.

i.e. $f(t) = (-)\, f(t - \tfrac{T}{2}) = (-)\, f(t + \tfrac{T}{2})$

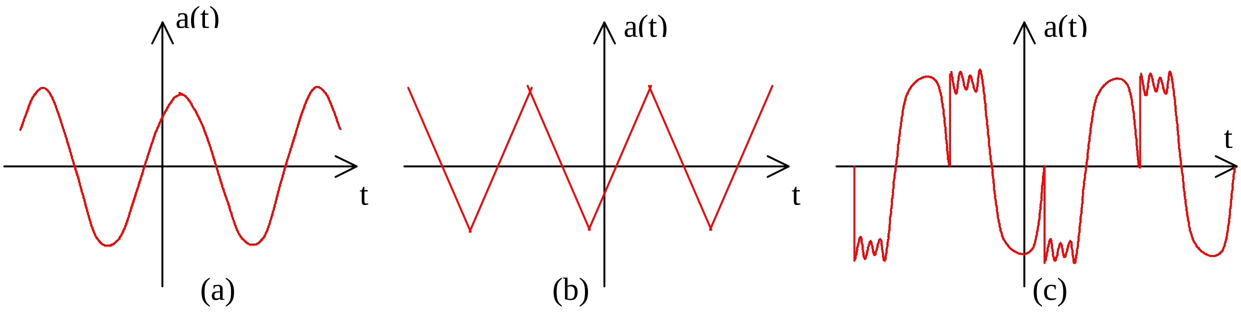

Figure 5 – Waveforms with Half-wave symmetry

The simplest form of *Half-wave Symmetry* is the *sinusoidal waveform* as shown in figure 5(a).

It can also be seen from the waveforms in the figure 5 that half-wave symmetry can only exist in periodic waveforms, and that only varying terms can exist in such waveforms. Note that direct terms cannot exist in half-wave symmetrical waveforms.

It is also evident, that if the waveform is defined for only **one half cycle**, not necessarily starting from t=0, the remaining half of the waveform is automatically known by the properties of symmetry.

Some useful Trigonometric Properties

The sinusoidal waveform being symmetrical does not have a mean value, and thus when integrated over a complete cycle or integral number of cycles will have zero value. From this the following properties follow. [Note: $\omega_o T = 2\pi$]

$$\int_{t_o}^{t_o+T} \sin \omega_o t.dt = 0 \qquad \int_{t_o}^{t_o+T} \cos \omega_o t.dt = 0$$

$$\int_{t_o}^{t_o+T} \sin n\omega_o t.dt = 0 \qquad \int_{t_o}^{t_o+T} \cos n\omega_o t.dt = 0$$

$$\int_{t_o}^{t_o+T} \sin n\omega_o t.\cos m\omega_o t.dt = 0$$ for all values of **m** and **n**

$$\int_{t_o}^{t_o+T} \sin n\omega_o t.\sin m\omega_o t.dt \begin{cases} = 0 & \text{when n} \neq \text{m} \\ = \frac{T}{2} & \text{when n} = \text{m} \end{cases}$$

$$\int_{t_o}^{t_o+T} \cos n\omega_o t.\cos m\omega_o t.dt \begin{cases} = 0 & \text{when n} \neq \text{m} \\ = \frac{T}{2} & \text{when n} = \text{m} \end{cases}$$

Evaluation of Coefficients A_n and B_n

$$f(t) = \frac{A_o}{2} + \sum_{n=1}^{\infty} (A_n \cos n\omega_o t + B_n \sin n\omega_o t)$$

You will notice that the first term of the Fourier Series is written as $A_o/2$ rather than A_o. This is because it can be shown that A_o can also be evaluated using the same general expression as for A_n with n=0. It is also worth noting that $A_o/2$ also corresponds to the direct component of the waveform and may be obtained directly as the mean value of the waveform.

Let us now consider the general method of evaluation of coefficients.

Consider the integration of both sides of the Fourier series as follows.

$$\int_{t_0}^{t_0+T} f(t)\cdot dt = \int_{t_0}^{t_0+T} \frac{A_o}{2}\cdot dt + \int_{t_0}^{t_0+T} \sum_{n=1}^{\infty}(A_n \cos n\omega_o t + B_n \sin n\omega_o t)\cdot dt$$

using the properties of trigonometric functions derived earlier, it is evident that only the first term on the right hand side of the equation can give a non zero integral.

i.e. $$\int_{t_0}^{t_0+T} f(t)\cdot dt = \int_{t_0}^{t_0+T} \frac{A_o}{2}\cdot dt + 0 = \frac{A_o}{2}\cdot T$$

$$\therefore A_o = \frac{2}{T}\int_{t_o}^{t_o+T} f(t)\cdot dt$$

or from mean value we have $\frac{A_o}{2} = \frac{1}{T}\int_{t_o}^{t_o+T} f(t)\cdot dt$ which gives the same result.

Consider the integration of both sides of the Fourier series, after multiplying each term by $\cos n\omega_o t$ as follows.

$$\int_{t_0}^{t_0+T} f(t)\cdot \cos n\omega_o t\cdot dt = \int_{t_0}^{t_0+T} \frac{A_o}{2}\cdot \cos n\omega_o t\cdot dt + \int_{t_0}^{t_0+T} \sum_{n=1}^{\infty}(A_n \cos n\omega_o t + B_n \sin n\omega_o t)\cdot \cos n\omega_o t\cdot dt$$

using the properties of trigonometric functions derived earlier, it is evident that only $\cos n\omega_o t$ term on the right hand side of the equation can give a non zero integral.

$$\therefore A_n = \frac{2}{T}\int_{t_o}^{t_o+T} f(t)\cdot \cos n\omega_o t\cdot dt$$

Similarly integration of both sides of the Fourier series, after multiplying by $\sin n\omega_o t$ gives

$$\therefore B_n = \frac{2}{T}\int_{t_o}^{t_o+T} f(t)\cdot \sin n\omega_o t\cdot dt$$

Analysis of Symmetrical Waveforms

Even Symmetry

When even symmetry is present, the waveform from 0 to T/2 also corresponds to the mirror image of the waveform from –T/2 to 0. Therefore it is useful to select t_o = –T/2 and integrate from t = –T/2.

f(t) = f(-t)

$$A_n = \frac{2}{T}\int_0^T f(t)\cdot \cos n\omega_o t\cdot dt$$

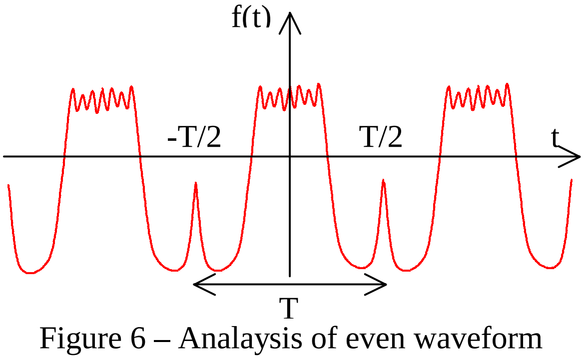

Figure 6 – Analaysis of even waveform

$$A_n = \frac{2}{T}\int_{-T/2}^{0} f(t)\cdot \cos n\omega_o t\cdot dt + \frac{2}{T}\int_0^{T/2} f(t)\cdot \cos n\omega_o t\cdot dt$$

If in the first part of the expression, if the variable 't' is replaced by the variable '-t' the equation may be re-written as

$$A_n = \frac{2}{T}\int_{T/2}^{0} f(-t)\cdot\cos(-n\omega_o t)\cdot(-dt) + \frac{2}{T}\int_{0}^{T/2} f(t)\cdot\cos n\omega_o t\cdot dt$$

Since the function is even, f(-t) = f(t), and $\cos(-n\omega_o t) = \cos(n\omega_o t)$.

Thus the equation may be simplified to

$$A_n = (-)\frac{2}{T}\int_{T/2}^{0} f(t)\cdot\cos(n\omega_o t)\cdot dt + \frac{2}{T}\int_{0}^{T/2} f(t)\cdot\cos n\omega_o t\cdot dt$$

The negative sign in front of the first integral can be replaced by interchanging the upper and lower limits of the integral. In this case it is seen that the first integral term and the second integral term are identical. Thus

$$A_n = \frac{2\times 2}{T}\int_{0}^{T/2} f(t)\cdot\cos n\omega_o t\cdot dt$$

Thus in the case of even symmetry, the value of A_n can be calculated as twice the integral over half the cycle from zero.

A similar analysis can be done to calculate B_n. In this case we would have

$$B_n = \frac{2}{T}\int_{T/2}^{0} f(-t)\cdot\sin(-n\omega_o t)\cdot(-dt) + \frac{2}{T}\int_{0}^{T/2} f(t)\cdot\sin n\omega_o t\cdot dt$$

Since the function is even, f(-t) = f(t), and $\sin(-n\omega_o t) = -\sin(n\omega_o t)$.

In this case the two terms are equal in magnitude but have opposite signs so that they cancel out.

Therefore $\mathbf{B_n = 0}$ for all values of **n** when the waveform has **even symmetry**.

Thus an ***even waveform*** will have only ***cosine*** terms and a direct term.

$$f(t) = \frac{A_o}{2} + \sum_{n=1}^{\infty} A_n \cos n\omega_o t$$

where $$A_n = \frac{4}{T}\int_{0}^{T/2} f(t)\cdot\cos n\omega_o t\cdot dt$$

Odd Symmetry

When odd symmetry is present, the waveform from 0 to T/2 also corresponds to the negated mirror image of the waveform from –T/2 to 0. Therefore as for even symmetry t_o = –T/2 is selected and integrated from t = –T/2.

f(t) = – f(-t)

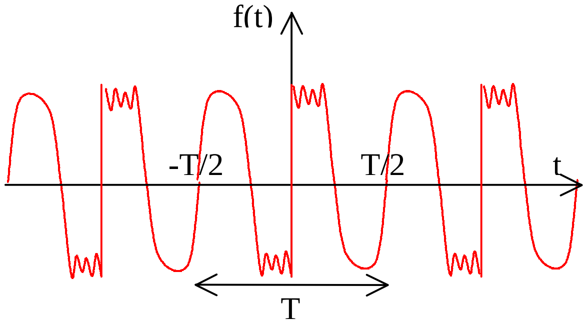

Figure 7 – Analaysis of odd waveform

$$A_n = \frac{2}{T}\int_0^T f(t)\cdot\cos n\omega_o t\cdot dt$$

$$A_n = \frac{2}{T}\int_{-T/2}^{0} f(t)\cdot\cos n\omega_o t\cdot dt + \frac{2}{T}\int_0^{T/2} f(t)\cdot\cos n\omega_o t\cdot dt$$

In the first part of the expression, if the variable 't' is replaced by the variable '-t' it can be easily seen that this part of the expression is exactly equal to the negative of the second part.

$$\therefore A_n = (-)\frac{2}{T}\int_0^{T/2} f(t)\cdot\cos n\omega_o t\cdot dt + \frac{2}{T}\int_0^{T/2} f(t)\cdot\cos n\omega_o t\cdot dt = 0$$ for all **n** for odd waveform

In a similar way, for B_n, the two terms can be seen to exactly add up.

Thus

$$B_n = \frac{2\times 2}{T}\int_0^{T/2} f(t)\cdot\sin n\omega_o t\cdot dt$$

Thus in the case of odd symmetry, the value of B_n can be calculated as twice the integral over half the cycle from zero.

Thus an ***odd waveform*** will have only ***sine*** terms and no direct term.

$$f(t) = \sum_{n=1}^{\infty} B_n \sin n\omega_o t$$

where $B_n = \frac{4}{T}\int_0^{T/2} f(t)\cdot\sin n\omega_o t\cdot dt$

Half-wave Symmetry

When half-wave symmetry is present, the waveform from **(t_o+T/2)** to **(t_o+T)** also corresponds to the negated value of the previous half cycle waveform from **t_o** to **(t_o+T/2)**.

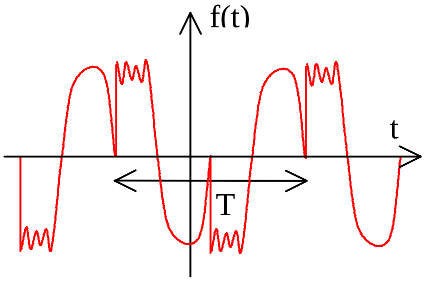

Figure 8 – Analysis of waveform with half wave symmetry

$$A_n = \frac{2}{T}\int_{t_o}^{t_o+T} f(t)\cdot\cos n\omega_o t\cdot dt$$

$$A_n = \frac{2}{T}\int_{t_o}^{t_o+T/2} f(t)\cdot\cos n\omega_o t\cdot dt + \frac{2}{T}\int_{t_o+T/2}^{t_o+T} f(t)\cdot\cos n\omega_o t\cdot dt$$

In the second part of the expression, the variable 't' is replaced by the variable 't-T/2'.

$$A_n = \frac{2}{T}\int_{t_o}^{t_o+T/2} f(t)\cdot\cos n\omega_o t\cdot dt + \frac{2}{T}\int_{t_o}^{t_o+T/2} f(t-T/2)\cdot\cos n\omega_o (t-T/2)\cdot d(t-T/2)$$

f(t-T/2) = – f(t) for half-wave symmetry, and

since $\omega_o T = 2\pi$, $\cos n\omega_o(t - T/2) = \cos(n\omega_o t - n\pi)$ which has a value of $(-)\cos n\omega_o t$ when n is odd and has a value of $\cos n\omega_o t$ when n is even.

From the above it follows that the second term is equal to the first term when n is odd and the negative of the first term when n is even.

Thus $$A_n = \frac{2\times 2}{T}\int_{t_o}^{t_o+T/2} f(t)\cdot \cos n\omega_o t\cdot dt$$ when n is odd

and $A_n = 0$ when n is even

Similarly it can be shown that

$$B_n = \frac{2\times 2}{T}\int_{t_o}^{t_o+T/2} f(t)\cdot \sin n\omega_o t\cdot dt$$ when n is odd

and $B_n = 0$ when n is even

Thus it is seen that in the case of ***half-wave symmetry***, even harmonics do not exist and that for the odd harmonics the coefficients A_n and B_n can be obtained by taking double the integral over any half cycle.

It is to be noted that many practical waveforms have half-wave symmetry due to natural causes.

Summary of Analysis of waveforms with symmetrical properties

1. With **even symmetry**, B_n is 0 for all n, and A_n is twice the integral over half the cycle from zero time.
2. With **odd symmetry**, A_n is 0 for all n, and B_n is twice the integral over half the cycle from zero time.
3. With **half-wave symmetry**, A_n and B_n are 0 for even n, and twice the integral over any half cycle for odd n.
4. If **half-wave symmetry** and either **even symmetry** or **odd symmetry** are present, then A_n and B_n are 0 for even n, and four times the integral over the quarter cycle for odd n for $\mathbf{A_n}$ or $\mathbf{B_n}$ respectively and zero for the remaining coefficient.
5. It is also to be noted that in any waveform, $\mathbf{A_o/2}$ corresponds to the mean value of the waveform and that sometimes a symmetrical property may be obtained by subtracting this value from the waveform.

Piecewise Continuous waveforms

Most waveforms occurring in practice are continuous and single valued (i.e. having a single value at any particular instant). However when sudden changes occur (such as in switching operations) or in square waveforms, theoretically vertical lines could occur in the waveform giving multi-values at these instants. As long as these multi-values occur over finite bounds, the waveform is single-valued and continuous in pieces, or said to be **Piecewise continuous.**

Figure 9 shows such a waveform. Analysis can be carried out using the Fourier Series for both continuous or piecewise continuous waveforms. However in the case of piecewise continuous waveforms, the value calculated from the Fourier Series for the waveform at the discontinuities would correspond to the mean value of the vertical region. However this is not a practical problem as practical waveforms will not have exactly vertical changes but those occurring over very small intervals of time.

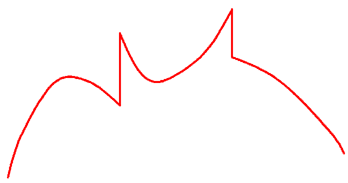

Figure 9 – Piecewise continuous waveform

Frequency Spectrum

The frequency spectrum is the plot showing each of the harmonic amplitudes against frequency. In the case of periodic waveforms, these occur at distinct points corresponding to d.c., fundamental and the harmonics. Thus the spectrum obtained is a line spectrum. In practical waveforms, the higher harmonics have significantly lower amplitudes compared to the lower harmonics. For smooth waveforms, the higher harmonics will be negligible, but for waveforms with finite discontinuities (such as square waveform) the harmonics do not decrease very rapidly. The harmonic magnitudes are taken as $\sqrt{A_n^2 + B_n^2}$ for the n^{th} harmonic and has thus a positive value. Each component also has a phase angle which can be determined.

Example 1

Find the Fourier Series of the piecewise continuous rectangular waveform shown in figure 10.

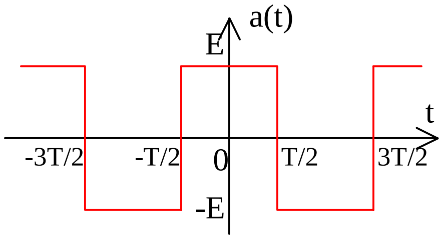

Figure 10 – Rectangular waveform

Solution

Period of waveform = 2T

Mean value of waveform = 0. $\therefore$ $A_o/2 = 0$

Waveform has even symmetry. $\therefore$ $B_n = 0$ for all n

Waveform has half-wave symmetry. $\therefore$ $A_n, B_n = 0$ for even n

Therefore, A_n can be obtained for odd values of n as 4 times the integral over quarter cycle as follows.

$$A_n = \frac{4 \times 2}{2T} \int_0^{2T/4} a(t) \cdot \cos n\omega_o t \cdot dt \quad \text{for odd n}$$

$$= \frac{4}{T} \int_0^{T/2} E \cdot \cos n\omega_o t \cdot dt = \frac{4}{n\omega_o T} \cdot E \cdot \sin n\omega_o t \Big|_0^{T/2} = \frac{4E}{n\omega_o T} \cdot \sin \frac{n\omega_o T}{2} = \frac{4E}{n\pi} \cdot \sin n\frac{\pi}{2} \quad \text{for odd n}$$

i.e. $A_1 = 4E/\pi$, $A_3 = -4E/3\pi$, $A_5 = 4E/5\pi$, $A_7 = -4E/7\pi$, ...

$$\therefore \text{a(t)} = \frac{4E}{\pi}\left[\cos\omega_o t - \frac{\cos 3\omega_o t}{3} + \frac{\cos 5\omega_o t}{5} - \frac{\cos 7\omega_o t}{7} + \ldots\ldots\right]$$

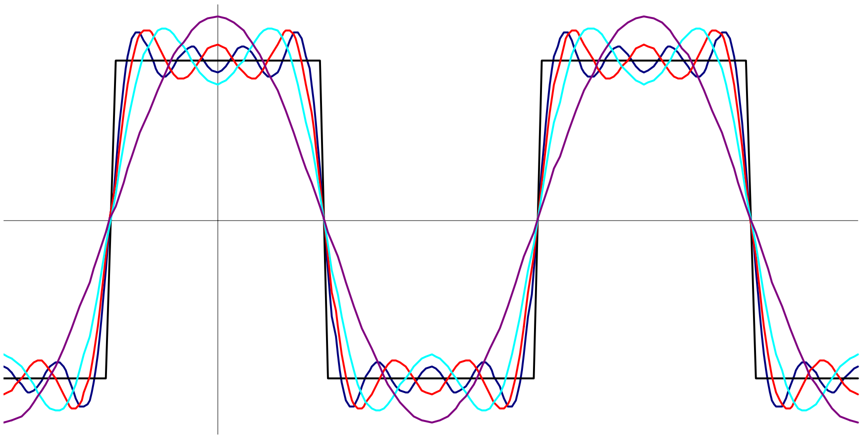

Figure 11 – Fourier Synthesis of Rectangular Waveform

Figure 11 shows the synthesis of the waveform using the Fourier components. The waveform shown correspond to the (i) original waveform, (ii) fundamental component only, (iii) fundamental component + third harmonic, (iv) fundamental component + third harmonic + fifth harmonic, and (v) fundamental, third, fifth and seventh harmonics.

You can see that with the addition of each component, the waveform approaches the original waveform more closely, however without an infinite number of components it will never become exactly equal to the original.

The frequency spectrum of the waveform is shown in figure 12.

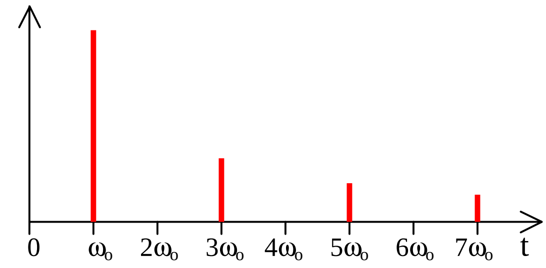

Figure 12 – Line Spectrum

Let us now consider the same rectangular waveform but with a few changes.

Example 2

Find the Fourier series of the waveform shown in figure 13.

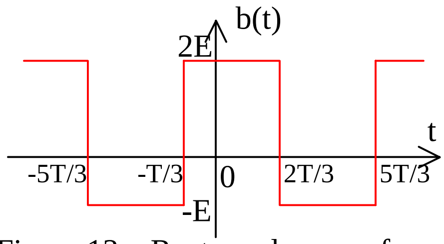

Figure 13 – Rectangular waveform

Solution

It is seen that the waveform does not have any symmetrical properties although it is virtually the same waveform that was there in example 1.

Period = 2T, mean value = E/2

It is seen that if E/2 is subtracted from the waveform b(t)

It is also seen that the waveform is shifted by T/6 to the right from the position for even symmetry.

Thus consider the waveform $b_1(t) = b(t - T/6) - E/2$.

This is shown in figure 14 and differs from the waveform a(t) in figure 10 in magnitude only (1.5 times). Therefore the analysis of $b_1(t)$ can be obtained directly from the earlier analysis.

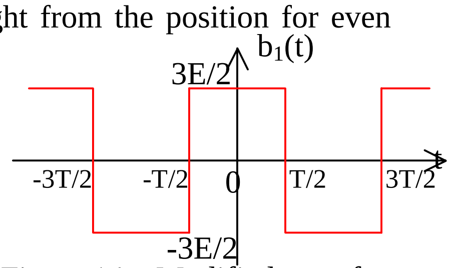

Figure 14 – Modified waveform

$$\therefore\ b_1(t) = 1.5 \times a(t) = \frac{6E}{\pi}\left[\cos\omega_o t - \frac{\cos 3\omega_o t}{3} + \frac{\cos 5\omega_o t}{5} - \frac{\cos 7\omega_o t}{7} + \ldots\ldots\right] \quad \text{where } \omega_o 2T = 2\pi$$

$$\therefore\ b(t) = b_1(t + T/6) + E/2, \quad \text{also } \omega_o T/6 = \pi/6$$

$$= \frac{E}{2} + \frac{6E}{\pi}\left[\cos(\omega_o t + \pi/6) - \frac{\cos(3\omega_o t + \pi/2)}{3} + \frac{\cos(5\omega_o t + 5\pi/6)}{5} - \frac{\cos(7\omega_o t + 7\pi/6)}{7} + \ldots\ldots\right]$$

If the problem was worked from first principles the series would first have been obtained as a sum of sine and cosine series whose resultant would be the above answer. Figure 15 shows the corresponding line spectrum. The only differences from the earlier one are that the amplitudes are 1.5 times higher and a d.c. term is present.

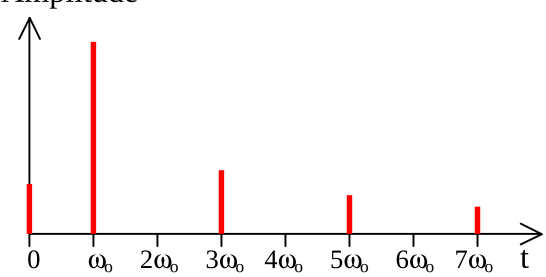

Figure 15 – Line Spectrum

Example 3

Find the Fourier Series of the triangular waveform shown in figure 16.

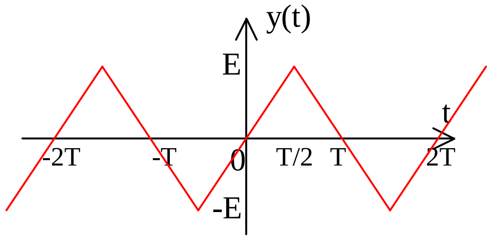

Figure 16 – Triangular waveform

Solution

Period of waveform = 2T, $\omega_o.2T = 2\pi$

Mean value of waveform = 0. $\therefore A_o/2 = 0$

Waveform has odd symmetry. $\therefore A_n = 0$ for all n

Waveform has half-wave symmetry. $\therefore A_n, B_n = 0$ for even n

$$B_n = \frac{4\times 2}{2T}\int_0^{2T/4}\frac{2E}{T}\cdot t\cdot \sin n\omega_o t\cdot dt \quad = \frac{8E}{T^2}\cdot t\cdot\frac{\cos n\omega_o t}{n\omega_o}\Bigg|_0^{\frac{T}{2}} - \frac{8E}{T^2}\int_0^{T/2}\frac{\cos n\omega_o t}{n\omega_o}\cdot dt \quad \text{for odd n}$$

$$= \frac{8E}{T^2}\cdot\frac{T}{2}\cdot\frac{\cos n\omega_o T/2}{n\omega_o} + \frac{8E}{T^2}\cdot\frac{(\sin n\omega_o T/2 - 0)}{(n\omega_o)^2}$$

$$= \frac{4E\cdot\cos n\pi/2}{n\pi} + \frac{8E\cdot\sin n\pi/2}{(n\pi)^2}$$

Substituting values

$B_1 = 8E/\pi^2$, $B_3 = -8E/(3\pi)^2$, $B_5 = 8E/(5\pi)^2$, $B_7 = -8E/(7\pi)^2$, ...

$$\therefore y(t) = \frac{8E}{\pi^2}\left[\sin\omega_o t - \frac{\sin 3\omega_o t}{3^2} + \frac{\sin 5\omega_o t}{5^2} - \frac{\sin 7\omega_o t}{7^2} + ...\right]$$

Consider the derivative of the original waveform y(t). This would have the waveform shown in figure 17 which corresponds to the same type of rectangular waveform that we had in example 1 except that the amplitude is 2/T times higher. Thus by using the integral of the analysed original waveform we should also be able to obtain the above result for y(t).

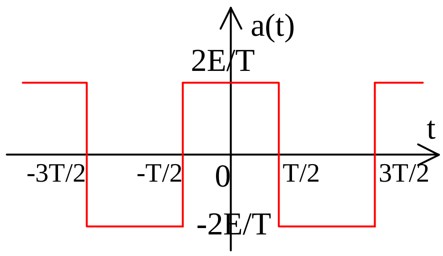

Figure 17 – Rectangular waveform

Using earlier solution, we have

$$a(t) \quad = \frac{2}{T}\cdot\frac{4E}{\pi}\left[\cos\omega_o t - \frac{\cos 3\omega_o t}{3} + \frac{\cos 5\omega_o t}{5} - \frac{\cos 7\omega_o t}{7} + ...\right]$$

$$\therefore y(t) \quad = \int a(t)\,dt = \int\frac{2}{T}\cdot\frac{4E}{\pi}\left[\cos\omega_o t - \frac{\cos 3\omega_o t}{3} + \frac{\cos 5\omega_o t}{5} - \frac{\cos 7\omega_o t}{7} + ...\right]\cdot dt$$

$$= \frac{2}{T}\cdot\frac{4E}{\pi}\left[\frac{\sin\omega_o t}{\omega_o} - \frac{\sin 3\omega_o t}{3^2\omega_o} + \frac{\sin 5\omega_o t}{5^2\omega_o} - \frac{\sin 7\omega_o t}{7^2\omega_o} + ...\right]$$

which when simplified is identical to the result obtained using the normal method.

Example 4

Find the Fourier series of the waveform shown in figure 18.

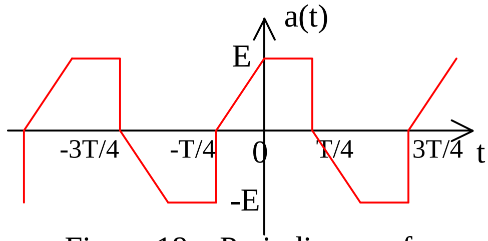

Figure 18 – Periodic waveform

Solution

Period = T, $\therefore \omega_o T = 2\pi$

Mean value = 0, $\therefore A_o/2 = 0$

Waveform possesses half-wave symmetry. ∴ even harmonics are absent.

$$\therefore\ A_n = \frac{4}{T}\int_0^{T/2} f(t)\cdot\cos n\omega_o t\cdot dt = \frac{4}{T}\int_0^{T/4} E\cdot\cos n\omega_o t\cdot dt + \frac{4}{T}\int_{T/4}^{T/2}(E-\frac{4E}{T}\cdot t)\cdot\cos n\omega_o t\cdot dt$$

$$= \left[\frac{4}{T}E\cdot\frac{\sin n\omega_o t}{n\omega_o}\right]_0^{\frac{T}{4}} + \frac{4}{T}\cdot(E-\frac{4E}{T}\cdot t)\cdot\frac{\sin n\omega_o t}{n\omega_o}\Bigg|_{\frac{T}{4}}^{\frac{T}{2}} - \frac{4}{T}\cdot\int_{T/4}^{T/2}(-\frac{4E}{T})\cdot\frac{\sin n\omega_o t}{n\omega_o}\cdot dt$$

$$= \left[\frac{4}{T}E\cdot\frac{\sin n\omega_o \frac{T}{4}}{n\omega_o}\right] + \frac{4}{T}\cdot(-E)\cdot\frac{\sin n\omega_o \frac{T}{2}}{n\omega_o} + \frac{16E}{T^2}\cdot\frac{(-\cos n\omega_o t)}{(n\omega_o)^2}\Bigg|_{\frac{T}{4}}^{\frac{T}{2}}$$

since $\omega_o T = 2\pi$, $\omega_o T/4 = \pi/2$ and $\omega_o T/2 = \pi$

$$\therefore\ A_n = 4E\cdot\frac{\sin\frac{n\pi}{2}}{n\cdot 2\pi} - 4E\cdot\frac{\sin n\pi}{n\cdot 2\pi} + 16E\cdot\frac{(-\cos n\pi + \cos\frac{n\pi}{2})}{(n\cdot 2\pi)^2} \quad \text{for odd n.}$$

Substituting different values of n, we have

$$A_1 = -\frac{2E}{\pi} - 0 - \frac{4E}{\pi^2} = 1.0419E$$

$A_3 = -0.1672E$, $A_5 = 0.1435E$, $A_7 = -0.08267E$

Similarly, the B_n terms for odd n are given as follows.

$$B_n = \frac{4}{T}\int_0^{T/2} f(t)\cdot\sin n\omega_o t\cdot dt = \frac{4}{T}\int_0^{T/4} E\cdot\sin n\omega_o t\cdot dt + \frac{4}{T}\int_{T/4}^{T/2}(E-\frac{4E}{T}\cdot t)\cdot\sin n\omega_o t\cdot dt$$

$$= \left[\frac{4}{T}E\cdot\frac{\cos n\omega_o t}{(-)n\omega_o}\right]_0^{\frac{T}{4}} + \frac{4}{T}\cdot(E-\frac{4E}{T}\cdot t)\cdot\frac{\cos n\omega_o t}{(-)n\omega_o}\Bigg|_{\frac{T}{4}}^{\frac{T}{2}} - \frac{4}{T}\cdot\int_{T/4}^{T/2}(-\frac{4E}{T})\cdot\frac{\cos n\omega_o t}{(-)n\omega_o}\cdot dt$$

$$= \left[\frac{4}{T}E\cdot\frac{(1-\cos n\omega_o \frac{T}{4})}{n\omega_o}\right] + \frac{4}{T}\cdot E\cdot\frac{\cos n\omega_o \frac{T}{2}}{n\omega_o} + \frac{16E}{T^2}\cdot\frac{(-)\sin n\omega_o t}{(n\omega_o)^2}\Bigg|_{\frac{T}{4}}^{\frac{T}{2}}$$

since $\omega_o T = 2\pi$, $\omega_o T/4 = \pi/2$ and $\omega_o T/2 = \pi$

$$\therefore\ B_n = 4E\cdot\frac{(1-\cos\frac{n\pi}{2})}{n\cdot 2\pi} + 4E\cdot\frac{\cos n\pi}{n\cdot 2\pi} + 16E\cdot\frac{(-\sin n\pi + \sin\frac{n\pi}{2})}{(n\cdot 2\pi)^2} \quad \text{for odd n.}$$

Substituting different values of n, we have

B_1 = 0.4053E, B_3 = -0.04503E, B_5 = 0.0162E, B_7 = -0.00827

The amplitudes can now be obtained for each frequency component from A_n and B_n.

d.c. term = 0, amplitude 1 = $\sqrt{1.0419^2 + 0.4053^2}$ = 1.1180, amplitude 3 = 0.1731, amplitude 5 = 0.1444, amplitude 7 = 0.08309, …….

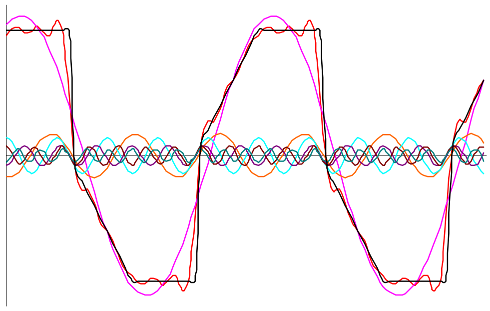

Figure 19 – Synthesised waveform

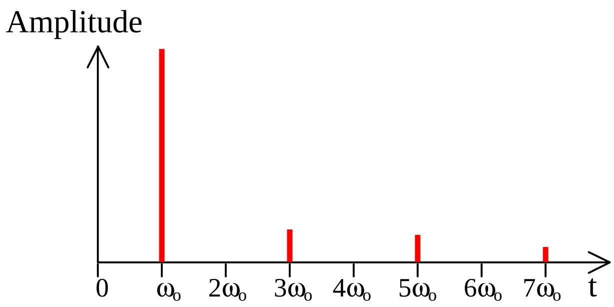

Figure 20 – Line Spectrum

Figure 19 shows the synthesised waveform (red) and its components up to the 29th harmonic (odd harmonics only) along with the original waveform (black). Figure 20 shows the line spectrum of the waveform of the first 7 harmonics.

Example 5

Figure 21 shows a waveform obtained from a power electronic circuit. Determine its Fourier Series if it is defined as follows for one cycle.

f(t) = 100 cos 314.16 t for –0.333 < t < 2.5 ms

f(t) = 86.6 cos (314.16 t – 0.5236) for 2.5 < t < 3.0 ms

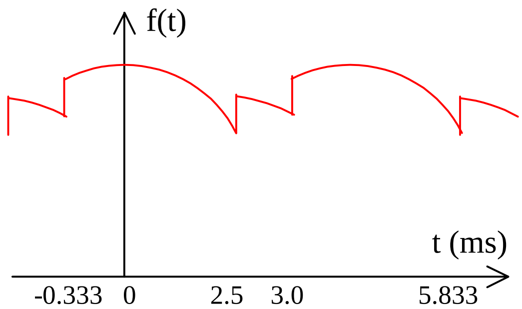

Figure 21 – Power electronics waveform

Solution

The waveform does not have any symmetrical properties. It has a period of 3.333 ms.

T = 0.003333 s, $\omega_o = 2\pi/T$ = 1885 rad/s

$$A_n = \frac{2}{T}\int_{-0.000333}^{0.0025} f(t)\cdot\cos n\omega_o t\cdot dt + \frac{2}{T}\int_{0.0025}^{0.003} f(t)\cdot\cos n\omega_o t\cdot dt$$

$$= \frac{2}{T}\int_{-0.000333}^{0.0025} 100\cdot\cos 314.16t\cdot\cos n\omega_o t\cdot dt + \frac{2}{T}\int_{0.0025}^{0.003} 86.6\cdot\cos(314.16t - 0.5236)\cdot\cos n\omega_o t\cdot dt$$

$$= \frac{100}{T}\int_{-0.000333}^{0.0025} (\cos(314.16t + n\omega_o t) + \cos(314.16t - n\omega_o t)\cdot dt$$

$$+ \frac{86.6}{T}\int_{0.0025}^{0.003} \cos(314.16t - 0.5236 + n\omega_o t) + \cos(314.16t - 0.5236 - n\omega_o t)\cdot dt$$

$$= \frac{100}{T}\left(\frac{\sin(314.16t + n\omega_o t)}{314.16 + n\omega_o} + \frac{\sin(314.16t - n\omega_o t)}{314.16 - n\omega_o}\right)_{-0.000333}^{0.0025}$$

$$+ \frac{86.6}{T}\left(\frac{\sin(314.16t - 0.5236 + n\omega_o t)}{314.16 + n\omega_o} + \frac{\sin(314.16t - 0.5236 - n\omega_o t)}{314.16 - n\omega_o}\right)_{0.0025}^{0.003}$$

$$= \frac{100}{0.003333}\left(\frac{\sin(314.16t + n1885t)}{314.16 + n1885} + \frac{\sin(314.16t - n1885t)}{314.16 - n1885}\right)_{-0.000333}^{0.0025}$$

$$+ \frac{86.6}{0.003333}\left(\frac{\sin(314.16t - 0.5236 + n1885t)}{314.16 + n1885} + \frac{\sin(314.16t - 0.5236 - n1885t)}{314.16 - n1885}\right)_{0.0025}^{0.003}$$

$$= 30\times10^3\left(\frac{\sin(0.7854 + 4.712n)}{314.16 + n1885} + \frac{\sin(0.7854 - 4.712n)}{314.16 - n1885} - \frac{\sin(-0.1046 + 0.6277n)}{314.16 + n1885} - \frac{\sin(-0.1046 + 0.6277n)}{314.16 - n1885}\right)$$

$$+ 25.98\times10^3\left(\begin{array}{l}\frac{\sin(0.9422 - 0.5236 + 5.655n)}{314.16 + n1885} + \frac{\sin(0.9422 - 0.5236 - 5.655n)}{314.16 - n1885} - \\ \frac{\sin(0.7854 - 0.5236 + 4.7125n)}{314.16 + n1885} - \frac{\sin(0.7854 - 0.5236 - 4.7125n)}{314.16 - n1885}\end{array}\right)$$

Substituting values, A_1, A_2, A_3, A_4, A_5, A_6, …. can be determined.

In a similar manner B_1, B_2, B_3, B_4, B_5, B_6, …. can be determined.

The Fourier Series of the waveform can then be determined.

The remaining calculations of the problem are left to the reader as an exercise.

Effective Value of a Periodic Waveform

The effective value of a periodic waveform is also defined in terms of power dissipation and is hence the same as the r.m.s. value of the waveform.

$$A_{effective} = \sqrt{\frac{1}{T}\int_{to}^{to+T} a^2(t)\cdot dt}$$

Since the periodic waveform may be defined as

$$a(t) = \frac{A_o}{2} + \sum_{n=1}^{\infty} A_n \cos n\omega_o t + \sum_{n=1}^{\infty} B_n \sin n\omega_o t$$

$$A_{eff} = \sqrt{\frac{1}{T}\int_{to}^{to+T}\left[\frac{A_o}{2} + \sum_{n=1}^{\infty} A_n \cos n\omega_o t + \sum_{n=1}^{\infty} B_n \sin n\omega_o t\right]^2 dt}$$

$$A_{eff} = \sqrt{\frac{1}{T}\int_{to}^{to+T}\left[\left(\frac{A_o}{2}\right)^2 + \sum_{n=1}^{\infty}\left(A_n \cos n\omega_o t\right)^2 + \sum_{n=1}^{\infty}\left(B_n \sin n\omega_o t\right)^2 + \sum product\ terms\right]dt}$$

Using the trigonometric properties derived earlier, only the square terms will give non-zero integrals. The *Product terms* will all give zero integrals.

$$A_{eff} = \sqrt{\frac{1}{T}\left[\left(\frac{A_o}{2}\right)^2\cdot T + \sum_{n=1}^{\infty} A_n^{\ 2}\cdot\frac{T}{2} + \sum_{n=1}^{\infty} B_n^{\ 2}\cdot\frac{T}{2}\right]} = \sqrt{\left(\frac{A_o}{2}\right)^2 + \sum_{n=1}^{\infty}\frac{A_n^{\ 2}}{2} + \sum_{n=1}^{\infty}\frac{B_n^{\ 2}}{2}}$$

$\frac{A_o}{2}$ is the d.c. term, and $\frac{\sqrt{(A_n^2 + B_n^2)}}{\sqrt{2}}$ is the r.m.s. value of the n^{th} harmonic.

Thus the effective value or r.m.s. value of a periodic waveform is the square root of the sum of the squares of the r.m.s. components.

Calculation of Power and Power Factor associated with Periodic Waveforms

Consider the voltage waveform and the current waveform to be available as Fourier Series of the same fundamental frequency ω_o as follows.

$$v(t) = V_{dc} + \sum_{n=1}^{\infty} V_n \sin(n\omega t + \alpha_n) \quad \text{and} \quad i(t) = V_{dc} + \sum_{n=1}^{\infty} I_n \sin(n\omega t + \beta_n)$$

$p(t) = v(t).i(t)$ and, average power P is given by

$$P = \frac{1}{T}\int_{to}^{to+T} v(t).i(t).dt = \frac{1}{T}\int_{to}^{to+T} \left[V_{dc} + \sum V_n \sin(n\omega\, t + \alpha_n)\right]\cdot\left[I_{dc} + I_n \sin(n\omega\, t + \beta_n)\right]dt$$

Using the trigonometric properties, it can be easily seen that only similar terms from v and I can give rise to non-zero integrals.

Thus $P = V_{dc}.I_{dc} + \Sigma\,(1/2)V_n I_n \cos(\alpha_n - \beta_n) = V_{dc}.I_{dc} + \Sigma\, V_{rms,n} I_{rms,n} \cos(\alpha_n - \beta_n)$

Thus the total power is given as the sum of the powers of the individual harmonics including the fundamental and the direct term.

Example 6

Determine the effective values of the voltage and the current, the total power consumed, the overall power factor and the fundamental displacement factor, if the Fourier series of the voltage and current are given as follows.

$v(t) = 5 + 8\sin(\omega t + \pi/6) + 2\sin 3\omega t$ volt

$i(t) = 3 + 5\sin(\omega t + \pi/2) + 1\sin(2\omega t - \pi/3) + 1.414\cos(3\omega t + \pi/4)$ ampere

Solution

$$V_{rms} = \sqrt{5^2 + \left(\frac{8}{\sqrt{2}}\right)^2 + \left(\frac{2}{\sqrt{2}}\right)^2} = 7.681 \text{ V}$$

$$I_{rms} = \sqrt{3^2 + \left(\frac{5}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1.414}{\sqrt{2}}\right)^2} = 4.796 \text{ A}$$

$$P = 5\times 3 + (8/\sqrt{2}).(5/\sqrt{2}).\cos(\pi/3) + 0 + (2/\sqrt{2}).(1.414/\sqrt{2}).\cos(\pi/2 + \pi/4)$$
$$= 15 + 10 - 1 = 24 \text{ W}$$

The ***overall power factor*** of a periodic waveform is defined as the ratio of the active power to the apparent power. Thus

Overall power factor = $24/(7.681 \times 4.796) = 0.651$

In the case of non-sinusoidal waveforms, the power factor is not associated with lead or lag as these no longer have any meaning.

The ***fundamental displacement factor*** corresponds to power factor of the fundamental. It tells us by how much the fundamental component of current is displaced from the fundamental component of voltage, and hence is also associated with the terms lead and lag.

Fundamental displacement factor (FDF) = $\cos\phi_1 = \cos(\pi/2 - \pi/6) = \cos\pi/3 = 0.5$ lead

Note that the term lead is used as the original current is leading the voltage by an angle $\pi/3$.

Analysis of Circuits in the presence of Harmonics in the Source

Due to the presence of non-linear devices in the system, voltages and currents get distorted from the sinusoidal. Thus it becomes necessary to analyse circuits in the presence of distortion in the source. This can be done by using the Fourier Series of the supply voltage and the principle of superposition.

For each frequency component, the circuit is analysed as for pure sinusoidal quantities using normal complex number analysis, and the results are summed up to give the resultant waveform.

Example 7

Determine the voltage across the load R for the supply voltage e(t) applied to the circuit shown in figure 22.

$e(t) = 100 + 30 \sin(300t + \pi/6) + 20 \sin 900t +$

$15 \sin(1500t - \pi/6) + 10 \sin 2100t$

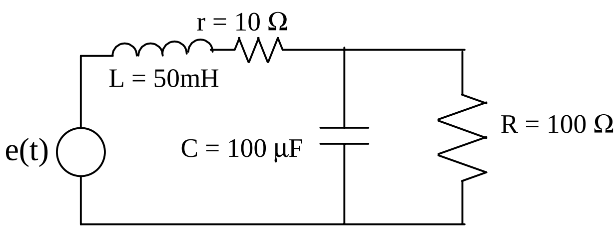

Figure 22 – Circuit with distorted source

Solution

For the d.c. term,

$$\frac{V_{dc}}{100} = \frac{100}{100+10}\,.\quad \therefore V_{dc} = 90.91 \text{ V}$$

For any a.c. term, if V_{nm} is the peak value of the n^{th} harmonic of the output voltage, then

$$\frac{V_{nm}/\sqrt{2}}{E_{nm}/\sqrt{2}} = \frac{C // R}{L + r + C // R} = \frac{R/(1+j\omega CR)}{j\omega L + r + R/(1+j\omega CR)} = \frac{R}{R + (j\omega L + r)(1+j\omega CR)}$$

$$V_{nm} = E_{nm} \cdot \frac{100}{100 + (j300n \times 0.050 + 10)(1 + j300n \times 100 \times 10^{-6} \times 100)}$$

$$V_{nm} = E_{nm} \cdot \frac{100}{100 + (j15n + 10)(1 + j3n)}$$

$$V_{nm} = E_{nm} \cdot \frac{100}{110 - 45n^2 + j45n}$$

for the fundamental

$V_{1m} = 30\times100/(65+j45) \qquad = 3000/79.06\angle 34.7^o = 37.95\angle -34.7^o$

$V_{3m} = 20\times100/(-295+j135) \qquad = 2000/324.42\angle 155.4^o = 6.16\angle -155.4^o$

$V_{5m} = 15\times100/(-1015+j225) \qquad = 1500/1039.64\angle 167.5^o = 1.44\angle -167.5^o$

$V_{7m} = 10\times100/(-2095+j315) \qquad = 1000/2118.5\angle 171.4^o = 0.47\angle -171.4^o$

$v(t) = 90.91 + 37.95 \sin(300t + 30^o - 34.7^o) + 6.16 \sin(900t - 155.4^o) + 1.44 \sin(1500t - 30^o - 167.5^o) + 0.47 \sin(2100t - 171.4^o)$

$v(t) = 90.91 + 37.95 \sin(300t - 4.7^o) + 6.16 \sin(900t - 155.4^o) + 1.44 \sin(1500t - 197.5^o)$

$+ 0.47 \sin(2100t - 171.4^o)$

Complex form of the Fourier Series

It would have been noted that the only frequency terms that were considered were positive frequency terms going up to infinity but that time was not limited to positive values. Mathematically speaking, frequency can have negative values, but as will be obvious, negative frequency terms would have a positive frequency term giving the same Fourier component. In the complex form, negative frequency terms are also defined.

Using the trigonometric expressions

$e^{j\theta} = \cos\theta + j\sin\theta$ and $e^{j\theta} = \cos\theta + j\sin\theta$

we may rewrite the Fourier series in the following manner.

$$f(t) = \frac{A_o}{2} + \sum_{n=1}^{\infty} (A_n \cos n\omega_o t + B_n \sin n\omega_o t)$$

$$f(t) = \frac{A_o}{2} + \sum_{n=1}^{\infty} A_n \cdot \left(\frac{e^{jn\omega_o t} + e^{-jn\omega_o t}}{2} \right) + B_n \cdot \left(\frac{e^{jn\omega_o t} - e^{-jn\omega_o t}}{2j} \right)$$

This can be re-written in the following form

$$f(t) = \frac{A_o - j0}{2} + \sum_{n=1}^{\infty} e^{jn\omega_o t} \cdot \left(\frac{A_n - jB_n}{2} \right) + e^{-jn\omega_o t} \cdot \left(\frac{A_n + jB_n}{2} \right)$$

It is to be noted that B_o is always 0, so that the ***j0*** with A_o may be written as jB_o. Also $e^{j0} = 1$

Thus defining $C_n = \frac{A_n - jB_n}{2}$, we have $C_0 = \frac{A_0 - j0}{2}$ and $C_{-n} = \frac{A_{-n} - jB_{-n}}{2}$

the term on the right hand side outside the summation can be written as $C_o\, e^{j0}$ and the first term inside the summation becomes $C_n\, e^{jn\omega_o t}$.

Since $A_n = \frac{2}{T} \int_{t_o}^{t_o+T} f(t) \cdot \cos n\omega_o t \cdot dt$, $A_{-n} = \frac{2}{T} \int_{t_o}^{t_o+T} f(t) \cdot \cos(-n\omega_o t) \cdot dt = A_n$

and $B_n = \frac{2}{T} \int_{t_o}^{t_o+T} f(t) \cdot \sin n\omega_o t \cdot dt$, $B_{-n} = \frac{2}{T} \int_{t_o}^{t_o+T} f(t) \cdot \sin(-n\omega_o t) \cdot dt = -B_n$

$$\therefore C_{-n} = \frac{A_{-n} - jB_{-n}}{2} = \frac{A_n + jB_n}{2}$$

That is, the second term inside the summation becomes $C_{-n}\, e^{-jn\omega_o t}$.

Thus the three sets of terms in the equation correspond to the zero term, the positive terms and the negative terms of frequency.

Therefore the Fourier Series may be written in complex form as

$$f(t) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{jn\omega_o t}$$

and the Fourier coefficient C_n can be calculated as follows.

$$C_n = \frac{A_n - jB_n}{2} = \frac{1}{2} \cdot \frac{2}{T} \int_{t_o}^{t_o+T} f(t) \cdot [\cos n\omega_o t - j \sin n\omega_o t] \cdot dt = \frac{1}{T} \int_{t_o}^{t_o+T} f(t) \cdot e^{-jn\omega_o t} \cdot dt$$